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Local Study of Singularities on an Equiform Motion

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Abstract: In this paper we investigate the local singularities of the configuration space corresponding to an equiform motion in the Euclidean space R^3. The chaotic behavior of singularities are displayed through figures.

Key words: Configuration space; Equiform motion

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1. INTRODUCTION

Recent advances in pattern recognition, computer vision, medical imaging, and free-from shape design inspired a fresh interest in surface features associated with singularities of the intrinsic geometric quantities on the surface. Intrinsic geometry has been proposed and studied for smoothing surfaces or getting a hierarchical description of surfaces. Therefore, in order to describe a shape (think of wrinkles on a face or think of the nose as a feature of facial shape) we use a characterization of a certain types of singularities of a shape [13]. The simplest example of singularities is given by the smoothing of a plane curve by its curvature. The main features of a plane curve are its points of inflections where the curvature is zero and the vertices where the curvature has a local maximum or minimum. For surfaces there are two principal curvatures and the features will be interested depend on the parabolic curves where one of these curvature is zero (the Gaussian curvature vanishes), the ridge or ravine curves where are them have a maximum or minimum on its corresponding line of curvature and umbilici points where they are equal. Parabolic points are associated with inflections on object contours. Ridge and ravine curves are very important for shape recognition. In particular, the principal curvatures are non differentiable functions at umbilici points, hence umbilics will become singular points depending on the variation of the principal curvatures. At parabolic points the Gaussian curvature of a surface vanishes. They are the boundaries between elliptic and hyperbolic regions. Alternatively, they are the points where the tangent planes have a specially higher order contact with the surface [4–6] and [8]. Also pattern recognition depends on the local investigation of the paths around singular points. For details of the configuration space and its dynamical system, we refer the reader to [3] and [16]. Our analysis here differs significantly from that in [11,12] and [14]. Instead of using the quantitative study for intrinsic properties of the configuration space corresponding to the equiform motion, we use the qualitative study to investigate the dynamics of the paths the motion.

The major part of this work is devoted to a study of dynamical systems qualitatively and geometrically for the configuration space corresponding to an equiform motion. The local phase portraits and the qualitative behavior in a neighborhood of the origen for the quadratic part of the vector fields define the motion are shown through figures [1,2,7] and [9].

The outline of remainder of this paper is as follows. In section 2, constructions of the configuration space for the equiform motion are present. In section 3, the local representation of the configuration space is obtained. In section 4, we investigate the height function for the configuration space. In section 5, special cases are investigated. Finally, section 6 is devoted to conclusions.

2. CONFIGURATION SPACE

It's well-known that the similar (equiform) motion is defined as Rigid motion with scaling. This motion can be represented by a translation vector T and a rotation matrix A as the following:

$$\overline{x} = \rho A x + T \tag{2.1}$$

where $A^t A = A A^t = I,\ x,\ \overline{x} \in R^3$ and ρ is the scaling factor [5] and [11]. Also the space of all possible rigid transformations of an object constitute the configuration space of the motion. Thus the configuration space is defined as the space of all directions of any system. This space has the structure of a manifold which is called configuration manifold of the motion. From (2.1), it is easy to see that the similar motion can be defined through a linear mapping as in the following:

$$\begin{pmatrix} \overline{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \rho A & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \tag{2.2}$$

The linear map (2.2) in question may be defined explicitly in the 3-dimensional Euclidean space as

$$\overline{x}(u^\alpha) = \rho(u^\beta) A(u^\beta) x(u^\beta) + T(u^\alpha) \tag{2.3}$$

for some fixed parameter u^β and $\alpha = 1, 2$.

Without loss of generality, we consider the following representation of (2.3) as follows:

$$\overline{x}(u^1, u^2) = \rho(u^2) R_z(u^2) x(u^1) + T(u^1, u^2) \tag{2.4}$$

where $x = x(u^1) = (f(u^1), 0, h(u^1))$ is a regular representation of a curve C (the profile curve) in the plane $xz(y = 0)$ in $R^3, R_z(u^2)$ is the rotation matrix around $z-$axis, $\rho(u^2)$ is the equiform factor and T is the translation vector.

The configuration space (2.1) has several forms as the following:

(i) Natural rigid motion, $\rho = 1,\ T = \text{const}$.

(ii) Natural rotation $\rho = 1,\ T = 0$.

(iii) Generalized rigid motion along rotation axis, $\rho = 1,\ T = T(u^2)$.

(iv) Helical motion $\rho = 1,\ T = au^1,\ a = \text{const}$.

Consider the motion for which the translation along $z-$axis, i.e., $T(u^1,\ u^2) = t(u^2)e_3$. Thus, as in [3] the parametric representation of the motion under consideration is represented locally by:

$$F(u) = F(u^1, u^2) = \rho(u^2) f(u^1)\cos u^2,\ f(u^1)\sin u^2,\ h(u^1) + \hat{T}(u^2),\ u \in D \subset R^2 \tag{2.5}$$

where $\rho \neq 0,\ \hat{T} = \dfrac{T(u^2)}{\rho(u^2)}$. Thus $F(u)$ defines some diffeomorphism 2-dimensional surface $F = F(u)$ in R^3.

Remark 1

The kinematics of the equiform motion is given through the configuration space governed by (2.5).

Here and in what follows we use the summation convention d'Einstein, i.e., we sum over the repeated $\alpha, \beta, \gamma, ...$ indices from 1 to 2 unless otherwise indicated. Also, we will use the same notation as in [10] and [11]. In particular, $(g_{\alpha\beta})$ and $(L_{\alpha\beta})$ denote respectively the metric and second fundamental form of the configuration space and the mean curvature H and Gauss curvature G are given by

$$H = g^{\alpha\beta} L_{\alpha\beta} = L^\alpha_\alpha,\quad G = \frac{L}{g},\quad L = Det(L_{\alpha\beta}) \tag{2.6}$$

where $g^{\alpha\beta}$ is the α, β-entry of the inverse of the matrix.

Consider Guass-Weingarten equations for the configuration space as follows

$$\begin{aligned} &X_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} X_\gamma + l_{\alpha\beta} N \\ &N_\alpha = -L^\beta_\alpha X_\beta,\quad X_\gamma = \frac{\partial X}{\partial u^\gamma},\quad X_{\alpha\beta} = \frac{\partial^2 X}{\partial u^\alpha \partial u^\beta} \end{aligned} \tag{2.7}$$

where$(L^\beta_\gamma) = (g^{\alpha\beta} L_{\alpha\gamma})$ is the matrix of Weingarten map of the configuration space, $L_{\alpha\beta} = \prec X_{\alpha\beta}, N \succ,\ N = \dfrac{X_1 \times X_2}{\sqrt{g}}$ and the induced connection on M is given via the Christoffel symbols

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\upsilon} \left(\frac{\partial g_{\beta\upsilon}}{\partial u^\alpha} + \frac{\partial g_{\alpha\upsilon}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\upsilon}\right) \tag{2.8}$$

3. LOCAL REPRESENTATION OF THE CONFIGURATION SPACE

For convenience, let $u = u^1,\ v = u^2$ and using the expansion of the regular functions $\rho(u^2),\ h(u^1),\ t(u^2)$ and $f(u^1)$ in power serves around the point $(u^1, u^2) = (0, 0)$ we have

$$\begin{aligned}
\rho(u^2) &= \rho(v) = c_i v^i,\ c_0 = 1,\ c_i = \frac{\rho^{(i)}(0)}{i!} \\
h(u^1) &= h(u) = a_i u^i,\ a_i = \frac{h^{(i)}(0)}{i!},\ a_0 \neq 0 \\
t(u^2) &= t(v) = b_i v^i,\ b_i = \frac{t^{(i)}(0)}{i!} \\
f(u^1) &= f(u) = f_i u^i,\ f_i = \frac{f^{i}(0)}{i!},\ f_0 \neq 0
\end{aligned} \tag{3.1}$$

where (i) denotes the ith derivative and without less of generality we take $f(u^1) = f(u) = u$.

The configuration space can be approximated by $X = (p_1,\ p_2,\ p_3)$, where $p_\gamma(\gamma = 1,\ 2,\ 3)$ are power series in the parameters u, v which are convergent uniformally to the coordinates X_γ. The series p_i are given by

$$\begin{aligned}
p_1 &= uc_i^1 v^i + o(v)^5 \\
p_2 &= uvc_i^2 v^i + o(v)^5 \\
p_3 &= c_{ij}^3 u^i v^j + o(u, v)^5
\end{aligned} \tag{3.2}$$

where $i, j = 1, 2, 3, 4$ and $c_i^1,\ c_i^2,\ c_{ij}^3$ are given through the constants $c_i,\ a_i,\ b_i$ as in the following

$$c_1^1 = c_1^2 = c_1,\ c_2^1 = \frac{1}{2}(c_2 - 1),\ c_2^2 = \frac{1}{6}(3c_2 - 1)$$

$$c_{00}^3 = a_0 + b_0,\ c_{10}^3 = a_1,\ c_{01}^3 = a_0 c_1 + b_1$$

$$c_{20}^3 = \frac{a_2}{2},\ c_{11}^3 = a_1 c_1$$

$$c_{02}^3 = \frac{1}{2}(a_0 c_2 + b_2),\ c_{30}^3 = \frac{a_3}{6}$$

and so on.

It is straightforward to compute the induced metric tensor locally as

$$g_{\alpha\beta} = \prec X_\alpha, X_\beta \succ = \sum_{i=1}^{n} P_{i,\alpha}\, P_{i,\beta} m^{\alpha\beta} + m_\gamma^{\alpha\beta} u^\gamma + m_{\gamma\upsilon}^{\alpha\beta} u^\gamma u^\upsilon + o(u^1, u^2)^3 \tag{3.3}$$

where the notation $\prec .,\ . \succ$ denotes the ordinary scalar product of vectors in R^3, and $m^{\alpha\beta},\ m_{\gamma}^{\alpha\beta},\ m_{\gamma\upsilon}^{\alpha\beta}$ are given through the constants $a^i,\ b^i,\ c^i$ as in the following

$$m^{11} = 1 + a_1^2,\ m^{21} = m^{12} = a_1(c_{301}) = a_1(a_0c_1 + b_1)$$

$$m^{22} = c_{32} = (c_{301})^2 = (a_0\,c_1 + b_1)^2$$

$$m_1^{11} = 2a_1a_2,\ m_2^{11} = 2c_1(1 + a_1^2)$$

$$m_{11}^{11} = a_2^2 + a_1a_3$$

$$m_{12}^{11} = 2c_1(2a_1a_2) = 4a_1c_1a_2$$

$$m_{22}^{11} = m_{11}(c_1^2 + c_2) = (1 + a_1^2)(c_1^2 + c_2)$$

and so on. The metric $\bar{g}(u^1,\, u^2)$ is given locally by

$$\bar{g}(u^1, u^2) = \text{Linear term} + \text{quadratic term} + o(u, v)^3 \tag{3.4}$$

However, the perturbation for the function $\rho,\ h,\ t$ simplify the subsequent analysis compared with that in [11] and [14].

4. HIGHT FUNCTION OF THE CONFIGURATION SPACE

In [3,4] we introduce the configuration space of an equiform motion globally, but in this work we study the same problem locally. For concreteness and easy of computation, in this section we will adopt a local graph representation of the configuration space under investigation as in the following

$$F(u, v) \approx (p_i(u, v)) \tag{4.1}$$

where $p_i(u, v)$ are polynomials defined locally near the point $(u, v) = (0, 0)$.

The perturbed normal vector field N is given by

$$N = N^i\ e_i,\ \ N^i = n_\alpha^i\, u^\alpha + n_{\alpha\beta}^i\, u^\alpha\, u^\beta \tag{4.2}$$

where n_α^i and $n_{\alpha\beta}^i$ are constants depend on the first derivatives of the polynomials p_i. Thus the perturbed 2^{nd} fundamental quantities $L_{\alpha\beta}$ are given by

$$L_{\alpha\beta} = P_{i,\alpha\beta}(n_\gamma^i\, u^\gamma + n_{\mu\upsilon}^i\, u^\mu\, u^\upsilon) = l_\gamma^{\alpha\beta}\, u^\gamma + l_{\mu\upsilon}^{\alpha\beta}\, u^\mu\, u^\upsilon \tag{4.3}$$

Using the well known formula (2.6) for the Gauss and mean curvatures we have the perturbations

$$\begin{aligned} G &= G_\alpha\, u^\alpha + G_{\alpha\beta}\, u^\alpha\, u^\beta, \\ H &= H_\alpha\, u^\alpha + H_{\alpha\beta}\, u^\alpha\, u^\beta \end{aligned} \tag{4.4}$$

where G_α, $G_{\alpha\beta}$, H_α and $H_{\alpha\beta}$ are constants depends on the perturbed invariants $g_{\alpha\beta}$, $L_{\alpha\beta}$. Thus the Guass and mean curvatures are computed locally from the pertured Weingarten matrix $g^{\gamma\beta}\, L_{\beta\alpha}$ in the case where the quantities $g_{\alpha\beta}$, $g^{\alpha\beta}$, $L_{\alpha\beta}$ are perturbed to quadratic polynomials. The surface and its Guass image are displayed locally in Figures 1(a) and 1(b) respectively. From these figures it follows the chaotic behavior of the singularities of the configuration space. Also the curved regions on the configuration space are displayed from the perturbed Gauss and mean curvatures through the Figures 2(a), 2(b).

Definition 1

Consider the immersion $F:\ M \to R^3$ of the surface M in the space R^3. For any unit vector field $v \in R^3$. We define the function $\Pi_v : M \to R$. Which is called the orthogonal projection on the line generated by the unit vector v as the following

$$\Pi_v(p) = \prec F(p), v \succ,\ p \in U \subset R^3.$$

This projection is called the height function in the direction v.

Using this definition, we get the height functions

$$\prec F(p), e_i \succ = p_i(u^1, u^2) \tag{4.5}$$

along the directions $e_i(\{e_i\}$ is the orthogonal base for $R^3)$. Thus

$$p_i(u^1, u^2) = c^i \qquad \text{(constants)} \tag{4.6}$$

represent the level sets on the coordinate surfaces

$$y_1 = (p_1(u^1, u^2),\ u^1,\ u^2) \tag{4.7a}$$

$$y_2 = (u^1,\ p_2(u^1, u^2),\ u^2) \tag{4.7b}$$

$$y_3 = (u^1,\ u^2,\ p_3(u^1, u^2)) \tag{4.7c}$$

The perturbed surfaces along the line e_i and their level sets c^i corresponding to the height functions $p_i = p_i(u^1, u^2)$ are displayed in Figures 3(a), 3(b), 4(a), 4(b) and 5(a), 5(b) respectively. These figures represent the approximation of the given surface $X = X(u^1, u^2)$ near the origin. These approximations characterize the qualitative behavior near the origin for the equiform motion.

5. CONFIGURATION SPACE OF THE NATURAL ROTATION

In this case $\rho = 1,\ T = 0,$ so $c_i = 0,\ \ c \succ 0,\ b_i = 0,\ \forall\ i.$ Thus the perturbed height functions p_i are given as

$$p_1 = u(1 - \frac{v^2}{2} + \frac{v^4}{4} + o(v^6))$$

$$p_2 = u(v - \frac{v^3}{6} + o(v^5))$$

$$p_3 = a_i\, u^i,\ \ a_0 \neq 0$$

The perturbed surfaces and level sets corresponding to the height functions p_1, p_2, p_3 are displayed through the Figures 6(a), 6(b), 7(a), 7(b) and 8(a), 8(b) respectively. The perturbed height functions P_3 is a power series in one variable, thus the corresponding perturbed level set is degenerate to a family of curves as we see in Figure 8. From these figures it follows that these surfaces of type ruled and translation surface.

Remark 2

For surfaces of revolution every plane containing the z-axis, is a plane of symmetry, so all the meridians are ridges. Along a circle of latitude corresponding to inflection of the profile curve, the principal curvature associated to the meridian ridges is constant nonzero. Along a latitude corresponding to an extremum of the profile curve the principal curvature associated to the meridian ridges is identically zero.

6. CONCLUSION

In the local study, the singularities on the surface and its approximation also the projections and its level sets reflects the significant of the singularities on the surface itself. In this study it follows that the paths of the equiform motion on the first projection consist of 1-period stable path and a fixed point as in Figure 3(b). In the second direction e_2, there exists a periodic bifurcation stable path, where the paths consist of 2-island as in Figure 4(b). But in the third direction e_3, it is easy to see that the paths of the equiform motion consist of 1-island, i.e., i-period stable path as in Figure 5(a). The perturbation of the configuration space in case of natural rotation (surface of revolution) are displayed through Figures 6, 7 and 8.

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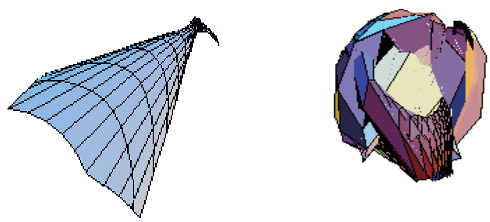
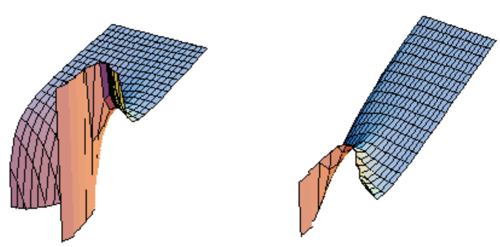
APPENDIX



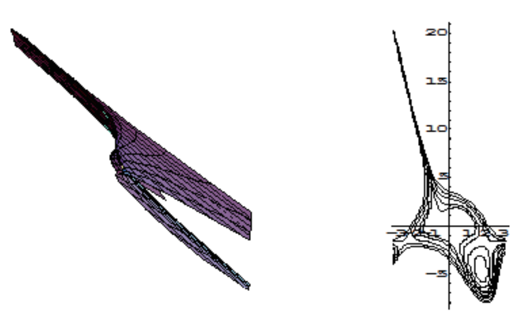
(a) (b)

Figure 1
Perturbed Surface and Its Gauss Image



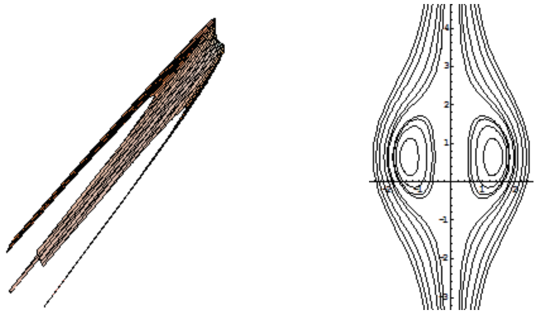
(a) (b)

Figure 2
Perturbed Gauss and Mean Curvatures



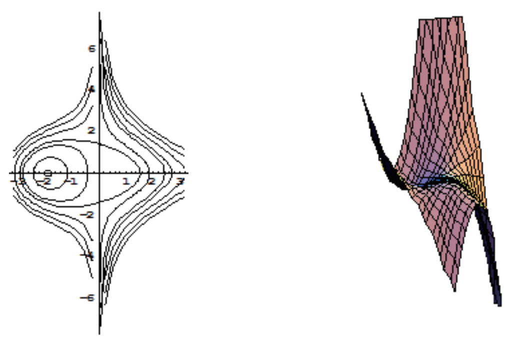
(a) (b)

Figure 3
Perturbed Surface Along e_1



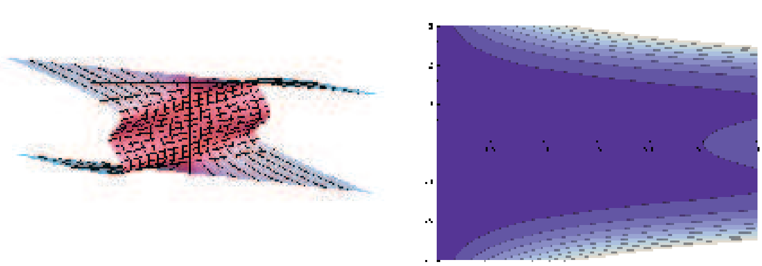
(a) (b)

Figure 4
Perturbed Surface Along e_2



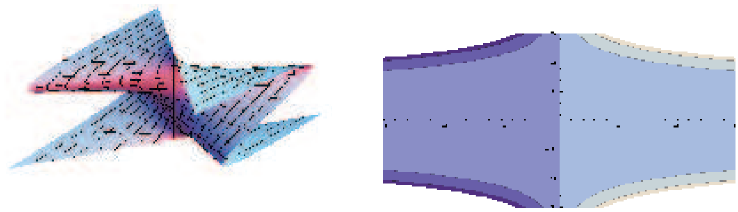
(a) (b)

Figure 5
Perturbed Surface Along e_3



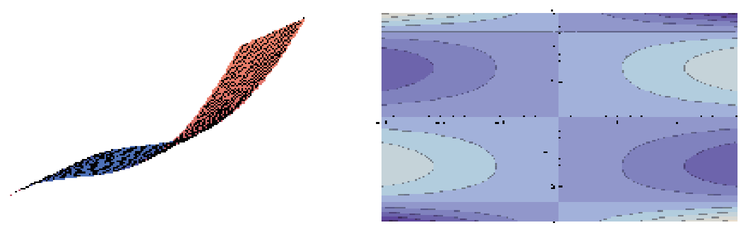
(a) (b)

Figure 6
Perturbed Ruled Surface P_1



(a) (b)

Figure 7
Perturbed Ruled Surface P_2



(a) (b)

Figure 8
Perturbed Translation Surface P_3