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Countable Semiadditive Functionals and the Hardy–Littlewood Maximal Operator

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Abstract: We describe the continuity of nonlinear Hardy–Littlewood maximal operator in nonmetricable function space $X^{p_0} = \bigcup_{p>p_o} L^p(\Omega)$, $1 < p_0 < \infty$, Ω is a measurable subset of R^n with finite measure.

Key words: Hardy–Littlewood; Maximal function; Countably semiadditive; Inductive limit of Lebesgue spaces

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1. INTRODUCTION

The present article consists of two parts. The first part is devoted to the classical operator defined by the Hardy–Littlewood maximal function in the spaces $L^p(\Omega)$, where Ω is a measurable subset of R^n. The maximal function operator was introduced in the one-dimensional case by Hardy and Littlewood in 1930. The boundedness of this operator in $L^p(R)$ $(1<p\leq\infty)$ was actually established by them. The fundamental multi-dimensional inequality of weak type for the maximal function was obtained in the late 1930s by N. Wiener. Various modifications of the maximal function operator have been studied intensively also in recent

times. This is connected with the fact that the maximal function operator is a very important operator in analysis [01-02]. It appears in the theory of Fourier series and in measure theory, in the theory of differentiation of integrals with respect to various collections of sets, in the theory of singular integrals, and so on (see, e.g., [03]). Therefore interest in it has not diminished even at the present time. Along the lines of the classical bounds of this operator we provide theorems concerning the boundedness of the Hardy–Littlewood maximal function operator in the inductive limit of Lebesgue spaces. These theorems are useful for the investigation of extremal function spaces (not necessarily Banach spaces), in which the given operator is bounded or continuous.

As usual we will denote Lebesgue measure by μ and by $S(\mu)=S(\mathrm{R}^n, \Sigma, \mu)$ the space of functions on R^n which are measurable with respect to Σ, the algebra of μ -measurable sets in R^n. We will denote by $\| f \,|\, X \|$ the norm of f in a normed space $(X,\|.\|)$.

2. HARDY–LITTLEWOOD MAXIMAL FUNCTION OPERATOR

For a function $f \in L^{1,loc}$ Hardy and Littlewood introduced a new function *Mf*, which plays an important role in the theory of functions of a real variable and in all harmonic analysis. This function can be defined for each $x\in\mathrm{R}^n$ by means of the identity:

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(t)|dt$$

Here, as usual, $B(x,r)$ is the open ball of radius r with centre at the point x. The function $Mf(x)$ is measurable since the set $\{x{:}Mf(x)>a\}$ is open for any $a>0$. The operator $Mf(x)$ is called the *Hardy–Littlewood maximal operator*.

This operator $Mf(x){:}L^{1,loc} \to S(\mu)$ has the following properties:

(i) $Mf(x)\geq 0$ for any $f(x)\in L^{1,loc}$ and $x\in\mathrm{R}^n$;

(ii) for any $f_1(x), f_2(x)\in L^{1,loc}$, we have the inequality:

$$M(f_1+f_2)(x) \leq Mf_1(x) + Mf_2(x)$$

(iii) for any $f(x)\in L^{1,loc}$, $a\in\mathrm{R}$ and $x\in\mathrm{R}^n$, we have the identity:

$$M(\alpha f)(x) = |\alpha|\, Mf(x)$$

Thus we can say that the operator $Mf(x)$ is positive (property (i)), subadditive (property (ii)), and positively homogeneous (property (iii)).

The operator $Mf(x)$ clearly satisfies the inequality:

$$\| Mf \,|\, L^\infty \| \leq \| f \,|\, L^\infty \|$$

Because of subadditivity the last inequality implies the continuity of the operator $M{:}L^\infty \to L^\infty$ and also the continuity and countable semiadditivity of the functional $\rho(f)=\|Mf|L^\infty\|$ on L^∞ [04]. As we will now see, this operator also satisfies another important inequality.

We say that a positively homogeneous subadditive operator T is *of weak type* (p,p) $(1\le p<\infty)$if for any $(a>0)$ the inequality

$$\mu\{x:|Tf(x)|>\alpha\}\le\left(\frac{c(p)\|f\,|\,L^p\|}{\alpha}\right)^p$$

holds, with constant $c(p)$ independent of f and $a>0$.

For the proof of the weak bound for the operator Mf we require the classical lemma of Vitali.

Lemma 1. (Vitali) Let E be a measurable set in R^n and suppose there is a covering of this set by a family of balls $\{B(x,r(x)):x\in G\}$. Then a sequence (finite or infinite) of nonintersecting balls $\{B(x_i,r(x_i))\}$ can be chosen from this family in such a way that the inequality

$$\sum_i \mu(B(x_i,r(x_i)))\ge c_n\mu(E) \tag{1}$$

is satisfied, where c_n is a positive constant depending only on the dimension; we may put $c_n=5^n$.

Proof. Let us describe the selection of the balls $B(x_i,r(x_i))$. First we choose as $B(x_1,r(x_1))$ a ball which is "almost" the biggest possible; this means that

$$r(x_1)\ge\frac{1}{2}\sup_{x\in G}r(x).$$

Let us suppose that $B(x_1,r(x_1))$, ... , $(x_{k-1},r(x_{k-1}))$ have already been chosen. Again we choose as $B(x_k,r(x_k))$ a ball which is "almost" the biggest possible among those remaining and which is disjoint from each of the balls $B(x_1,r(x_1))$, ... , $B(x_{k-1},r(x_{k-1}))$, that is to say,

$$r(x_k)\ge\frac{1}{2}\sup\{r(x):B(x,r(x))\cap(\bigcup_{j=1}^{k-1}B(x_j,r(x_j)))=\varnothing\}$$

Thus the desired sequence of balls $\{B(x_i,r(x_i))\}$ is constructed by induction.

If the selection process terminates at the N -th step, then the inequality

$$\mu(E)\le\sum_{i=1}^{N}\mu(B(x_i,r(x_i)))$$

is satisfied and the lemma is proved with $c_n=1$.

Suppose that the selection process does not terminate after a finite number of steps. If $\sum_i\mu(B(x_i,r(x_i)))=\infty$, then (1) is established. Now let us consider the case where $\sum_i\mu(B(x_i,r(x_i)))<\infty$ We will denote by $B^*(x_i,r^*(x_i))$ the ball with the same centre as $B(x_i,r(x_i))$ but expanded five times, i.e. $r^*(x_i)=5r(x_i)$. It will be established that the inclusion

$$E\subseteq\bigcup_i B^*(x_i,r^*(x_i)) \tag{2}$$

holds. For the proof of (2) it is enough to show that if $x_0\in G$ then

$$B(x_0, r(x_0)) \subseteq \bigcup_i B^*(x_i, r^*(x_i))$$

Let us suppose that $B(x_0,r(x_0))$ is not a member of $\{B(x_i,r(x_i))\}$. It follows from the convergence of the series $\sum_i \mu(B(x_i,r(x_i)))$ that $\lim_i r(x_i)=0$. Therefore we may choose the first k such that $r(x_{k+1})< \frac{1}{2} r(x_0)$. Then the ball $B(x_0,r(x_0))$ intersects with one of the balls $\{B(x_i,r(x_i))\}_{i=1}^{k}$. It is clear that for all $x_0 \in G$

$$B(x_0, r(x_0)) \subseteq \bigcup_{i=1}^{k} B^*(x_i, r^*(x_i))$$

Therefore

$$\mu(E) \le \sum_i \mu(B^*(x_i, r^*(x_i))) = 5^n \sum_i \mu(B(x_i, r(x_i)))$$

The lemma is proved.

Theorem 1. The Hardy–Littlewood maximal operator Mf has weak type (1,1).

Proof. Suppose that $f(x) \in L^{1,ioc}$ and $a>0$ are given and let us consider the set

$$U(\alpha, f) = \{x \in \mathrm{R}^n : Mf(x) > \alpha\}$$

For each $x \in U(a,f)$ we can choose a ball $B(x,r)$ such that

$$\frac{1}{\mu(B(x,r(x)))} \int_{B(x,r)} |f(t)|dt > \alpha$$

It is clear that

$$U(\alpha, f) \subseteq \bigcup_{x \in U(\alpha, f)} B(x, r(x))$$

We apply Vitali's covering lemma for the collection $\bigcup_{x \in U(a,f)} B(x,r(x))$ and obtain a sequence of balls $\{B(x_i,r(x_i))\}$ with nonintersecting interiors, for which the following relation holds:

$$\mu(U(\alpha, f)) \le c_0 \sum_i \mu(B(x_i, r(x_i))) \le \frac{c_0}{\alpha} \int_{B(x_i, r(x_i))} |f(t)|dt$$

$$= \frac{c_0}{\alpha} \int_{\bigcup_i B(x_i, r(x_i))} |f(t)|dt \le \frac{c_0}{\alpha} \int_{\mathrm{R}^n} |f(t)|dt = \frac{c_0}{\alpha} \| f \mid L^1 \|$$

The theorem is proved.

Theorem 2. (See, e.g., [03]) The Hardy–Littlewood maximal operator Mf is bounded in any L^p for $p \in (1,\infty)$.

Proof. The case $p=\infty$ is obvious. Let $p \in (1,\infty)$ and let $f(x) \in L^p$ be given. We apply the technique of partition of a function to its "big" and "small" parts. Fix $a>0$ and put $f_1(x)=f(x)$ if $|f(x)| \ge \frac{\alpha}{2}$ and $f_1(x)=0$ in the contrary case. Then we have successively the inequalities:

$$|f(x)| \le |f_1(x)| + \frac{\alpha}{2}, \quad Mf(x) \le Mf_1(x) + \frac{\alpha}{2}$$

Hence we obtain with the aid of Theorem 1

$$\mu(U(\alpha, Mf)) \leq \frac{2c_0}{\alpha} \int_{\{x:|f(x)|\geq \frac{\alpha}{2}\}} |f(x)|dx$$

We put g(x)=Mf(x) and let $\lambda(a,\mathrm{g})=\mu(\{x:|\mathrm{g}(x)|\geq a\})$, namely its distribution function. Then we will have

$$\int_{\mathrm{R}^n} (Mf)^p(x)\,dx = -\int_0^\infty \alpha^p\, d\lambda(\alpha, g) = p\int_0^\infty \alpha^{p-1}\lambda(\alpha, g)\,d\alpha$$

$$= p\int_0^\infty \alpha^{p-1}\mu(U(\alpha, Mf))\,d\alpha$$

$$\leq p\int_0^\infty \alpha^{p-1}\left(\frac{2c_0}{\alpha}\int_{\{x:|f(x)|\geq\frac{\alpha}{2}\}} |f(x)|\,dx\right)d\alpha$$

On reversing the order of integration in the last integral the inner integral will be equal to

$$\int_0^{2|f(x)|} \alpha^{p-2}d\alpha = \frac{|2f(x)|^{p-1}}{p-1}$$

and consequently the double integral becomes

$$\frac{2pc_0}{p-1}\int_{\mathrm{R}^n} |f(x)||2f(x)|^{p-1}\,dx = \frac{2^p\,pc_0}{p-1}\int_{\mathrm{R}^n} |f(x)|^p\,dx$$

The theorem is proved.

3. CONTINUITY OF HARDY–LITTLEWOOD MAXIMAL OPERATOR

Let Ω be a measurable subset of R^n with finite measure $\mu(\Omega)>0$. Then for $p>1$ the maximal operator M maps $L^p(\Omega)$ to $L^p(\Omega)$ and is continuous. It is clear that $L^{p_1}(\Omega)\subset L^{p_0}(\Omega)$, when $1<p_0<p_1$, and the inclusion is continuous. Therefore the inductive limit

$$X^1 = \bigcup_{p>1} L^p(\Omega)$$

with nonmetrizable topology τ_1 is defined, and moreover (X,τ) is continuously embedded in L^1. More generally, locally convex (nonmetrizable) spaces

$$X^{p_0} = \bigcup_{p>p_0} L^p(\Omega)$$

can be defined in such a way that (X^{p_0},τ_{p_0}) is continuously embedded in $L^{p_0}(\Omega)$, $(1<p_0<\infty)$.

Theorem 3. For $p_0\in(1,\infty)$ the Hardy–Littlewood maximal operator is continuous as an operator $M{:}(X^{p_0},\tau_{p_0}\to(X^{p_0},\tau_{p_0})$.

Proof. It is clear that because of the subadditivity of the operator M it is sufficient to establish its continuity at zero. Let U be an absolutely convex neighbourhood of zero in (X^{p_0},τ_{p_0}). This means that in each $L^P(\Omega)$ a ball $B_p(0,r_p)$ can be found such that

$$MB_p(0,r_p) \subset U \quad (p > p_0)$$

The absolutely convex envelope $V=\mathrm{co}\left(\bigcup_{p>p_o} B_p(0,r_p)\right)$ is a neighbourhood of zero in (X^{p_0},τ_{p_0}),. We will show that $MV \subset U$. In fact, if $f \in V$ we can write

$$f = \sum_{i=1}^{n} \lambda_i f_i \text{, where} \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, f_i \in B_{p_i}(0, r_{p_i})(i = 1,2,\ldots,n)$$

We have by the subadditivity of the operator M

$$0 \leq Mf = M(\sum_{i=1}^{n} \lambda_i f_i) \leq \sum_{i=1}^{n} \lambda_i Mf_i \in U$$

and for $p^* = \min_{1 \leq i \leq n} p_i$ we obtain from monotonicity of the norm in $L^{P^*}(\Omega)$

$$\| Mf \mid L^{p^*}(\Omega) \| \leq \| \sum_{i=1}^{n} \lambda_i Mf_i \mid L^{p^*}(\Omega) \|$$

It follows from this that $Mf \in U$. The theorem is proved.

4. CONCLUSION

The development of harmonic analysis and interpolation theory shows the importance of the continuity of Hardy–Littlewood maximal function operator, especially, for partial differential equations. In this case the function spaces can be nonmetricable and the concept of boundadness of operator should be replaced by the concept of continuity what is possible for Hardy–Littlewood maximal function operator.

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