CSCanada
Studies in Mathematical Sciences
Vol. 5, No. 2, 2012, pp. [55–65]
DOI: 10.3968/j.sms.1923845220120502.1016

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
www.cscanada.org

Characterizations of Mappings Via Z-Open Sets

Ahmad I. EL-Maghrabi[a],* and Ali M. Mubarki[a]

[a] Department of Mathematics, Faculty of Science, Taibah University, AL-Madinah AL-Munawarah, K.S.A.

* Corresponding author.
Address: Department of Mathematics, Faculty of Science, Taibah University, PO Box 344, AL-Madinah AL-Munawarah, K.S.A.; E-Mail: amaghrabi@taibahu.edu.sa

Received: June 11, 2012/ Accepted: September 3, 2012/ Published: November 30, 2012

Abstract: The aim of this paper we introduce Z-irresolute, Z-open, Z-closed, pre-Z-open and pre-Z-closed mappings and investigate properties and characterizations of these new types of mappings.

Key words: Z-irresolute mapping; Z-open; Z-closed mapping; Pre-Z-open; Pre-Z-closed mappings

EL-Maghrabi, A. I., & Mubarki, A. M. (2012). Characterizations of Mappings Via Z-Open Sets. *Studies in Mathematical Sciences, 5*(2), 55–65. Available from http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220120502.1016 DOI: 10.3968/j.sms.1923845220120502.1016

1. INTRODUCTIONS AND PRELIMINARIES

The concept of Z-open sets in topological spaces was introduced by EL-Magharabi and Mubarki [1,2]. We continue to explore further properties and characterizations of Z-irresolute and Z-open mappings. We also introduce and study properties and characterizations of Z-closed, pre-Z-open and pre-Z-closed mappings.

A subset A of a topological space (X, τ) is called regular open (resp. regular closed) [3] if

$$\mathrm{A} = \mathrm{int}(\mathrm{cl}(\mathrm{A}))(\mathrm{resp.A} = \mathrm{cl}(\mathrm{int}(\mathrm{A}))).$$

The delta interior [4] of a subset A of X is the union of all regular open sets of X contained in A is denoted by δ-int(A). A subset A of a space X is called δ-open if it is the union of regular open sets. The complement of δ-open set is called δ-closed. Alternatively, a set A of (X,τ) is called δ-closed [4] if A = δ-cl(A), where

$$\delta\text{−cl(A)} = \{x \in X : A \cap \text{int(cl(U))} \neq \emptyset,\ \ U \in \tau \text{ and } \ x \in U\}.$$

Throughout this paper (X, τ) and (Y, σ) (simply X and Y) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and X\A denote the closure of A, the interior of A and the complement of A respectively. A subset A of a space X is called δ-semiopen [2] (resp. Z-open [1]) if

$$A \subseteq cl(\delta - \text{int}(A)) \text{ (resp.} A \subseteq cl(\delta - \text{int}(A)) \cup \text{int}(cl(A))).$$

The complement of a Z-open set is called Z-closed. The intersection of all Z-closed sets containing A is called the Z-closure of A and is denoted by Z-cl(A). The union of all Z-open sets contained in A is called the Z-interior of A and is denoted by Z-int(A). The Z-boundary [1] of A (briefly, Z-b(A)) is defined by

$$\text{Z} - \text{b(A)} = \text{Z} - \text{cl(A)} \cap \text{Z} - \text{cl(X\textbackslash A)}.$$

Z-Bd(A) = A\Z-int(A) is said to be Z-border of A. A point $p \in X$ is called a Z-limit point of a set A $\subseteq$X [1] if every Z-open set G $\subseteq$ X containing p contains a point of A other than p. The set of all Z-limit points of A is called a Z-derived set of A and is denoted by Z-d(A). The family of all Z-open (resp. Z-closed) is denoted by ZO(X) (ZC(X)).

2. Z-IRRESOLUTE MAPPING

Definition 2.1. A mapping f : (X, τ) $\rightarrow$ (Y, σ) is called Z-irresolute if

$$f^{-1}(\text{U}) \in \text{ZO(X)},$$

for each U $\in$ ZO(X).

Theorem 2.1. Let f : (X, τ) $\rightarrow$ (Y, σ) be a mapping, then the followings are equivalent:

(1) f is Z-irresolute,
(2) The inverse image of each Z-closed in (Y, σ) is Z-closed in (X, τ),
(3) Z-cl(f^{-1}(B)) $\subseteq$ f^{-1}(Z-cl(B))$\subseteq$ f^{-1}(cl(B)), for each B $\subseteq$ Y,
(4) f(Z-cl(A)) $\subseteq$ Z-cl(f(A)) $\subseteq$ cl(f(A)), for each A $\subseteq$ X,
(5) f^{-1}(Z-int(B)) $\subseteq$ Z-int(f^{-1}(B)), for each B $\subseteq$ Y,
(6) Z-Bd(f^{-1}(B)) $\subseteq$ f^{-1}(Z-Bd(B)), for each B $\subseteq$ Y,
(7) Z-b(f^{-1}(B)) $\subseteq$ f^{-1}(Z-b(B)), for each B $\subseteq$ Y,
(8) f(Z-b(A)) $\subseteq$ Z-b(f(A)), for each A $\subseteq$ X,
(9) f(Z-d(A)) $\subseteq$ Z-cl(f(A)), for each A $\subseteq$ X.

Proof. (1)$\rightarrow$(2). Obvious.

(2)$\rightarrow$(3). Let B $\subseteq$ Y and B $\subseteq$ Z-cl(B) $\subseteq$ cl(B). Then by (2)

$$\text{Z} - \text{cl}(f^{-1}(\text{B})) \subseteq \text{Z} - \text{cl}(f^{-1}(\text{Z} - \text{cl(B)})) = f^{-1}(\text{Z} - \text{cl(B)}) \subseteq f^{-1}(\text{cl(B)}),$$

(3)$\rightarrow$(4). Immediately by replacing B by f(A) in (3), (4)→(1). Let W ∈ ZO(Y) and F = Y\W ∈ ZC(Y). Then by (4),

$$f(\mathrm{Z-cl}(f^{-1}(\mathrm{F}))) \subseteq \mathrm{Z-cl}(f(f^{-1}(\mathrm{F})) \subseteq \mathrm{Z-cl}(\mathrm{F}) = \mathrm{F}.$$

So Z-cl(f^{-1}(F)) ⊆ f^{-1}(F) and hence, f^{-1}(F) = X\f^{-1}(W) ∈ ZC(X), thus f^{-1}(W) ∈ ZO(X). Therefore f is Z-irresolute,

(1)→(5). Let B ⊆ Y. Then Z-int(B) is Z-open in Y. By (1), f^{-1}(Z-int(B)) is Z-open in X. Hence f^{-1}(Z-int(B)) = Z-int(f^{-1}(Z-int(B))) ⊆ Z-int(f^{-1}(B)),

(5)→(6). Let B ⊆ Y. Then by (5), f^{-1}(Z-int(B)) ⊆ Z-int(f^{-1}(B)) we have f^{-1}(B)\Z-int(f^{-1}(B)) ⊆ f^{-1}(B)\f^{-1}(Z-int(B)). Therefore, Z-Bd(f^{-1}(B)) ⊆ f^{-1}(Z-Bd(B)),

(6)→(5). Let B ⊆ Y. Then by (6), Z-Bd(f^{-1}(B)) = f^{-1}(B)\Z-int(f^{-1}(B)) ⊆ f^{-1}(Z-Bd(B)) = f^{-1}(B\Z-int(B)) = f^{-1}(B)\f^{-1}(int(B)) this implies f^{-1}(Z-int(B)) ⊆ Z-int(f^{-1}(B)),

(5)→(1). Let B ⊆ Y be Z-open. Then B = Z-int(B). Hence by (5) we have f^{-1}(B) = f^{-1}(Z-int(B)) ⊆ Z-int(f^{-1}(B)). Thus f^{-1}(B) is Z-open in X. So, f is Z-irresolute,

(1)→(7). Let B ⊆ Y, by (3), we have Z-b(f^{-1}(B)) = Z-cl(f^{-1}(B))\Z-int(f^{-1}(B)) ⊆ f^{-1}(Z-cl(B))\Z-int(f^{-1}(B)) ⊆ f^{-1}[Z-b(B) ∪ Z-int(B)]\Z-int(f^{-1}(B)) ⊆ f^{-1}[Z-b(B) ∪ Z-int(B)]\Z-int(f^{-1}(Z-int(B))). By (1) we have Z-b(f^{-1}(B)) ⊆ (f^{-1}(Z-b(B)) ∪f^{-1}(Z-int(B)))\f^{-1}(Z-int(B)) = f^{-1}(Z-b(B)),

(7)→(1). Let B ∈ ZC(Y) and Z-b(f^{-1}(B)) ⊆ f^{-1}(Z-b(B)). Then, Z-b (f^{-1}(B)) ⊆ f^{-1}(Z-cl(B)\Z-int(B)) = f^{-1}(B\Z-int(B)) = f^{-1}(Z-Bd(B)) ⊆ f^{-1}(B) by Theorem 4.2 [1], we have, f^{-1}(B)∈ZC(X). Therefore f is Z-irresolute,

(7)→(8). Follows by replacing f(A) instead of B in (7),

(8)→(7). Let B ⊆ Y, by(8), we have f(Z-b(f^{-1}(B))) ⊆ Z-b(f(f^{-1}(B))) ⊆ Z-b(B) and therefore Z-b(f^{-1}(B)) ⊆ f^{-1}(Z-b(B)),

(1)→(9). Let A ⊆ X. Then by (4), f(Z-d(A)) ⊆ f(Z-cl(A)) ⊆ Z-cl(f(A)),

(9)→(1). Let F be a Z-closed set in Y, by (7), f(Z-d(f^{-1}(F)) ⊆ Z-cl(f(f^{-1}(F))) ⊆ Z-cl(F) = F, then Z-d(f^{-1}(F)) ⊆ f^{-1}(F) by Theorem 4.4 [1], we have, f^{-1}(F) ∈ ZC(X). Therefore f is Z-irresolute. □

Theorem 2.2. A mapping f : (X, τ) → (Y, σ) is Z-irresolute if and only if for each x in X, the inverse image of every Z-neighbourhood of $f(x)$ is a Z-neighbourhood of x.

Proof. Necessity. Let x ∈ X and let B be Z-neighbourhood of $f(x)$. Then there exists U ∈ ZO(Y) such that $f(x)$ ∈ U ⊆ B. This implies that $x \in f^{-1}$(U) ⊆ f^{-1}(B). Since f is Z-irresolute, so f^{-1}(U) ∈ ZO(X). Hence f^{-1}(B) is a Z-neighbourhood of x.

Sufficiency. Let B ∈ ZO(Y). Put A = f^{-1}(B). Let x ∈ A. Then $f(x)$ ∈ B. But B being Z-open set is a Z-neighbourhood of $f(x)$. So by hypothesis, A = f^{-1}(B) is a Z-neighbourhood of x. Hence there exists A_x ∈ ZO(X) such that $x \in \mathrm{A}_x$ ⊆ A. Thus A = ∪ { $\mathrm{A}_x : x \in$ A}. Therefore f is Z-irresolute. □

Theorem 2.3. A function f : (X, τ) → (Y, σ) is Z-irresolute if and only if f(Z-d(A)) ⊆ f(A) ∪ Z-d(f(A)), for each A ⊆ X.

Proof. Necessity. Let f: X → Y be Z-irresolute. Let A ⊆ X and a_0 ∈ Z-d(A). Assume that $f(\mathrm{a}_0) \notin f$(A) and let V denote a Z-neighbourhood of $f(\mathrm{a}_0)$. Since f is Z-irresolute, so by Theorem 2.2, there exists a Z-neighbourhood U of a_0 such that

$f(\mathrm{U}) \subseteq \mathrm{V}$. From $\mathrm{a}_0 \in$ Z-d(A), it follows that $\mathrm{U} \cap \mathrm{A} \neq \varphi$, therefore, at least one element $\mathrm{a} \in \mathrm{U} \cap \mathrm{A}$ such that $f(\mathrm{a}) \in f(\mathrm{A})$ and $f(\mathrm{a}) \in \mathrm{V}$. Since $f(\mathrm{a}_0) \notin f(\mathrm{A})$, we have $f(\mathrm{a}) \neq f(\mathrm{a}_0)$. Thus every Z-neighbourhood of $f(\mathrm{a}_0)$ contains an element of $f(\mathrm{A})$ different from $f(\mathrm{a}_0)$, consequently, $f(\mathrm{a}_0) \in$ Z-d(f(A)). This proves necessity of the condition.

Sufficiency. Assume that f is not Z-irresolute. Then by Theorem 2.2, there exists $\mathrm{a}_0 \in \mathrm{X}$ and a Z-neighbourhood V of $f(\mathrm{a}_0)$ such that every Z-neighbourhood U of a_0 contains at least one element $\mathrm{a} \in \mathrm{U}$ for which $f(\mathrm{a}) \notin \mathrm{V}$. Put $\mathrm{A} = \{\mathrm{a} \in \mathrm{X}: f(\mathrm{a}) \notin \mathrm{V}\}$. Then $\mathrm{a}_0 \notin \mathrm{A}$ since $f(\mathrm{a}_0) \in \mathrm{V}$, and therefore $f(\mathrm{a}_0) \notin f(\mathrm{A})$; also $f(\mathrm{a}_0) \notin$ Z-d(f(A)), since $f(\mathrm{A}) \cap (\mathrm{V}\backslash\{f(\mathrm{a}_0)\}) = \varphi$. It follows that $f(\mathrm{a}_0) \in f(\text{Z-d(A)})\backslash(f(\mathrm{A}) \cup \text{Z-d}(f(\mathrm{A}))) \neq \varphi$, which is a contradiction to the given condition. ☐

Theorem 2.4. Let $f : (\mathrm{X}, \tau) \to (\mathrm{Y}, \sigma)$ be a 1 - 1 mapping. Then f is Z-irresolute if and only if $f(\text{Z-d(A)}) \subseteq \text{Z-d}(f(\mathrm{A}))$, for each $\mathrm{A} \subseteq \mathrm{X}$.

Proof. Necessity. Let f be Z-irresolute. Let $\mathrm{A} \subseteq \mathrm{X}$, $\mathrm{a}_0 \in$ Z-d(A) and V be a Z-neighbourhood of $f(\mathrm{a}_0)$. Since f is Z-irresolute, so by Theorem 2.2, there exists a Z-neighbourhood U of a_0 such that $f(\mathrm{U}) \subseteq \mathrm{V}$. But $\mathrm{a}_0 \in$ Z-d(A), hence there exists an element $\mathrm{a} \in \mathrm{U} \cap \mathrm{A}$ such that $\mathrm{a} \neq \mathrm{a}_0$, then $f(\mathrm{a}) \in f(\mathrm{A})$ and, since f is 1 - 1, $f(\mathrm{a}) \neq f(\mathrm{a}_0)$. Thus every Z-neighbourhood V of $f(\mathrm{a}_0)$ contains an element of $f(\mathrm{A})$ different from $f(\mathrm{a}_0)$, consequently $f(\mathrm{a}_0) \in$ Z-d(f(A)). We have $f(\text{Z-d(A)}) \subseteq \text{Z-d}(f(\mathrm{A}))$.

Sufficiency. It follows from Theorem 2.3. ☐

Definition 2.2. [1] A mapping $f : (\mathrm{X}, \tau) \to (\mathrm{Y}, \sigma)$ is called Z-continuous if the inverse image of each open set of (Y, σ) is Z-open in (X, τ).

Theorem 2.5. Let $f : (\mathrm{X}, \tau) \to (\mathrm{Y}, sigma)$ be a mapping, then the following statements holds:

(1) $g \circ f$ is Z-irresolute if both f and g are Z-irresolute.

(2) $g \circ f$ is Z-continuous if f is Z-irresolute and g is Z-continuous.

Definition 2.3. A space (X, τ) is called:

(1) Z-T_1- Space if for any pair of distinct points x, y of X, there is a Z-open set $\mathrm{U} \subseteq \mathrm{X}$ such that $\mathrm{x} \in \mathrm{U}$ and $\mathrm{y} \notin \mathrm{U}$ and there is a Z-open set $\mathrm{V} \subseteq \mathrm{X}$ such that $\mathrm{y} \in \mathrm{V}$ and $\mathrm{x} \notin \mathrm{V}$.

(2) Z-T_2- Space if for each two distinct points $\mathrm{x}, \mathrm{y} \in \mathrm{X}$, there exists two disjoint Z-open sets U, V with $\mathrm{x} \in \mathrm{U}$, $\mathrm{y} \in \mathrm{V}$.

Theorem 2.6. Let $f : (\mathrm{X}, \tau) \to (\mathrm{Y}, \sigma)$ be injective and Z-irresolute mapping. Then the followings are hold:

(1) If Y is Z-T_1-Space, then X is Z-T_1-Space,

(2) If Y is Z-T_2-Space, then X is Z-T_2-Space.

Proof. (1) Let x, y be any distinct points in X. Since f is injective and Y is a Z-T_1-Space, there exists two Z-open sets U and V in Y such that $f(x) \in \mathrm{U}$, $f(y) \notin \mathrm{U}$ or $f(y) \in \mathrm{V}$, $f(x) \notin \mathrm{V}$ with $f(x) \neq f(y)$. By using Z-irresoluteness of f, then $f^{-1}(\mathrm{U})$ and $f^{-1}(\mathrm{V})$ are Z-open sets in X such that $x \subseteq f^{-1}(\mathrm{U})$, $y \notin f^{-1}(\mathrm{U})$ or $x \notin f^{-1}(\mathrm{V})$, $y \in f^{-1}(\mathrm{V})$. Therefore, X is a Z-T_1- Space.

(2) similar to (1). ☐

Definition 2.4. A space (X, τ) is said to be Z-compact (resp. Z-Lindelöf) if every Z-open cover of X has a finite (resp. countable) subcover.

Theorem 2.7. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be surjection and Z-irresolute mapping. Then the followings are hold:

(1) If X is Z-compact, then Y is Z-compact.

(2) If X is Z-Lindelöf, then Y is Z-Lindelöf.

Proof. (1) Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be surjection and Z-irresolute mapping and let $U = \{U_i\text{: } i \in I\}$ be a cover of Y by $U_i \in ZO(Y, \sigma)$, for each $i \in I$. Then $O = \{f^{-1}(U_i)\text{: } i \in I\}$ is a cover of X. Since f is Z-irresolute, then O is a Z-open cover of X which is Z-compact. Hence, there exists a finite subset I_o of I such that $X = \underset{i}{\cup} \{f^{-1}(U_i)\text{: } i \in I_o\}$ which implies $X = f^{-1}(\underset{i}{\cup} U_i)$ and therefore $Y = \cup \{U_i : i \in I_o\}$. This shows that Y is Z-compact.

(2) similar to (1). □

Definition 2.5. A space (X, τ) is said to be Z-connected if it cannot be written as a union of two non-empty disjoint Z-open sets.

Theorem 2.8. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be Z-irresolute and X is Z-connected. Then Y is Z-connected.

Proof. Suppose that Y is not connected. Then there exist two non-empty disjoint Z-open sets U and V in X such that $Y = U \cup V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty disjoint Z-open sets in X with $X = f^{-1}(U) \cup f^{-1}(V)$ which contradicts the fact that X is Z-connected. □

3. Z-OPEN AND Z-CLOSED MAPPINGS

Definition 3.1. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

(1) Z-open if the image of each open set in (X, τ) is Z-open sets in (Y, σ).

(2) Z-closed if the image of each closed set in (X, τ) is Z-closed sets in (Y, σ).

Theorem 3.1. For a Z-open (resp. Z-closed) mapping. If $W \subseteq Y$ and $F \subseteq X$ is a closed (resp. open) set containing $f^{-1}(W)$, then there exists Z-closed (resp. Z-open) set $H \subseteq Y$ containing W such that $f^{-1}(H) \subseteq F$.

Proof. Let $H = Y \backslash f(X \backslash F)$. Since $f^{-1}(W) \subseteq F$ which is a closed set and $W \subseteq H$, $X \backslash F$ is an open set. Since f is Z-open mapping, then $f(X \backslash F)$ is Z-open set. Therefore H is Z-closed and $f^{-1}(H) = X \backslash f^{-1} f(X \backslash F) \subseteq F$.

While the second side of the theorem can be proved in the same manner. □

Theorem 3.2. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be Z-open and let $B \subseteq Y$. Then $f^{-1}(\text{Z-cl}(\text{Z-int}(\text{Z-cl}(B)))) \subseteq \text{cl}(f^{-1}(B))$.

Proof. Since $\text{cl}(f^{-1}(B))$ is closed in X containing $f^{-1}(B)$, then by Theorem 3.1, there exists a Z-closed set $B \subseteq H \subseteq Y$, such that $f^{-1}(H) \subseteq \text{cl}(f^{-1}(B))$. Thus, $f^{-1}(\text{Z-cl}(\text{Z-int}(\text{Z-cl}(B))) \subseteq f^{-1}(\text{Z-cl}(\text{Z-int}(\text{Z-cl}(H)))) \subseteq f^{-1}(H) \subseteq \text{cl}(f^{-1}(B))$. □

Theorem 3.3. For a mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ the following statements are equivalent:

(1) f is Z-open,

(2) For each $x \in X$ and each nbd U of X, there exists $W \in ZO(X)$ containing $f(x)$ such that $W \subseteq f(U)$,

(3) $f^{-1}(\text{int}(\delta\text{-cl}(B))) \cap f^{-1}(\text{cl}(\text{int}(B))) \subseteq \text{cl}(f^{-1}(B))$, for each $B \subseteq Y$,

(4) If f is bijective, then int(δ-cl(f(A))) $\cap$ cl(int(f(A)))$\subseteq$ f(cl(A)), for each A $\subseteq$ X.

Proof. (1) $\leftrightarrow$ (2). is immediately,

(1) $\rightarrow$ (3). Let B $\subseteq$ Y and f is Z-open mapping. Then by Theorem 3.1, there exists Z-closed set V $\subseteq$ Y containing B such that cl(f^{-1}(B)) $\supseteq$ f^{-1}(B) $\supseteq$ f^{-1}(int(δ-cl(V))) $\cap f^{-1}$(cl(int(V))) $\supseteq$ f^{-1}(int(δ-cl(B))) $\cap f^{-1}$(cl(int(B))) and therefore f^{-1}(int(δ-cl(B))) $\cap f^{-1}$(cl(int(B))) $\subseteq$ cl(f^{-1}(B)),

(3) $\rightarrow$ (4). Let f be a bijective mapping and f(A) $\subseteq$ Y by (3) f^{-1}(int(δ-cl(f(A)))) $\cap f^{-1}$(cl(int(f(A)))) $\subseteq$ cl(f^{-1}(f(A))) = cl(A). Hence int(δ-cl(f(A))) *cap* cl(int(f(A))) $\subseteq$ f(cl(A)),

(4) $\rightarrow$ (1). Let V $\in \tau$, by (4), f(cl(X$\backslash$V)) = f(X$\backslash$V) $\supseteq$ int(δ-cl(f(X$\backslash$V))) $\cap$ cl(int(f(X$\backslash$V))). By bijection f, we have f(V) $\subseteq$ cl(δ-int(f(V))) $\cup$ int(cl(f(V))) and so f is Z-open. $\square$

Remark 3.1. The bijection condition in Theorem 3.3 (4) is necessary as shown by the following example.

Example 3.1. Let Y be the usual space of real numbers and X = (0,1) be an open sub space of Y. The mapping f : X $\rightarrow$ Y defined by $f(x) = x$, for every $x \in$ X. Put M = X, then cl(δ-int(f(M))) = [0,1] and f(cl(M)) = X. Therefore cl(δ-int(f(M))) $\cap$ int(cl(f(M))) $\subseteq$ cl(δ-int(f(M))) $\nsubseteq$ f(cl(M)).

Theorem 3.4. A mapping f : (X, τ) $\rightarrow$ (Y, σ) is Z-open if and only if f(int(A)) $\subseteq$ Z-int(f(A)), for each A $\subseteq$ X.

Proof. (1) Let f be a Z-open mapping and A $\subseteq$ X, then Z-int(f(int(A))) = f(int(A)) $\in$ ZO(Y). Therefore Z-int(f(int(A))) = f44(int(A)) $\subseteq$ Z-int(f(A)).

Conversely. Let U $\in \tau$, and f(U) = f(int(U)) $\subseteq$ Z-int(f(U)). Then f(U) = Z-int(f(U)). Thus, f(U) is Z-open in Y. Therefore, f is Z-open.

We remark that the equality does not hold in the preceding theorem as the following example. $\square$

Example 3.2. Let X = Y = {1, 2}. Suppose is the indiscrete topology on X and is the discrete topology on Y. Let f : (X, τ) $\rightarrow$ (Y, σ) the identity mappings and A = {1}. Then φ = f(int(A)) $\subseteq$ Z-int(f(A)) = {1}.

Theorem 3.5. A mapping f : (X, τ) $\rightarrow$ (Y, σ) is Z-open if and only if int(f^{-1}(B)) $\subseteq$ f^{-1}(Z-int(B)), for each B $\subseteq$ Y.

Proof. Necessity. Let B $\subseteq$ Y. Since int(f^{-1}(B) is open in X and f is Z-open, then f(int(f^{-1}(B))) is Z-open in Y. Also, we have f(int(f^{-1}(B))) $\subseteq$ f(f^{-1}(B)) $\subseteq$ B. Hence, f(int(f^{-1}(B))) $\subseteq$ Z-int(B). Therefore, int(f^{-1}(B)) $\subseteq$ f^{-1}(Z-int(B)).

Sufficiency. Let A $\subseteq$ X. Then f(A) $\subseteq$ Y. Hence by hypotheses, we obtain int(A) $\subseteq$ int(f^{-1}(f(A))) $\subseteq$ f^{-1}(Z-int(f(A))). Thus f(int(A)) $\subseteq$ Z-int(f(A)), for each A $\subseteq$ X. Hence by Theorem 3.4, we have f is Z-open. $\square$

Theorem 3.6. A mapping f : (X, τ) $\rightarrow$ (Y, σ) is Z-open if and only if $^{-1}$(Z-Bd(A)) $\subseteq$ Bd(f^{-1}(A)), for each A $\subseteq$ X.

Proof. It follows from Theorem 3.5. $\square$

Theorem 3.7. A mapping $f : (X, \tau) \to (Y, \sigma)$ is Z-open if and only if

$$f^{-1}(Z - \mathrm{cl}(B)) \subseteq \mathrm{cl}(f^{-1}(B)),$$

for each $B \subseteq Y$.

Proof. Necessity. Let $B \subseteq Y$ and let f be Z-open. Let $x \in f^{-1}$(Z-cl(B)). Then $f(x) \in$ Z-cl(B). Assume that $U \in \tau$ such that $x \in U$. Since f is Z-open, then $f(U)$ is a Z-open set in Y. Hence, $B \cap f(U) \neq \varphi$. Thus $U \cap f^{-1}(B) \neq \varphi$. Therefore $x \in \mathrm{cl}(f^{-1}(B))$. f^{-1}(Z-cl(B)) $\subseteq \mathrm{cl}(f^{-1}(B))$.

Sufficiency. Let $B \subseteq Y$. Then $Y\backslash B \subseteq Y$. By hypotheses, f^{-1}(Z-cl($Y\backslash B$)) $\subseteq \mathrm{cl}(f^{-1}(Y\backslash B))$ and hence $X\backslash \mathrm{cl}(X\backslash f^{-1}(B)) \subseteq X\backslash f^{-1}$(Z-cl($Y\backslash B$)) $= f^{-1}(Y\backslash$(Z-cl($Y\backslash B$))) we obtain $\mathrm{int}(f^{-1}(B)) \subseteq f^{-1}$(Z-int(B)). By Theorem 3.5, we have f is Z-open. □

Theorem 3.8. A mapping $f : (X, \tau) \to (Y, \sigma)$ is Z-closed if and only if

$$Z - \mathrm{cl}(f(A)) \subseteq f(\mathrm{cl}(A)),$$

for each $A \subseteq X$.

Proof. Necessity. Let f be Z-closed mapping and $A \subseteq X$. Then $f(A) \subseteq f(\mathrm{cl}(A))$. But $f(\mathrm{cl}(A))$ is a Z-closed in Y. Therefore, Z-cl($f(A)$) $\subseteq f(\mathrm{cl}(A))$.

Conversely, suppose that Z-cl($f(A)$) $\subseteq f(\mathrm{cl}(A))$, for each $A \subseteq X$. Let $A \subseteq X$ be a closed. Then Z-cl($f(A)$) $\subseteq f(\mathrm{cl}(A)) = f(A)$. Hence $f(A)$ is Z-closed in Y. Therefore, f is Z-closed. □

Theorem 3.9. Let $f : (X, \tau) \to (Y, \sigma)$ be Z-closed. Then

$$Z - \mathrm{int}(Z - \mathrm{cl}(f(A))) \subseteq f(\mathrm{cl}(A)),$$

for each $A \subseteq X$.

Proof. Suppose f is a Z-closed mappings and $A \subseteq X$. Then $f(\mathrm{cl}(A))$ is Z-closed in Y. Then Z-int(Z-cl(cl($f(A)$))) $\subseteq f(\mathrm{cl}(A))$. But Z-int(Z-cl($f(A)$)) $\subseteq$ Z-int(Z-cl(cl($f(A)$))). Therefore, Z-int(Z-cl($f(A)$)) $\subseteq f(\mathrm{cl}(A))$. □

Theorem 3.10. Let $f : (X, \tau) \to (Y, \sigma)$ be Z-closed and $B, C \subseteq Y$.

(1) If U is an open neighbourhood of $f^{-1}(B)$, then there exists a Z-open neighbourhood V of B such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

(2) If f is onto, then if $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint open neighbourhood so have B and C.

Proof. (1) Let $V = Y\backslash f(X\backslash U)$. Then $V^c = Y\backslash V = f(U^c)$. Since f is Z-closed, so V is a Z-open set. Since $f^{-1}(B) \subseteq U$, we have $V^c = f(U^c) \subseteq f(f^{-1}(B^c)) \subseteq B^c$. Hence, $B \subseteq V$ and thus V is a Z-open neighbourhood of B. Further $U^c \subseteq f^{-1}(f(U^c)) = f^{-1}(V^c) = (f^{-1}(V))^c$. Therefore, $f^{-1}(V) \subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint open neighbourhood M and N, then by (1), we have Z-open neighbourhoods U and V of B and C respectively such that $f^{-1}(B) \subseteq f^{-1}(U) \subseteq$ Z-int(M) and $f^{-1}(C) \subseteq f^{-1}(V) \subseteq$ Z-int(N). Since M and N are disjoint, so are Z-int(M) and Z-int(N) and hence so $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint as well. It follows that U and V are disjoint too as f is onto. □

Theorem 3.11. For a bijective mapping $f : (\mathrm{X}, \tau) \to (\mathrm{Y}, \sigma)$ the following are equivalent:

(1) f^{-1} is Z-continuous,

(2) f is Z-open,

(3) f is Z-closed.

Proof. (1) $\to$ (2). Let $\mathrm{V} \in \tau$ and f^{-1} be Z-continuous, by bijective of f. Then $(f^{-1})^{-1}(\mathrm{V}) = f(\mathrm{V}) \in \mathrm{ZO(Y)}$ and therefore f is Z-open mapping,

(2) $\to$ (3). Let V be closed in X. Then $\mathrm{X}\backslash\mathrm{U}$ is open in X, by (2), $f(\mathrm{X}\backslash\mathrm{U})$ is Z-open in Y. But $f(\mathrm{X}\backslash\mathrm{U}) = f(\mathrm{X})\backslash f(\mathrm{U}) = \mathrm{Y}\backslash f(\mathrm{U})$. Thus $f(\mathrm{U})$ is Z-closed in Y. Therefore f is Z-closed.

(3) $\to$ (1). Let $\mathrm{V} \in \tau$, by (3), we have $f(\mathrm{X}\backslash\mathrm{V})$ is Z-closed in Y and hence, $f(\mathrm{V}) = (f^{-1})^{-1}(\mathrm{V}) \in \mathrm{ZO(Y)}$. Therefore, f^{-1} is Z-continuous. □

Remark 3.2. The composition of two Z-open (resp. Z-closed) mapping may not be Z-open (resp. Z-closed). The following example shows this fact.

Example 3.3. Let X = {a, b, c, d, e, h} and Y = Z = {a, b, c, d, e} with topology $\tau_x = \{\varphi, \{a, e\}, X\}$, an indiscrete topology $(Y, \Im)$ and $\tau_z = \{\varphi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, Z\}$. A mapping $f : (X, \tau_x) \to (Y, \Im)$ defined as $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = f(h) = d$, $f(e) = e$ and the identity mapping $g : (Y, \Im) \to (Z, \tau_z)$. It is clear of f and g is Z-open but $g \circ f$ is not Z-open.

Theorem 3.12. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ and $g : (Y, \tau_Y) \to (Z, \tau_Z)$ be two mappings. Then the following statements hold:

(1) If f is surjective open (resp. closed) and g is Z-open, then $g \circ f$ is Z-open (resp. Z-closed),

(2) If $g \circ f$ is Z-open (resp. Z-closed) and f is surjective continuous, then g is Z-open (resp. Z-closed),

(3) If $g \circ f$ is open (resp. closed) and g is injective Z-continuous, then f is Z-open (resp. Z-closed).

Theorem 3.13. Let $f : (X, \tau) \to (Y, \sigma)$ be a Z-open bijection. Then the following are hold

(1) If X is Z-T_1-Space, then Y is Z-T_1-Space,

(2) If X is Z-T_2-Space, then Y is Z-T_2-Space.

Theorem 3.14. Let $f : (X, \tau) \to (Y, \sigma)$ be a Z-open bijection. Then the following are hold

(1) If Y is Z-compact, then X is Z-compact.

(2) If Y is Z-Lindelöf, then X is Z-Lindelöf.

Theorem 3.15. Let $f : (X, \tau) \to (Y, \sigma)$ be a Z-open surjection and Y is Z-connected. Then X is Z-connected.

4. PRE-Z-OPEN AND PRE-Z-CLOSED MAPPING

Definition 4.1. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be pre-Z-open (resp. pre-Z-closed) if $f(V) \in ZO(Y, \sigma)$ (resp. $ZC(Y, \sigma)$), for each $V \in ZO(X, \tau)$ (resp. $ZC(X, \tau)$).

Theorem 4.1. A mapping $f : (X, \tau) \to (Y, \sigma)$ is pre-Z-closed if and only if for each $S \subseteq Y$ and each $U \subseteq ZO(X, \tau)$ containing $f^{-1}(S)$, there exists $V \in ZO(Y, \sigma)$ containing S such that $f^{-1}(V) \subseteq U$.

Proof. Let S $\subseteq$ Y and f^{-1}(S) $\subseteq$ U. Put V = Y\\$f$(X\\U), then V $\in$ ZO(Y, σ). Since f^{-1}(S) $\subseteq$ U, then f(X\\U) $\subseteq$ ff^{-1}(Y\\S) $\subseteq$ Y\\S and therefore f^{-1}(V) $\subseteq$ U. Conversely, let F be a Z-closed in (X, τ). For any y $\in$ Y\\$f$(F), then f^{-1}(y) $\in$ X\\F $\in$ ZO(X, τ). Hence there exists $V_y \in$ ZO(Y, σ) containing y such that $f^{-1}(V_y)$ $\subseteq$ X\\F, which implies y $\in V_y \subseteq$ Y\\$f$(F). So Y\\$f$(F) = $\cup$ {V_y: y $\in$ Y\\$f$(F)} and therefore f(F), F is Z-closed in (Y, σ). □

Theorem 4.2. A mapping f : (X, τ) $\rightarrow$ (Y, σ) is pre-Z-open if and only if f(Z-int(A)) $\subseteq$ Z-int(f(A)), for each A $\subseteq$ X.

Proof. The proof is similar as Theorem 3.4. □

We remark that the equality does not hold in Theorem 4.3, as the following example shows.

Example 4.1. Let X = {1, 2}. suppose that (X, τ_1) is the indiscrete space and (X, τ_2) is the discrete space. Let f = Id : (X, τ_1) $\rightarrow$ (X, τ_1). Let A = {1}. Then $\emptyset$ = f(Z-int(A)) $\neq$ Z-int(f(A)) = {1}.

Theorem 4.3. A mapping f : (X, τ) $\rightarrow$ (Y, σ) is pre-Z-open if and only if Z-int(f^{-1}(B)) $\subseteq$ f^{-1}(Z-int(B)), for all B $\subseteq$ Y.

Proof. The proof is similar as Theorem 3.5. □

Theorem 4.4. A mapping f : (X, τ) $\rightarrow$ (Y, σ) is pre-Z-open if and only if f^{-1}(Z-Bd(B)) $\subseteq$ Z-Bd(f^{-1}(B)), for all B $\subseteq$ Y.

Proof. It follows from Theorem 4.3. □

Theorem 4.5. A mapping f : (X, τ) $\rightarrow$ (Y, σ) is pre-Z-open if and only if f^{-1}(Z-cl(B)) $\subseteq$ Z-cl(f^{-1}(B)), for all B $\subseteq$ Y.

Proof. The proof similar as Theorem 3.7. □

Theorem 4.6. Let f : (X, τ) $\rightarrow$ (Y, σ) be a mapping such that f(Z-int(A)) $\subseteq$ cl(δ-int(f(A))), for every subset A of X. Then f is pre-Z-open.

Proof. Suppose A is an Z-open set in X. Then by hypothesis, we have

$$f(\mathrm{A}) = f(\mathrm{Z}-\mathrm{int}(\mathrm{A})) \subseteq \mathrm{cl}(\delta-\mathrm{int}(f(\mathrm{A}))).$$

Take B = δ-int(f(A)). Then B is δ-open in Y. Also it implies that B $\subseteq$ f(A) $\subseteq$ cl(B). Hence f(A) is δ-semiopen in Y. Since δSO(Y) $\subseteq$ ZO(Y). Thus f(A) is Z-open in Y. This implies that f is pre-Z-open. □

Theorem 4.7. A mapping f : (X, τ) $\rightarrow$ (Y, σ) is pre-Z-closed if and only if Z-cl(f(A)) $\subseteq$ f(Z-cl(A)), for all A $\subseteq$ X.

Proof. Necessity. Suppose f is a pre-Z-closed mapping and A is an arbitrary subset of X. Then f(Z-cl(A)) is Z-closed in Y. Since f(A) $\subseteq$ f((Z-cl(A)), we obtain

$$\mathrm{Z}-\mathrm{cl}(f(\mathrm{A})) \subseteq f(\mathrm{Z}-\mathrm{cl}(\mathrm{A})).$$

Sufficiency. Suppose F is an arbitrary Z-closed set in X. By hypothesis, we obtain f(F) $\subseteq$ Z-cl(f(F)) $\subseteq$ f(Z-cl(F)) = f(F). Hence f(F) = Z-cl(f(F)). Thus f(F) is Z-closed in Y. It follows that f is pre-Z-closed. □

Theorem 4.8. Let $f : (X, \tau) \to (Y, \sigma)$ be a pre-Z-closed function, and B, C $\subseteq$ Y.

(1) If U is a Z-open neighbourhood of f^{-1}(B), then there exists a Z-open neighbourhood V of B such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

(2) If f is also onto, then if f^{-1}(B) and f^{-1}(C) have disjoint neighborurhoods, so have B and C.

Proof. The proof is similar as Theorem 3.9. □

Theorem 4.9. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection. Then the following are equivalent:

(1) f is pre-Z-closed,

(2) f is pre-Z-open,

(3) f^{-1} is Z-irresolute.

Proof. (1) $\to$ (2) Let U $\in$ ZO(X, τ). Then X\U is Z-closed in X. By (1), f(X\U) is Z-closed in Y. But $f(X\backslash U) = f(X)\backslash f(U) = Y\backslash f(U)$. Thus f(U) is Z-open in Y .

(2) $\to$ (3) Let A $\subseteq$ X. Since f is pre-Z-open, so by Theorem 4.2, f^{-1}(Z-cl(f(A))) $\subseteq$ Z-cl (f^{-1}(f(A))). It implies that Z-cl(f(A)) $\subseteq$ f(Z-cl(A)). Thus Z-cl($(f^{-1})^{-1}$(A)) $\subseteq$ $(f^{-1})^{-1}$(Z-cl(A)), for all A $\subseteq$ X. Then by Theorem 4.2, it follows that f^{-1} is Z-irresolute.

(3) $\to$ (1) Let A be an arbitrary Z-closed set in X. Then X\A is Z-open in X. Since f^{-1} is Z-irresolute, $(f^{-1})^{-1}$(X\A) is Z-open in Y. But $(f^{-1})^{-1}(X\backslash A) = f(X\backslash A) = Y\backslash f(A)$. Thus f(A) is Z-closed in Y. □

Theorem 4.10. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ and $g : (Y, \tau_Y) \to (Z, \tau_Z)$ be two mappings such that $g \circ f : (X, \tau_X) \to (Z, \tau_Z)$ is Z-irresolute. Then:

(1) If g is a pre-Z-open injection, then f is Z-irresolute.

(2) If f is a pre-Z-open surjection, then g is Z-irresolute.

Proof. (1) Let U $\in$ ZO(Y, τ_Y). Then g(U) $\in$ ZO(Z, τ_Z), since g is pre-Z-open. Also $g \circ f$ is Z-irresolute. Therefore, $(g \circ f)^{-1}(g(U)) \in$ ZO(X, τ_X). Since g is an injection, so we have $(g \circ f)^{-1}(g(U)) = (f^{-1} \circ g^{-1})(g(U)) = f^{-1}(g^{-1}(g(U))) = f^{-1}(U)$. Consequently f^{-1}(U) is Z-open in X. This proves that f is Z-irresolute.

(2) Let V $\in$ ZO(Z, τ_Z). Then $(g\circ f)^{-1}(V) \in$ ZO(X, τ_X), since $g\circ f$ is Z-irresolute. Also f is pre-Z-open, $f((g \circ f)^{-1}(V))$ is Z-open in Y. Since f is surjective, we note that $f((g\circ f)^{-1}(V)) = (f\circ(g\circ f)^{-1})(V) = (f\circ(f^{-1}\circ g^{-1}))(V) = ((f\circ f^{-1})\circ g^{-1})(V) = g^{-1}(V)$. Hence g is Z-irresolute. □

Theorem 4.11. For a mappings $f : (X, \tau_X) \to (Y, \tau_Y)$ and $g : (Y, \tau_Y) \to (Z, \tau_Z)$, then

(1) $g \circ f$ is pre-Z-open (resp. pre-Z-closed) if both f and g are pre-Z-open (resp. pre-Z-closed).

(2) $g \circ f$ is Z-open (resp. Z-closed) if f is Z-open (resp. Z-closed) and g are pre-Z-open (resp. pre-Z-closed).

(3) If f is Z-continuous surjection and $g \circ f$ is pre-Z-open (resp. pre-Z-closed), then g is Z-open (resp. Z-closed).

Proof. It is clear. □

Theorem 4.12. Let $f : (X, \tau) \to (Y, \sigma)$ be a pre-Z-open bijection. Then the following are hold

(1) If X is Z-T_1-Space, then Y is Z-T_1-Space,

(2) If X is Z-T_2-Space, then Y is Z-T_2-Space.

Theorem 4.13. Let $f : (X, \tau) \to (Y, \sigma)$ be a pre-Z-open bijection. Then the following are hold

(1) If Y is Z-compact, then X is Z-compact.

(2) If Y is Z-Lindelöf, then X is Z-Lindelöf.

Theorem 4.15. Let $f : (X, \tau) \to (Y, \sigma)$ be be a pre-Z-open bijection and Y is Z-connected. Then X is Z-connected.

REFERENCES

[1] EL-Magharabi, A. I., & Mubarki, A. M. (2011). Z-open sets and Z-continuity in topological spaces. *International Journal of Mathematical Archive (IJMA), 2*(10), 1819-1827.

[2] Park, J. H., Lee, B. Y., & Son, M. J. (1997). On δ-semiopen sets in topological spaces. *J. Indian Acad. Math., 19*(1), 59-67.

[3] Stone, N. V. (1937). Application of the theory of Boolean rings to general topology. *TAMS, 41*, 375-381.

[4] Velicko, N. V. (1968). H-closed topological spaces. *Amer. Math. Soc. Transl., 78*, 103-118.