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Fritz John Type Duality in Nondifferentiable Continuous Programming with Equality and Inequality Constraints

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Abstract: A Fritz John type dual for a nondifferentiable continuous programming problem with equality and inequality constraints which represent many realistic situations is formulated using Fritz John type optimality conditions instead of Karush-Kuhn-Tucker type conditions and thus does not require a regularity condition. Various duality results under suitable generalized convexity assumptions are derived. A pair of Fritz John type dual continuous programming with natural boundary conditions rather than fixed end points is also presented. Finally, it is pointed that our duality results can be considered as dynamic generalizations of those of a nondifferentiable nonlinear programming problem in the presence of equality and inequality constraints recently treated in the literature.

Key words: Fritz John type duality; Semi-strict pseudoconvexity; Nonlinear programming; Natural boundary conditions

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1. INTRODUCTION

Optimality conditions form the foundations of mathematical programming both theoretically and computationally. The best known necessarily optimality condition for a mathematical programming problem is the Karush-Kuhn-Tucker optimality condition. However, the Fritz John optimality conditions which appeared before the Karush-Kuhn-Tucker optimality conditions by about three years, is regarded more general in a sense. In order for Karush-Kuhn-Tucker optimality criterion to hold, a constraint-qualification on the constraint functions of the problem is required to be imposed while no such qualification need be imposed on the constraints for the Fritz John optimality conditions to hold. In literature of mathematical programming, Karush-Kuhn-Tucker optimality conditions are generally used to formulate for Wolfe and Mond-Weir type duals, thus a constraint qualification is needed in order to eliminate the requirement of constraint qualification, Weir and Mond [1] used Fritz John optimality condition instead of Karush-Kuhn-Tucker optimality conditions to study duality for a constrained nonlinear programming optimization problem.

Hanson [2] pointed out that some of the duality theorems of mathematical programming have analogues in the variational calculus. This relationship between mathematical programming and the classical calculus of variations is explored and extended. Since mathematical programming and the classical calculus of variations have undergone independent development, it is felt that the mutual adaption of ideas and techniques may prove useful. Mond and Hanson [3] were the first two formulate a dual problem for constrained variational problem and established duality results using a regularity condition. Subsequently Chandra *et al.* [4] studied Wolfe type duality for a class of nondifferentiable continuous programming problems while Bector *et al.* [5] studied Mond-Weir type duality for the problem of [4]. In order to derive duality results for the variational problem, they required a constraint qualification.

Motivated with the results of Weir and Mond [1], in this paper we study Fritz john type duality for a class of nondifferentiable continuous programming problems with equalities and inequalities together which represent many realistic situations. A pair of Fritz John type dual continuous programming with natural boundary conditions rather than fixed end points is also presented. The relationship between our results and those of [1] is also indicated.

2. STATEMENT OF THE PROBLEMS AND PRELIMINARIES

Consider the following non-differentiable continuous programming problems treated in [4]:

(CEP): Minimize

$$\int_I \left\{ f\left(t, x(t), \dot{x}(t)\right) + \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}} \right\} \mathrm{d}t$$

subject to

$$x(a) = \alpha, \quad x(b) = \beta$$

$$g\left(t, x(t), \dot{x}(t)\right) \leq 0, \qquad t \in I \tag{1}$$

$$h\left(t, x(t), \dot{x}(t)\right) = 0, \qquad t \in I \tag{2}$$

(CP): Minimize

$$\int_I \left\{ f\left(t, x(t), \dot{x}(t)\right) + \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}} \right\} \mathrm{d}t$$

subject to

$$x(a) = \alpha, \quad x(b) = \beta$$
$$g\left(t, x(t), \dot{x}(t)\right) \leq 0, \qquad t \in I$$

where

(i) $I = [a, b] \subset R$ is a real interval and piecewise smooth $x : I \to R^n$ with derivative $\dot{x}$.

(ii) $f : I \times R^n \times R^n \to R$, $g : I \times R^n \times R^n \to R^m$ and $h : I \times R^n \times R^n \to R^k$ are continuously differentiable.

(iii) $B(t)$ is positive semi-definite $n \times n$ matrix with continuous on I.

Chandra *et al.* [4] derived the following Fritz John type optimality condition for (CEP):

Theorem 1 (Necessary optimality conditions): If x is an optimal solution of (CEP) and $h_x\left(., x(.), \dot{x}(.)\right)$ maps onto a closed subspace of $C(I, R^k)$ then there exists Lagrange multipliers $\tau \in R$ piecewise smooth $y : I \to R^m$, $z : I \to R^k$ and $\omega : I \to R^n$

$$\begin{aligned}
\tau\left(f_x\left(t, \overline{x}(t), \dot{\overline{x}}(t)\right) + B(t)\bar{\omega}(t)\right) + y(t)^T g_x\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \\
+ z(t)^T h_x\left(t, \overline{x}(t), \dot{\bar{x}}(t)\right) - D\left(\tau f_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))\right) \\
+ y(t)^T g_{\dot{x}}\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) + \bar{z}(t)^T h_{\bar{x}}\left(t, \overline{x}(t), \dot{\bar{x}}(t)\right) = 0, &\quad t \in I \\
y(t)^T g\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) = 0, &\quad t \in I \\
x(t)^T B(t)\omega(t) = \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}}, &\quad t \in I \\
\omega(t)^T B(t)\omega(t) \leq 1, &\quad t \in I \\
(\tau, y(t)) \geq 0, &\quad t \in I \\
(\tau, y(t), z(t)) \neq 0, &\quad t \in I
\end{aligned}$$

where

(i) $C(I, R^k)$ denotes the space of continuous functions from I into R^k.

(ii) x is a vector space of piecewise smooth $x : I \to R^n$ equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$ with the extended differentiable operator D for x smooth, by $u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)ds$;

Thus $D = \frac{\mathrm{d}}{\mathrm{d}t}$ expect at discontinuities. Theorem1, gives Karush-Kuhn type optimality conditions if $\tau = 1$, the optimum may be then called normal if Robinson condition [4] or Slater condition [4] is assumed.

We give the following definitions which will be required in proving duality results.

(i) The functional $\int_I f \; \mathrm{d}t$ is strictly pseudo convex if all $x \neq u$,

$$\int_I \left\{ (x(t) - u(t))^T f_u\left(t, u(t), \dot{u}(t)\right) + (\dot{x}(t) - \dot{u}(t))^T f_{\dot{u}}\left(t, u(t), \dot{u}(t)\right) \right\} \mathrm{d}t \geq 0$$

$$\int_I f(t, x(t), \dot{x}(t))\mathrm{d}t > \int_I f(t, u(t), \dot{u}(t))\mathrm{d}t.$$

. (ii) If $\int_I y(t)^T g\mathrm{d}t$ will be said to be semi-strictly pseudoconvex if $\int_I y(t)^T g\mathrm{d}t$ is pseudoconvex for all $y(t) \geq 0, t \in I$, $y(t) \neq 0, t \in I$.

The generalized Schwartz inequality states that

$$x(t)^T B(t)w(t) \leq \left(x(t)^T B(t)x(t)\right)^{1/2}\left(w(t)^T B(t)w(t)\right)^{1/2},$$

with equality if $B(t)\left[x(t) - q(t)\ z(t)\right] = 0$ for some $q(t) \in R$.

3. FRITZ JOHN TYPE DUALITY

Using Karush-Kuhn-Tuck optimality conditions, Chandra *et al.* [4] have formulated the following Wolfe type dual continuous programming problem for the problem (CP):

(WCED): Maximize

$$\int_j \left(f(t, u(t), \dot{u}(t) + u(t)^T B(t)\omega(t) + y(t)^T g(t)u(t), \dot{u}(t)\right)\mathrm{d}t$$

subject to

$$\begin{aligned} u(a) = \alpha, \quad u(b) = \beta,& \\ f_x(t, x(t), \dot{x}(t)) + B(t)\omega(t) + y(t)^T g_x(t, \bar{x}(t), \dot{\bar{x}}(t))& \\ -D\left(f_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))\right) + y(t)^T g_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) = 0,& \quad t \in I \\ \omega(t)^T B(t)\omega(t) \leq 1,& \quad t \in I \\ y(t) \geq 0,& \quad t \in I. \end{aligned}$$

Bector *et al.* [5] formulated the Mond–Weir type dual for (CP) as the following:

(M-WCED): Maximize

$$\int_j \left(f(t, u(t), \dot{u}(t)) + u(t)^T B(t)\omega(t)\right)\mathrm{d}t$$

subject to

$$\begin{aligned} u(a) = \alpha, \quad u(b) = \beta,& \\ f_x(t, x(t), \dot{x}(t)) + B(t)\omega(t) + y(t)^T g_x((t)\bar{x}(t), \dot{\bar{x}}(t))& \\ -D\left(f_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))\right) + y(t)^T g_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) = 0,& \quad t \in I \\ \omega(t)^T B(t)\omega(t) \leq 1,& \quad t \in I \\ y(t) \geq 0,& \quad t \in I. \end{aligned}$$

Strong duality for both the Wolfe type dual and Mond–Weir type dual is established using Karush-Kuhn-Tucker necessary optimality conditions at the optimum of (CP).

Here we consider (CEP) and propose a different dual to establish strong duality using Fritz John type necessary optimality conditions. Consequently no regularity condition is required.

The following is the Fritz John type dual to the problem (CEP):

($\mathrm{F_rCED}$): Maximize

$$\int_I \left\{f\left(t, u(t), \dot{u}(t)\right) + u(t)^T B(t)w(t)\right\} \mathrm{d}t$$

subject to

$$u(a) = \alpha, \quad u(b) = \beta \tag{3}$$

$$\begin{aligned} &r\left(f_u(t, u(t), \dot{u}(t)) + B(t)\omega(t)\right) + y(t)^T g_u(t, u(t), \dot{u}(t)) + z(t)^T h_u(t, u(t), \dot{u}(t)) \\ &-D\left(r f_{\dot{u}}(t, u(t), \dot{u}(t)) + y(t)^T g_{\dot{u}}(t, u(t), \dot{u}(t)) + z(t)^T h_u(t, u(t), \dot{u}(t))\right) = 0, \quad t \in I \end{aligned} \tag{4}$$

$$\int_I y(t)^T g\left(t, x(t), \dot{u}(t)\right)\mathrm{d}t \geq 0 \tag{5}$$

$$\int_I z(t)^T h\left(t, u(t), \dot{u}(t)\right)\mathrm{d}t \geq 0 \tag{6}$$

$$\omega(t)^T B(t)\omega(t) \leq 1 \quad t \in I \tag{7}$$

$$(r, y(t)) \geq 0 \quad t \in I \tag{8}$$

$$(r, y(t), z(t)) \neq 0 \quad t \in I \tag{9}$$

Theorem 2 (Weak Duality): Let x is feasible for (CEP) and (u, r, y, ω, z) feasible for ($\mathrm{F_rCED}$). If $\int_I \left(f\left(t, ..,\right) + (.)^T B\left(t\right) w\left(t\right)\right) \mathrm{d}t$ pseudo convex, $\int_I y(t)^T g\left(t, ., .\right)\mathrm{d}t$ is semi-strictly pseudo convex and $\int z(t)^T h\left(t, ., .\right) \mathrm{d}t$ is quasiconvex, then

$$\inf(\mathrm{CEF}) \geq \sup(\mathrm{F_rCED}).$$

Proof. Let x is feasible for (CEP) and (u, r, y, ω, z) feasible for ($\mathrm{F_rCED}$). Suppose

$$\int_I \left\{f(t, x(t), \dot{x}(t)) + \left(x(t)^T B(t)x(t)\right)^{\frac{1}{2}}\right\}\mathrm{d}t < \int_I \left\{f(t, u(t), \dot{u}(t)) + \left(u(t)^T B(t)\omega(t)\right)\right\}\mathrm{d}t$$

Using $x(t)^T B(t)x(t) \leq \left(x(t)^T B(t)x(t)\right)^{\frac{1}{2}} \left(\omega(t)^T B(t)\omega(t)\right)^{\frac{1}{2}}$, $t \in I$ with $\omega(t)^T B(t)\omega(t) \leqslant 1$, $t \in I$, this yields

$$\int_I \left\{f(t, x(t), \dot{x}(t)) + x(t)^T B(t)\omega(t)\right\}\mathrm{d}t < \int_I \left\{f(t, u(t), \dot{u}(t)) + u(t)^T B(t)\omega(t)\right\}\mathrm{d}t$$

By the pseudoconvexity of $\int\limits_I \left\{f(t,x(t),\dot{x}(t)) + x(t)^T B(t)\omega(t)\right\}\mathrm{d}t$, this yields

$$\int\limits_I \left\{(x(t)-u(t))^T \left(f_x(t,u(t),\dot{u}(t)) + B(t)\omega(t)\right) + (\dot{x}(t)-\dot{u}(t))^T f_{\dot{u}}(t,u(t),\dot{u}(t))\right\}\mathrm{d}t \leq 0 \tag{10}$$

implies

$$\int\limits_I \left\{(x-u)^T r\left(f_x(t,u,\dot{u}) + B(t)\omega(t)\right) + (\dot{x}(t)-\dot{u}(t))^T r f_{\dot{u}}(t,u(t),\dot{u}(t))\right\}\mathrm{d}t \leq 0$$

with strict inequality in (10) if $r > 0$.

From the constraints of (CEP) and ($\mathrm{F_r}$CED),

$$\int\limits_I y(t)^T g(t,x(t),\dot{x}(t))\mathrm{d}t \leq \int\limits_I y(t)^T g(t,u(t),\dot{u}(t))\mathrm{d}t$$

Since $\int\limits_I y(t)^T g(t,x(t),\dot{x}(t))\mathrm{d}t$ is semi-strictly pseudoconvex, this yields

$$\int\limits_I \left\{(x(t)-u(t))^T y(t)^T g_u(t,u(t),\dot{u}(t)) + (\dot{x}(t)-\dot{u}(t))^T y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t))\right\}\mathrm{d}t < 0 \tag{11}$$

with strict-inequality in the above inequality if some $y(t) > 0, \quad i \in \{1,2,3,...,m\}$. Also we have

$$\int\limits_I z(t)^T h(t,x(t),\dot{x}(t))\mathrm{d}t \leq \int\limits_I z(t)^T h(t,u(t),\dot{u}(t))\mathrm{d}t$$

By the quasi convexity of $\int z(t)^T h\mathrm{d}t$, this implies

$$\int\limits_I \left\{(x(t)-u(t))^T \left(z(t)^T h_u(t,u(t),\dot{u}(t)\right) + (\dot{x}(t)-\dot{u}(t))^T \left(z(t)^T h_{\dot{u}}(t,u(t),\dot{u}(t)\right)\right\}\mathrm{d}t \leq 0 \tag{12}$$

Combining (10), (11) and (12), we have

$$0 > \int_I (x(t) - u(t))^T \Big\{ (rf_u(t, u(t), \dot{u}(t)) + B(t)\omega(t)) + y(t)^T g_u(t, u(t), \dot{u}(t))$$
$$+ z(t)^T h_u(t, u(t), \dot{u}(t)) \Big\} + (\dot{x}(t) - \dot{u}(t))^T \,(rf_{\dot{u}}(t, u(t), \dot{u}(t))$$
$$+ y(t)^T g_{\dot{u}}(t, u(t), \dot{u}(t)) + z(t)^T h_{\dot{u}}(t, u(t), \dot{u}(t)) \Big) \,\mathrm{d}t$$
$$= \int_I \Big[(x(t) - u(t))^T \left\{ r\,(f_u(t, u(t), \dot{u}(t)) + B(t)\omega(t)) + y(t)^T g_u(t, u(t), \dot{u}(t)) \right.$$
$$+ z(t)^T h_u(t, u(t), \dot{u}(t)) - D\,(rf_{\dot{u}}(t, u(t), \dot{u}(t))) + y(t)^T g_{\dot{u}}(t, u(t), \dot{u}(t))$$
$$\left. + \; z(t)^T h_u(t, u(t), \dot{u}(t)) \right\} \Big] \,\mathrm{d}t$$
$$+ (x(t) - u(t))^T \Big(rf_{\dot{u}}(t, u(t), \dot{u}(t)) + y(t)^T g_{\dot{u}}(t, u(t), \dot{u}(t)) + z(t)^T h_{\dot{u}}(t, u(t), \dot{u}(t)) \Big) \Big|_a^b$$
$$= \int_I \Big[(x - u)^T \left\{ r\,(f_u(t, u(t), \dot{u}(t)) + B(t)\omega(t)) + y(t)^T g_u(t, u(t), \dot{u}(t)) + z(t)^T h_u(t, u(t), \dot{u}(t)) \right.$$
$$\left. - D \Big(rf_{\dot{u}}(t, u(t), \dot{u}(t)) + y(t)^T g_{\dot{u}}(t, u(t), \dot{u}(t)) z(t)^T h_{\dot{u}}(t, u(t), \dot{u}(t)) \Big) \right\} \Big] \,\mathrm{d}t < 0$$

contradicting equality constraint of ($\mathrm{F_r}$CED).

Hence

$$\int_I \left\{ f(t, x(t), \dot{x}(t)) + (x(t)^T B(t) x(t))^{\frac{1}{2}} \right\} \mathrm{d}t \geq \int_I \left\{ f(t, u(t), \dot{u}(t) + u(t)^T B(t)\omega(t) \right\} \mathrm{d}t$$

That is,

$$\inf(\mathrm{CEF}) \geq \sup(\mathrm{F_rCED}).$$

□

Theorem 3 (Strong Duality): If $\bar{x}$ is an optimal solution of (CEP), then there exist $x \in R$, piecewise smooth $y : I \to R^m$, $z : I \to R^p$ and $\omega : I \to R^n$, such that $(r, x(t), y(t), z(t), \omega(t))$ is feasible for ($\mathrm{F_r}$CED) and the corresponding value of (CEP) and ($\mathrm{F_r}$CED) are equal. If also, the hypotheses of Theorem 2 hold then $(r, x(t), y(t), z(t), \omega(t))$ is an optimal solution of ($\mathrm{F_r}$CED).

Proof. Since $\bar{x}$ is an optimal solution of ($\mathrm{F_r}$CED), by Theorem 1 there exist $r \in R$, piecewise smooth $y : I \to R^m$, $z : I \to R^p$ and $\omega : I \to R^n$ such that

$$\begin{aligned} &rf_x\,(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_x\,(t, \bar{x}, \dot{\bar{x}}) + \bar{z}(t)^T h_x\,(t, \bar{x}, \dot{\bar{x}}) \\ &- D\left(rf_{\dot{x}}\,(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}\,(t, \bar{x}, \dot{\bar{x}}) + \bar{z}(t)^T h_{\dot{x}}\,(t, \bar{x}, \dot{\bar{x}}) \right) = 0, \quad t \in I \end{aligned} \tag{13}$$

$$\bar{y}(t)^T g\,(t, \bar{x}, \dot{\bar{x}}) = 0, \quad t \in I \tag{14}$$

$$w(t)^T B(t) w(t) \leq 1, \quad t \in I \tag{15}$$

$$\left(\bar{x}(t)^T B\,(t)\,\bar{x}\,(t) \right)^{1/2} = w(t)^T B\,(t)\,\bar{x}\,(t)\,, \quad t \in I \tag{16}$$

$$(r, \bar{y}(t)) \geq 0, \quad t \in I \tag{17}$$

$$(r, \bar{y}(t), \bar{z}(t)) \neq 0, \quad t \in I \tag{18}$$

From (14), it implies $\int_I \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}})\,\mathrm{d}t = 0$ and from $h(t, \bar{x}, \dot{\bar{x}}) = 0,\ t \in I$, $\int_I \bar{z}(t)^T h(t, \bar{x}, \dot{\bar{x}})\,\mathrm{d}t = 0$. Consequently these along with (13), (15), (17) and (18) yield the feasibility of (r, y, z, ω) for ($\mathrm{F_r}$CED). In view of (16), we have

$$\begin{aligned}&\int_I \left(f(t, \bar{x}(t), \dot{\bar{x}}(t)) + \left(\bar{x}(t)^T B(t)\,\bar{x}(t)\right)^{1/2}\right)\mathrm{d}t\\ =&\int_I \left(f(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{x}(t)^T B(t)\,w(t)\right)\mathrm{d}t, \quad t \in I\end{aligned}$$

That is, the equality of objective functionals is obtained. Optimality of $(\bar{r}, \bar{x}, \bar{y}, \bar{z}, \bar{w})$ for ($\mathrm{F_r}$CED) follows, given pseudoconvexity of $\int_I \left(f(t, ..,) + (.)^T B(t)\,z(t)\right)\mathrm{d}t$, quasiconvexity of $\int_I z(t)^T h(t, ., .)\,\mathrm{d}t$ and semi-strict pseudoconvexity of $\int_I y(t)^T g(t, ., ,)\,\mathrm{d}t$ from Theorem 2. □

Theorem 4 (Strict-Converse duality): Assume that

(i) $\int_I \left(f(t, ., .) + (.)^T B(t) z(t)\right)\mathrm{d}t$ is strictly pseudoconvex,

(ii) $\int_I y(t)^T g(t, ., .)\mathrm{d}t$ is semi- strictly pseudoconvex, and

(iii) $\int_I z(t)^T h(t, ., .)\mathrm{d}t$ is quasiconvex.

(iv) $\bar{x}$ is an optimal solution of (CEP).

Assume also that (CEP) has an optimal solution $\bar{x}$. If $(r, \bar{u}, \bar{y}, \bar{z}, w)$ is an optimal solution of ($\mathrm{F_r}$CED), then $\bar{x}(t) = \bar{u}(t),\ t \in I$, i.e., $\bar{u}(t)$ is an optimal solution of (CEP).

Proof. We assume that $\bar{x}(t) \neq \bar{u}(t)$, $t \in I$ and obtain a contradiction. Since $\bar{x}$ is an optimal solution of (CEP), piecewise smooth Theorem 3 implies that there exist $r \in R$, $y : I \to R^m$, $z : I \to R^p$ and $\omega : I \to R^n$ such that $(r, \bar{u}, \bar{y}, \bar{z}, \bar{\omega})$ is an optimal solution of ($\mathrm{F_r}$CED). Since $(r, \bar{u}, \bar{y}, \bar{z}, \bar{\omega})$ is also an optimal solution of ($\mathrm{F_r}$CED), it implies that

$$\int_I \left(f(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T B(t)\bar{w}(t)\right) dt = \int_I \left(f(t, \bar{u}, \dot{\bar{u}}) + \bar{u}(t)^T B(t)\bar{w}(t)\right) dt$$

This, in view of the hypothesis (i), we have

$$\int_I \left\{(\bar{x} - \bar{u})^T \left(f_u(t, u, \dot{u}) + B(t)\,\bar{w}(t)\right) + (\dot{\bar{x}} - \dot{\bar{u}})^T f_{\dot{u}}(t, u, \dot{u})\right\}\mathrm{d}t < 0 \tag{19}$$

From the constraint of (CEP) and ($\mathrm{F_r}$CED), we obtain

$$\int_I \bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}\right)\mathrm{d}t \le \int_I \bar{y}(t)^T g\left(t, \bar{u}, \dot{\bar{u}}\right)\mathrm{d}t \tag{20}$$

and

$$\int_I \bar{z}(t)^T h\left(t, \bar{x}, \dot{\bar{x}}\right)\mathrm{d}t \le \int_I \bar{z}(t)^T h\left(t, \bar{u}, \dot{\bar{u}}\right)\mathrm{d}t \tag{21}$$

The inequality (20) in view of hypothesis (ii), implies

$$\int_I \left\{(\bar{x}-\bar{u})^T\left(\bar{y}(t)^T g_u\left(t,\bar{u},\dot{\bar{u}}\right)\right) + (\dot{\bar{x}}-\dot{\bar{u}})^T\left(\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right\}\mathrm{d}t \le 0 \tag{22}$$

with strict inequality for $y^i\left(t\right) > 0,\ t \in I.$

The inequality (21), because of the hypothesis (iii), yields

$$\int_I \left\{(\bar{x}-\bar{u})^T\left(z(t)^T h_u\left(t,\bar{u},\dot{\bar{u}}\right)\right) + (\dot{\bar{x}}-\dot{\bar{u}})^T\left(z(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right\}\mathrm{d}t \le 0 \tag{23}$$

Combining (19), (22) and (23), we have

$$\begin{aligned}
0 > &\int_I \Big\{(\bar{x}-\bar{u})^T\Big(rf_u\left(t,\bar{u},\dot{\bar{u}}\right) + B\left(t\right)\, w\left(t\right) + \ y(t)^T g_u\left(t,\bar{u},\dot{\bar{u}}\right) + z(t)^T h_u\left(t,\bar{u},\dot{\bar{u}}\right)\Big)\\
&+(\dot{\bar{x}}-\dot{\bar{u}})^T\Big(rf_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right) + y(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right) + z(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right)\Big)\Big\}\,\mathrm{d}t\\
= &\int_I \Big[(\bar{x}-\bar{u})^T\Big\{rf_u\left(t,\bar{u},\dot{\bar{u}}\right) + B\left(t\right) w\left(t\right) + y(t)^T g_u\left(t,\bar{u},\dot{\bar{u}}\right) + z(t)^T h_u\left(t,\bar{u},\dot{\bar{u}}\right)\\
&-D\Big(rf_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right) + y(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right) + z(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right)\Big)\Big\}\Big]\,\mathrm{d}t\\
&+(\bar{x}-\bar{u})^T\Big(rf_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right) + \ y(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right) + z(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right)\Big)\Big|_{t=a}^{t=b}
\end{aligned}$$

Using the boundary conditions of the problem (CEP) and ($\mathrm{F_r}$CED) we have

$$\begin{aligned}
&\int_I \Big[(\bar{x}-\bar{u})^T\Big\{rf_u\left(t,\bar{u},\dot{\bar{u}}\right) + B\left(t\right)\, w\left(t\right) + \ y(t)^T g_u\left(t,\bar{u},\dot{\bar{u}}\right) + z(t)^T h_u\left(t,\bar{u},\dot{\bar{u}}\right)\\
&\quad -D\Big(rf_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right) + \ y(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right) + z(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}}\right)\Big)\Big\}\Big]\,\mathrm{d}t < 0
\end{aligned}$$

Contradicting the feasibility of $(r, \bar{x}, \bar{y}, \bar{z}, \omega)$ for ($\mathrm{F_r}$CED).

Hence $\bar{x}\left(t\right) = \bar{u}\left(t\right)$, $t \in I$, i.e $\bar{u}\left(t\right)$ is an optimal solution.

To establish the converse duality theorem, we may write the problem ($\mathrm{F_r}$CED) in the form following form:

Maximize

$$\int_I \left(f\left(t,x,\dot{x}\right) + x(t)^T B\left(t\right) w\left(t\right)\right)\mathrm{d}t$$

subject to

$$x\left(a\right) = 0 = x\left(b\right)$$

$$\theta\left(t, x\left(t\right), \dot{x}\left(t\right), \ddot{x}\left(t\right), y\left(t\right), \dot{y}\left(t\right), z\left(t\right), \dot{z}\left(t\right), w\left(t\right), r\right) = 0,\ t \in I$$

$$\int_I y(t)^T g\left(t, x, \dot{x}\right)\mathrm{d}t \geq 0,\ t \in I$$

$$\int_I z(t)^T h\left(t, x, \dot{x}\right)\mathrm{d}t \geq 0,\ t \in I$$

$$w(t)^T B\left(t\right) w\left(t\right) \leq 1,\ t \in I$$

$$\left(r, y\left(t\right)\right) \geq 0,\ t \in I$$

$$\left(r, y\left(t\right), z\left(t\right)\right) \neq 0,\ t \in I$$

$$\begin{aligned}\theta =& \theta\left(r, t, x\left(t\right), \dot{x}\left(t\right), \ddot{x}\left(t\right), y\left(t\right), \dot{y}\left(t\right), z\left(t\right), \dot{z}\left(t\right), w\left(t\right)\right)\\ =& r\left(f_x\left(t, x, \dot{x}\right) + B\left(t\right) w\left(t\right) - D f_{\dot{x}}\left(t, x, \dot{x}\right)\right) + y(t)^T g_x\left(t, x, \dot{x}\right)\\ &+ z(t)^T h_x\left(t, x, \dot{x}\right) - D\left(y(t)^T g_{\dot{x}} + z(t)^T h_{\dot{x}}\right)\end{aligned}$$

Consider $\theta\left(r, t, x\left(t\right), \dot{x}\left(t\right), \ddot{x}\left(t\right), y\left(t\right), \dot{y}\left(t\right), z\left(t\right), \dot{z}\left(t\right), w\left(t\right)\right)$ as defining a mapping where $Q : R_+ \times X \times Y \times Z \times W \to U$, and Y and Z are spaces of differentiable functions y and z, W is the space of piecewise smooth functions w and R_+ is the set of nonnegative real numbers and U is a Banach space.

In the proof the following converse duality theorem, we have some restriction on the equality constraint $\theta\left(.\right) = 0$ that appears in the dual ($\mathrm{F_r}$CED). If suffices if the Frechet derivative $Q' = \left[Q_\gamma, Q_x, Q_y, Q_z, Q_w\right]$ has weak (*) closed range [4]. □

Theorem 5 (Converse Duality): Let $(r, \bar{x}, \bar{y}, \bar{z}, \bar{w})$ be an optimal solution of ($\mathrm{F_r}$CED). Assume that (H_1): $\int_I \left(f\left(t, , ..,\right) + (.)^T B\left(t\right) w\left(t\right)\right) \mathrm{d}t$ is pseudoconvex,

(H_2): $\int_I y(t)^T g\left(t, ..,\right) \mathrm{d}t$ is semi-strictly pseudoconvex,

(H_3): $\int_I z(t)^T h\left(t, ..,\right) \mathrm{d}t$ is quasiconvex,

(H_4): $Q' = \left[Q_\gamma, Q_x, Q_y, Q_z, Q_w\right]$ has closed range,

(H_5): $\int_I \left(\sigma(t)^T \theta_x - D\sigma(t)^T \theta_{\dot{x}} + D^2 \sigma(t)^T \theta_{\ddot{x}}\right) \sigma\left(t\right) \mathrm{d}t = 0$ implies $\sigma\left(t\right) = 0$,

(H_6): The set $\left\{y(t)^T g_x - Dy(t)^T g_{\dot{x}}, z(t)^T h_x - Dz(t)^T h_{\dot{x}}\right\}$ is linearly independent, and

(H_6): $y\left(a\right) = 0 = y\left(b\right);\ z\left(a\right) = 0 = z\left(b\right)$.

Then $\bar{x}$ is an optimal solution of (CEP).

Proof. Since $(r, \bar{x}, \bar{y}, \bar{z}, w)$ is an optimal solution of ($\mathrm{F_r}$CED), and Q' has closed range, the Fritz John optimality theorem (Theorem 1), show that there exist Lagrange multipliers $\tau \in R$, $\alpha \in R$, $\beta \in R$, and $\xi \in R$ piecewise smooth $\mu : I \to R^n$, $\eta : I \to R^m$ and $\phi : I \to R$ such that

$$\begin{aligned}&\tau\left[\left(f_x + B\left(t\right) w\left(t\right)\right) - D f_{\dot{x}}\right] + \mu(t)^T \theta_x - D\mu(t)^T \theta_{\dot{x}} + D^2 \mu(t)^T \theta_{\ddot{x}}\\ &+ \alpha\left(y(t)^T g_x - Dy(t)^T g_{\dot{x}}\right) + \beta\left(z(t)^T h_x - Dz(t)^T h_{\dot{x}}\right) = 0\end{aligned} \tag{24}$$

$$\alpha g + \mu(t) + \mu(t)\theta_y - D\mu(t)\theta_{\dot{y}} = 0 \tag{25}$$

$$\beta h + \mu(t)\theta_z - D\mu(t)\theta_{\dot{z}} = 0 \tag{26}$$

$$\mu(t)^T\left[(f_u + B(t)w(t)) - Df_{\dot{u}}\right] + \xi = 0 \tag{27}$$

$$\tau B(t)x(t) - 2\phi(t)B(t)w(t) = 0 \tag{28}$$

$$\phi(t)\left(1 - w(t)^T B(t)w(t)\right) = 0 \tag{29}$$

$$\alpha\int_I y(t)^T g\mathrm{d}t = 0 \tag{30}$$

$$\beta\int_I z(t)^T h\mathrm{d}t = 0 \tag{31}$$

$$\xi r = 0 \tag{32}$$

$$\eta(t)^T y(t) = 0 \tag{33}$$

$$(\tau, \alpha, \beta, \phi(t), \eta(t), \xi) \geq 0 \tag{34}$$

$$(\tau, \alpha, \beta, \phi(t), \mu(t), \eta(t), \xi) \neq 0 \tag{35}$$

Multiplying (25) by $y(t)$ and using (30), we get

$$\begin{aligned}
0 &= \int_I \left(\eta(t)^T\theta_y\right) y(t)\mathrm{d}t - \int_I y(t)^T D\mu(t)^T\theta_{\dot{y}}\mathrm{d}t \\
&= \int_I \mu(t)^T\left(y(t)^T(g_u - Dg_{\dot{u}})\right)\mathrm{d}t + \int_I y(t)^T D\mu(t)g_{\dot{u}}\mathrm{d}t \\
&= \int_I \mu(t)^T\left(y(t)^T g_x - Dy(t)^T g_{\dot{x}}\right)\mathrm{d}t + \int_I \dot{y}(t)^T\mu(t)^T g_{\dot{x}}\mathrm{d}t + \int_I y(t)^T D\mu(t)^T g_{\dot{x}}\mathrm{d}t \\
&= \int_I \mu(t)^T\left(y(t)^T g_x - Dy(t)^T g_{\dot{x}}\right)\mathrm{d}t + y(t)(\mu(t)g_{\dot{x}})\Big|_{t=a}^{t=b} \\
&\quad - \int_I y(t)D\mu(t)g_{\dot{x}} + \int_I y(t)D\mu(t)g_{\dot{x}}
\end{aligned}$$

Using $y(a) = 0 = y(b)$, we have

$$\int_I \mu(t)^T\left(y(t)^T g_u - Dy(t)^T g_{\dot{u}}\right)\mathrm{d}t = 0 \tag{36}$$

Multiplying (26) by $z(t)$ and then using (31), we have

$$\int_I z(t)^T \left(\mu(t)^T\theta_z - D\mu(t)^T\theta_{\dot{z}}\right)\mathrm{d}t = 0$$

This, in view of $z\,(a) = 0 = z\,(b)$, as earlier, reduces to

$$\int_I \mu(t)^T \left(z(t)^T h_z - Dz(t)^T h_{\dot{z}}\right)\mathrm{d}t = 0 \tag{37}$$

Multiplying (4) by $\mu(t)^T$ and using (36) and (37), we have

$$\int_I \mu(t)^T r\,(f_x + B\,(t)\,w\,(t) - Df_{\dot{x}})\mathrm{d}t = 0 \tag{38}$$

Multiplying (24) by $\mu(t)^T$ and using (36) (37) and (38), we have

$$\int_I \left[\left(r\mu(t)^T\theta_x - D\left(r\mu(t)^T\right)\right)\theta_{\dot{x}} + D^2\left(r\mu(t)^T\right)\theta_{\ddot{x}}\right] r\mu\,(t)\,\mathrm{d}t = 0$$

This because of the hypothesis (H_5), yields $\sigma\,(t) = r,\ \mu\,(t) = 0,\ t \in I.$

In view of (H_6), (4) implies $r \neq 0$. Hence $r > 0$ and so $\mu\,(t) = 0$, $t \in I$. Using $\mu\,(t) = 0$, $t \in I$ from (24) along with (4), we have

$$\left(\frac{\tau}{r} - \alpha\right)\left(y(t)^T g_x - Dy(t)^T g_{\dot{x}}\right) + \left(\frac{\tau}{r} - \beta\right)\left(z(t)^T h_x - Dz(t)^T h_{\dot{x}}\right) = 0$$

In view of the hypothesis (H_6), this yields

$$\frac{\tau}{r} - \alpha = 0 = \frac{\tau}{r} - \beta$$

If $\tau = 0$, it can be easily seen that $\alpha = 0$, $\beta = 0$, $\eta\,(t) = 0$, $t \in I$, $\xi = 0$, $\phi\,(t) = 0$, i.e $(\tau, \alpha, \beta, \phi\,(t)\,, \mu\,(t)\,, \eta\,(t)\,, \xi) = 0$, contradicting (35). Hence $\tau > 0$.

The relation (28) with $\tau > 0$ yields

$$B(t)x(t) = \frac{2\phi(t)}{\tau}B(t)w(t),\ t \in I \tag{39}$$

yielding

$$x(t)^T B(t)w(t) = \left(x(t)^T B(t)x(t)\right)^{1/2}\left(w(t)^T B(t)w(t)\right)^{1/2} \tag{40}$$

If $\phi(t) > 0$, $t \in I$, then (29) gives $w(t)^T B(t)w(t) = 1$ and so (40) yields

$$x(t)^T B(t)w(t) = \left(x(t)^T B(t)x(t)\right)^{1/2} \tag{41}$$

If $\phi(t) = 0$, then (39) gives $B(t)w(t) = 0$. So we still obtain

$$x(t)^T B(t)w(t) = \left(x(t)^T B(t)x(t)\right)^{1/2},\quad t \in I$$

That is, in either case, we have

$$\bar{x}(t)^T B(t)\bar{w}(t) = \left(\bar{x}(t)^T B(t)\bar{x}(t)\right)^{1/2}$$

Consequently (25) and (26) imply

$$g(t, \bar{x}, \dot{\bar{x}}) \leq 0,\ t \in I \quad \text{and} \quad h(t, \bar{x}, \dot{\bar{x}}) = 0,\ t \in I$$

yielding the feasibility of $\bar{x}$ for (CEP). Also using (41), we have

$$\int_I \left(f(t, \bar{x}(t), \dot{\bar{x}}(t)) + \left(x(t)^T B(t) x(t)\right)^{1/2} \right) \mathrm{d}t = \int_I \left(f(t, \bar{x}(t), \dot{\bar{x}}(t)) + x(t)^T B(t) w(t) \right) \mathrm{d}t.$$

This because of weak duality theorem (Theorem2) the optimality of $\bar{x}(t)$ of (CEP) follows. □

4. PROBLEMS WITH NATURAL BOUNDARY VALUES

The duality results can be extended to the corresponding problems, ($\mathrm{CEP_N}$), omitting the boundary conditions, and ($\mathrm{CEP_N}$) with natural boundary values. These problems are given as follows:

($\mathrm{CEP_N}$): Minimize

$$\int_I \left\{ f\left(t, u(t), \dot{u}(t)\right) + \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}} \right\} \mathrm{d}t$$

subject to

$$g\left(t, x(t), \dot{x}(t)\right) \leq 0, \quad t \in I$$
$$h\left(t, x(t), \dot{x}(t)\right) = 0, \quad t \in I$$

($\mathrm{CED_N}$): Maximize

$$\int_I \left\{ f\left(t, u(t), \dot{u}(t)\right) + \left(u(t)^T B(t) \omega(t)\right) \right\} \mathrm{d}t$$

subject to

$$u(a) = \alpha, \quad u(b) = \beta$$

$$r\left(f_u\left(t, u(t), \dot{u}(t)\right) + B(t)\omega(t)\right) + y(t)^T g_u\left(t, u(t), \dot{u}(t)\right) + z(t)^T h_u\left(t, u(t), \dot{u}(t)\right)$$
$$-D\left(\left(r f_{\dot{u}}(t, u(t), \dot{u}(t))\right) + y(t)^T g_{\dot{u}}\left(t, u(t), \dot{u}(t)\right) + z(t)^T h_u\left(t, u(t), \dot{u}(t)\right)\right) = 0,\ t \in I$$

$$\int_I y(t)^T g\left(t, x(t), \dot{u}(t)\right) \mathrm{d}t \geq 0$$

$$\int_I z(t)^T h\left(t, u(t), \dot{u}(t)\right) \mathrm{d}t \geq 0$$

$$\omega(t)^T B(t) \omega(t) \leq 1, \quad t \in I$$

$$(r, y(t)) \geq 0, \quad t \in I$$

$$(r, y(t)) \neq 0, \quad t \in I$$

$$\left(rf_{\dot{u}}(t, u(t), \dot{u}(t)) + y(t)^T g_{\dot{u}}\left(t, u(t), \dot{u}(t)\right) + z(t)^T h_u\left(t, u(t), \dot{u}(t)\right)\right) \Bigg|_{t=a}^{t=b} = 0.$$

5. NONLINEAR PROGRAMMING PROBLEMS

If functions f, g and h in the problem ($\mathrm{CEP_N}$) and ($\mathrm{CED_N}$) are independent of t and $B = 0$, then these problems reduce to the following problems treated by Husain and Srivastav [6].

($\mathrm{CEP_1}$): Minimize $f(x)$
subject to

$$g(x) \leq 0,$$
$$h(x) = 0.$$

($\mathrm{CED_1}$): Maximize $f(u)$
subject to

$$\left(rf_u(u) + y^T g_u(u) + z^T h_u(u)\right) = 0$$

$$y^T g(u) \geq 0$$

$$z^T h(u) \geq 0$$

$$(r, y) \geq 0$$

$$(r, y, z) \neq 0.$$

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