CSCanada
Studies in Mathematical Sciences
Vol. 5, No. 2, 2012, pp. [66–71]
DOI: 10.3968/j.sms.1923845220120502.983

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
www.cscanada.org

Nonoscillatory Solutions for System of Delay Difference Equations on Time Scales

LIU Lanchu[a],* and GAO Youwu[a]

[a] College of Science, Hunan Institute of Engineering, Xiangtan, Hunan, China.

* Corresponding author.
Address: College of Science, Hunan Institute of Engineering, 88 East Fuxing Road, Xiangtan, Hunan 411104, China; E-Mail: llc0202@163.com

Received: September 14, 2012/ Accepted: November 9, 2012/ Published: November 30, 2012

Abstract: In this paper, we consider certain system of delay difference equations

$$\Delta y_1(n) = p(n)y_2(n)$$
$$\Delta y_2(n) = -f(n, y_1(g(n)))$$

where $p(n) \in C[N_0, R^+]$, $yf(n, y) \geq 0$, $f \in C[N_0 \times R, R]$, $y \sup\limits_{n \geq n_0} | f(n, y) |> 0$ for any $y \neq 0$, $g(n) \in C[N_0, R]$, $g(n) \leq n$.

Key words: Nonoscillation; System; Difference equations; Time scales

LIU, L., & GAO, Y. (2012). Nonoscillatory Solutions for System of Delay Difference Equations on Time Scales. *Studies in Mathematical Sciences, 5*(2), 66–71. Available from http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220120502.983
DOI: 10.3968/j.sms.1923845220120502.983

1. INTRODUCTION

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. Thesis in 1988 in order to unify continuous and discrete analysis [1]. A time scale $\mathbb{T}$, is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications [9].

On any time scale $\mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\}, \quad \rho(t) := \sup\{s < t : s \in \mathbb{T}\}.$$

A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous function provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nondecreasing, and non-increasing on $[a, b]$, if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)$, respectively.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the delta derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \tag{1}$$

if f is continuous at t and t is right-scattered. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient $\frac{f}{g}$ (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma} \tag{2}$$

$$(\frac{f}{g})^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}} \tag{3}$$

For $t_0, b \in \mathbb{T}$, and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_{t_0}^{b} f^{\Delta}(t) \Delta t = f(b) - f(t_0).$$

An integration by parts formula reads

$$\int_{t_0}^{b} f(t) g^{\Delta}(t) \Delta t = [f(t) g(t)]_{t_0}^{b} - \int_{t_0}^{b} f^{\Delta}(t) g^{\sigma}(t) \Delta t. \tag{4}$$

and infinite integral is defined as

$$\int_{t_0}^{\infty} f(t) \Delta t = \lim_{b \to \infty} \int_{t_0}^{b} f(t) \Delta t \tag{5}$$

Our aim in this paper is to obtain sufficient conditions for existence of positive solutions of system of delay difference equations

$$\begin{cases} \Delta y_1(n) = p(n) y_2(n) \\ \Delta y_2(n) = -f(n, y_1(g(n))) \end{cases} \tag{6}$$

where $p(n) \in C[N_0, R^+]$, $yf(n, y) \geq 0$, $f \in C[N_0 \times R, R]$, $y \sup\limits_{n \geq n_0} \mid f(n, y) \mid > 0$ for any $y \neq 0$, $g(n) \in C[N_0, R]$, $g(n) \leq n$. $\Delta y(n) = y(n+1) - y(n)$ $N_0 = \{1, 2, \ldots\}$, $P(n) = \sum\limits_{i=n_0}^{n} p(i)$. First, we need the definition to use it for the general case.

Definition 1. A solution of (1) is said to be oscillatory (resp. weakly oscillatory) if each component (resp. at least one component) has arbitrarily large zeros.

A solution of (1) is said to be nonoscillatory (resp. weakly nonoscillatory) if each component (resp. at least one component) is eventually of constant sign.

Lemma 1. Let $(y_1(n), y_2(n))$ be a weakly nonoscillatory of (1). Then it is nonoscillatory and there exist constants $N > n_0$, $k_1 > 0$, $k_2 > 0$ such that

$$y_1(n)y_2(n) > 0 \quad for\ n \geq N,$$

$$k_1 P(n) y_2(n) \leq |y_1(n)| \leq k_2 P(n).$$

2. MAIN RESULTS

Theorem 1. Assume that f be either superlinear or sublinear, and

$$\sum_{n=n_0}^{\infty} \mid f(n, kP(g(n))) \mid < \infty$$

for some $k \neq 0$, then (1) has a nonoscillatory solution $(y_1(n), y_2(n))$ with the properties.

$$\lim_{n\to\infty} \frac{y_1(n)}{P(n)} = k, \qquad \lim_{n\to\infty} y_2(n) = k.$$

Proof. We give a proof for the case where f is sublinear and $k > 0$. The remaining cases can treated similarly.

Take $n_1 > n_0$ so large that

$$\sum_{n_1}^{\infty} f(n, kP(g(n))) \leq \frac{k}{2}$$

and

$$n_* = \inf_{n \geq n_1} g(n) > n_0$$

Let C_p denote the linear space of all continuous vector functions

$$\zeta(n) = (y_1(n), y_2(n))$$

on $[n_*, \infty)$. Such that

$$\|\zeta\| = max\{ \sup_{n\geq n_*} P^{-2}(n) \mid y_1(n) \mid, \sup_{n \geq n_*} \mid y_2(n) \mid \} < \infty \tag{7}$$

It is dear that C_p becomes a Banach space under the norm defined by (7). Define a set F by

$$\begin{aligned} F =& \{(y_1, y_2) \in C_p : kP(n) \leq y_1(n) \\ &\leq 2kP(n), k \leq y_2(n) \leq 2k, n \geq n_*\} \end{aligned}$$

Obviously, F is a bounded, convex, and closed subset of C_p.

Let Φ designate the operator which assigns to every element $\zeta = (y_1, y_2)$ of F a vector function $\Phi\zeta = (\Phi y_1, \Phi y_2)$ defined by

$$(\Phi y_1)(n) = y_2(n_0) \sum_{s=n_*}^{n_1-1} p(s) + \sum_{s=n_1}^{n-1} p(s) y_2(s) s \geq n_*$$

$$(\Phi y_2)(n) = \begin{cases} k + \sum\limits_{s=n}^{\infty} f(s, y_1(g(s))), & s \geq n_1; \\ k + \sum\limits_{s=n_1}^{\infty} f(s, y_1(g(s))), & n_* \leq s \leq n_1. \end{cases}$$

(i) Φ maps F into F.

The following inequalities are obvious:

$$kP(n) \leq (\Phi y_1)(n) \leq 2kP(n)$$
$$(\Phi y_2)(n) \geq k$$

Using the sublinearity of f, we see that

$$\begin{aligned}(\Phi y_2)(n) &\leq k + \sum_{n_1}^{\infty} \frac{y_1(g(n))f(n, y_1 g(n))}{y_1 g(n)} \\ &\leq k + \sum_{n_1}^{\infty} 2kP(n)\frac{f(n, kP(n))}{kP(n)} \\ &\leq k + 2\sum_{n_1}^{\infty} f(n, kP(n)) \\ &\leq 2k, \qquad n \geq n_*.\end{aligned}$$

(ii) Φ is continuous.

Let $\zeta_n = (y_{1n}, y_{2n})$ be a sequence of elements of F converging to an element $\zeta = (y_1, y_2)$ of F. $\lim\limits_{n\to\infty} \|\zeta_n - \zeta\| = 0$. It is easy to verify that for $n \geq n_*$,

$$P^{-2}(n) \mid (\Phi y_{1n})(n) - (\Phi y_1)(n) \mid \;\leq\; P^{-1}(n_0) \sup_{s\geq n_*} \mid y_{2n}(s) - y_2(s) \mid \tag{8}$$

$$\mid (\Phi y_{2n})(n) - (\Phi y_2)(n) \mid \leq \sum_{n_*}^{\infty} F_n(s) \tag{9}$$

where

$$F_n(s) = \mid f(s, y_{1n}(g(s)) - f(s, y_1(g(s)) \mid .$$

Evidently, the right-hand side of (3) tends to zero as $n \to \infty$. Since $F_n(s) \leq 4f(s, kP(g(s)))$, $F_n(s) \to 0$ as $n \to \infty$ for $s \geq n_*$, the Lebesgue dominated convergence theorem implies that the right side of (4) tends to zero as $n \to \infty$ and it follows that $\lim\limits_{n\to\infty} \|\Phi\zeta_n - \Phi\zeta\| = 0$.

(iii) ΦF is precompact.

By a theorem of Levitan, it's sufficient to show that when (y_1, y_2) ranges over F, the family of functions $\{P^{-2}\Phi y_1\}$ and $\{\Phi y_2\}$ are uniformly bound and equicauchy on $[n_*, \infty)$, since the uniform boundedness is clear, we need only to demonstrate the equicauchy. This will be done if it is shown that, for any given $\varepsilon > 0$. Let $(y_1, y_2) \in F$, then, we have for $n_2 > n_1 \geq n^*$.

$$\begin{aligned}&\mid (P^{-2}\Phi y_1)(n_2) - (P^{-2}\Phi y_1)(n_1) \mid \\ \leq\; & P^{-2}(n_2)\sum_{n}^{n_2} P(s)y_2(s) + P^{-2}(n_1)\sum_{n}^{n_1} P(s)y_2(s)\end{aligned}$$

$$\leq \quad 4kP^{-1}(n_1)$$

$$\begin{aligned}
\mid (\Phi y_2)(n_2) - (\Phi y_2)(n_1) \mid \quad &\leq \quad \sum_{n_2}^{n_1} f(s, y_1(g(s))) \\
&\leq \quad 2\sum_{n_1}^{\infty} f(s, kp(g(s))).
\end{aligned}$$

Therefore, for any given $\epsilon > 0$, there exists $n_2 > n_1 \geq n_*$, such that

$$\mid (P^{-2}\Phi y_1)(n_2) - (P^{-2}\Phi y_1)(n_1) \mid < \varepsilon$$

$$\mid (\Phi y_2)(n_2) - (\Phi y_2)(n_1) \mid < \varepsilon \tag{10}$$

The above inequalities ensure that there exists a $\delta = \delta(\varepsilon) > 0$, such that (10) holds for any $n_1, n_2 \in [n_*, \infty]$ with $0 < n_2 - n_1 < \delta$.

We now apply the Schaulder fixed point theorem to the operator Φ has a fixed point $\zeta = (y_1, y_2) \in F$. It is easily checked that this fixed point provides a solution of the system (1) with the asymptotic property (2). □

Theorem 2. Assume that f be either superlinear or sublinear, and

$$\sum_{n=n_0}^{\infty} P(n) \mid f(n,k) \mid < \infty$$

for some $k \neq 0$. Then (1) has a nonoscillatory solution $(y_1(n), y_2(n))$ with the properties.

$$\lim_{n\to\infty} y_1(n) = k \qquad \lim_{n\to\infty} y_2(n) = 0$$

Proof. The proof is similar to the proof of Theorem 1, as long as an operator Φ is defined which assigns to every $\xi(n) = (y_1(n), y_2(n)) \in F$ and $F = \{(y_1, y_2) \in C_p : k \leq y_1(n) \leq 2k, 0 \leq y_2(n) \leq \dfrac{k}{P(n)}, n \geq n_1\}$ a vector function $\Phi\xi = (\Phi y_1, \Phi y_2)$ given by

$$(\Phi y_1)(n) = \begin{cases} k + \sum\limits_{n}^{\infty} p(s) y_2(s) & n \geq n_1 \\ k + \sum\limits_{s=n^*}^{\infty} p(s) y_2(s) & n_* \leq n \leq n_1 \end{cases}$$

$$(\Phi y_2)(n) = \begin{cases} \sum\limits_{n}^{\infty} f(s, y_1(g(s))) & n \geq n^* \\ \sum\limits_{s=n^*}^{\infty} f(s, y_1(g(s))) & n_* \leq n \leq n_1 \end{cases}$$

Then there exists a fixed point $\zeta = (y_1, y_2) \in F$, which is a solution of (1). This completes the proof. □

REFERENCES

[1] Hilger, S. (1990). Analysis on measure chains a unified approach to continuous and discrete calculus. *Results in Matematics, 18*, 18-56.

[2] Hilger, S. (1997). Differential and difference calculus unified. *Nonlinear Analysis, 30*(5), 2683-2694.

[3] Bohner, M., & Castillo, J. E. (2001). Mimetic methods on measure chains. *Comput. Math. Appl., 42*, 705-710.

[4] Agarwal, R. P., & Bohner, M. (1999). Basic calculus on time scales and some of its applications. *Results Math., 35*, 3-22.

[5] Dosly, O., & Hilger, S. (2002). A necessary and sufficient condition for oscillation of the Sturm-Liouville dynamic equations on time scales. *Comput. Appl. Math., 141*, 147-158.

[6] Saker (2004). Oscillation of nonlinear differential equations on time scales. *Appl. Math. Comput., 148*, 81-91.

[7] Medico, A. D., & Kong, Q. K. (2004). Kamenev-type and interval oscillation criteria for second-order linear differential equations on a measure chain. *Math. Anal. Appl., 294*, 621-643.

[8] Tanigawa, T. (2003). Oscillation and nonoscillation theorems for a class of fourth order quasilinear functional differential equations. *Hiroshima Math., 33*, 297-316.

[9] Bohner, M., & Peterson, A. (2001). *Dynamic equations on time scales: an introduction with applications*. Boston: Birkhanser.

[10] Yu, J. S., & Zhang, B. G. (1994). Oscillation of delay difference equation. *Applicable Analysis, 53*, 118-224.

[11] Zhang, B. G., & Zhou, Y. (2001). Oscillation of a kind of two-variable function equation. *Computers and Mathematics with Application, 42*, 369-378.

[12] Liu, S. T., & Liu, Y. Q. (2002). Oscillation for nonlinear delay partial difference equations with positive and negative coefficients. *Computers and Mathematics with Applications, 43*, 1219-1230.

[13] Liu, G. H., & Liu, L. CH. (2011). Nonoscillation for system of neutral delay dynamic equation on time scales. *Studies in Mathematicsal Sciences, 3*, 16-23.