CSCanada
Studies in Mathematical Sciences
Vol. 5, No. 1, 2012, pp. [11–26]
DOI: 10.3968/j.sms.1923845220120501.Z0697

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
www.cscanada.org

Some Properties of the Generalized Stuttering Poisson Distribution and Its Applications

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Received: May 31, 2012/ Accept: July 25, 2012/ Published: August 31, 2012

Abstract: Based on the probability generating function of stuttering Poisson distribution (SPD), this paper considers some equivalent propositions of SPD. From this, we show that some distributions in the application of non-life insurance actuarial science are SPD, such as negative binomial distribution, compound Poisson distribution etc.. By weakening condition of equivalent propositions of SPD, we define the generalized SPD and prove that any non-negative discrete random variable X with $P\{X = 0\} > 0.5$ obey generalized SPD. Then, we discuss the waiting time distribution of generalized stuttering Poisson process. We consider cumulant estimation of generalized SPD's parameters. As an application, we use SPD with four parameters (4th SPD) to fit auto insurance claim data. The fitting results show that 4th SPD is more accurate than negative binomial and Poisson distribution.

Key words: Stuttering Poisson distribution; Probability generating function; F*aà* di Bruno's formula; Cumulant estimation; Generalized stuttering Poisson distribution; Non-life insurance actuarial science; Zero-inflated

Zhang, H., Chu, L., & Diao, Y. (2012). Some Properties of the Generalized Stuttering Poisson Distribution and Its Applications. *Studies in Mathematical Sciences, 5*(1), 11–26. Available from http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220120501.Z0697 DOI: 10.3968/j.sms.1923845220120501.Z0697

1. INTRODUCTION

Stuttering Poisson distribution (SPD) is a non-negative discrete compound Poisson distribution (see [1,2]), which has the feature that two or more events occur in a very short time (arrive in group or batches). For example, a man may claim for double or much compensation because he has two or more insurance policy from the insurance company. In inventory management, a customer may buy more than one goods of the same kind.

1.1. Stuttering Poisson Distribution

Definition 1.1 For a stochastic process $\{\xi(t), t \geq 0\}$, let

$$P_n(t) = P\{\,\xi(t) = n |\, \xi(0) = 0\}.$$

Similarly to some properties of Poisson process [3], stuttering Poisson process $\xi(t)$ satisfies the following properties:

Property 1: $\xi(0) = 0$.

Property 2: $\xi(t)$ has independent increments (i.e., the numbers of events that occur in disjoint time intervals are independent) and stationary increments (i.e., the distribution of the number of events that occur in any interval of time depends only on the length of the time interval).

Property 3: Counted occurrences may be simultaneous.

$$P_i(\Delta t) = \lambda \alpha_i \Delta t + o(\Delta t),\ (0 \leq \alpha_i \leq 1, i = 1, 2, \cdots, r),$$

where $1 \leq r \leq +\infty$. The r equals to finite or infinite in the following part of this paper.

Property 3 implies that stuttering Poisson process satisfies that the probability of two or more events occur at a very short time interval is non-zero. The probability is directly proportional to the length of time interval. When $i = 0$, according to the properties of independent increments, stationary increments and Chapman-Kolmogoroff equations, we have

$$P_0(t + \Delta t) = P_0(t) P_0(\Delta t)$$

implying that

$$P_0(\Delta t) \;=\; \mathrm{e}^{-\lambda \Delta t} \;=\; 1 - \lambda \Delta t + o(\Delta t), (\lambda > 0)$$

Add terms from $P_0(\Delta t)$ to $P_r(\Delta t)$, that is

$$1 = \sum_{i=1}^{r} P_i(\Delta t) = 1 - \lambda \Delta t + o(\Delta t) + \sum_{i=1}^{r} \lambda \alpha_i \Delta t + o(\Delta t),$$

As $\Delta t \to 0$, hence we obtain $\sum_{i=1}^{r} \alpha_i = 1$.

Definition 1.2 If $\xi(t)$ satisfies Property 1, 2 and 3, then we say that $\xi(t)$ obeys the rth stuttering Poisson distribution. Denote

$$\xi(t) \sim SP(\alpha_1 \lambda t, \cdots, \alpha_r \lambda t)$$

with parameters $(\alpha_1 \lambda t, \cdots, \alpha_r \lambda t) \in \mathbb{R}^r, (\alpha_r \neq 0)$.

1.2. PDF and PGF of Stuttering Poisson Distribution

Similar to Poisson law of small numbers, SPD can be deduced from the limiting distribution of univariate multinomial distribution [4]. Let $p_i = \frac{\alpha_i \lambda t}{N}$, then the probability generating fubction (PGF) of stuttering Poisson distribution

$$\begin{aligned} P(s) &= \lim_{\substack{Np_i=\alpha_i\lambda t \\ N\to\infty}} [(1-\sum_{i=1}^{r} p_i) + \sum_{i=1}^{r} p_i s^i]^N \\ &= \lim_{N\to\infty} [1+\frac{\lambda t}{N}(\sum_{i=1}^{r}\alpha_i s^i - \sum_{i=1}^{r}\alpha_i)]^N = e^{\lambda t \sum\limits_{i=1}^{r}\alpha_i(s^i-1)}. \end{aligned}$$

where $\alpha_i (i = 1, 2, \cdots)$ is probability density of an positive discrete distribution. With $P_n(t) = \frac{P^{(n)}(0)}{n!}$ and F*aà* di Bruno fomula [5]

$$\frac{d^n}{dt^n} g[f(t)] = \sum_{i=1}^{n}\left[\sum_{\substack{k_1+\cdots+k_u+\cdots k_n=i,k_u\in\mathbb{N} \\ 1\cdot k_1+\cdots+uk_u+\cdots+nk_n=n}} \frac{n! g^{(i)} f(t)}{k_1!k_2!\cdots k_n!}\left(\frac{f'(t)}{1!}\right)^{k_1}\left(\frac{f''(t)}{2!}\right)^{k_2}\cdots\left(\frac{f^{(n)}(t)}{n!}\right)^{k_n}\right]$$

we have

$$P_n(t) = \left[\alpha_n\lambda t + \cdots + \sum_{\substack{k_1+\cdots+k_u+\cdots k_n=i,k_u\in\mathbb{N} \\ 1\cdot k_1+\cdots+uk_u+\cdots+nk_n=n}} \frac{\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_n^{k_n}}{k_1!k_2!\cdots k_n!}(\lambda t)^i + \cdots + \frac{\alpha_1^n(\lambda t)^n}{n!}\right] e^{-\lambda t} \tag{1}$$

where $\lambda > 0, \sum_{i=1}^{r}\alpha_i=1, \alpha_i \geq 0$ $(i=1, 2, \cdots)$

Other methods to prove the expression of stuttering Poisson distribution can be obtained by solving system of differential equations [6] or system of functional equations [7]. The name of compound Poisson was used by W. Feller [8] and R. M. Adelson [1] to discuss distribution which PGF is $e^{\lambda t \Sigma_{i=1}^{r}\alpha_i(s^i-1)}$. When r=1, SPD degenerates to Poisson distribution. When $r \geq 2$, we call it non-degenerative SPD. When r=2, C.D. Kemp and A.W. Kemp [9] named it Hermite distribution owing to the PGF can be expanded in terms of Hermite polynomial. When r=3(4), Y. C. Patel [4] said it triple (quadruple) stuttering Poisson distributions. H.P. Galliher *et al.* [10] considered the demands $\xi(t)$ obey SPD with parameters

$$((1-\alpha)\lambda, (1-\alpha)\alpha\lambda, (1-\alpha)\alpha^2\lambda, \cdots)$$

of geometric distribution in inventory management theory and first name it stuttering Poisson. T.S. Moothathu, C.S. Kumar [11] considered SPD with parameters of binomial distribution. For more application in inventory management, see [12–14]. In queue theory, A. Kuczura [15] considered the situation that requests arrive in group or batches with constant service times. R. Mitchell [16] showed that SPD is a more exact model than the Poisson model to fit observed demand form actual historical data of several U.S. Air Force bases. D.E. Giles [17] used Hermite distribution to fit the data of the number of banking and currency crises in IMF-member countries.

2. EQUIVALENT PROPOSITIONS OF SPD

2.1. The Equivalent Proposition of Stuttering Poisson Distribution

Let $P(k,p)$ be a family of non-negative discrete distributions, which is closed under convolution operation, where p (denoting the mean value of the distribution) runs over all non-negative real numbers. L. Janossy [7] showed that $P(k,p)$ is SPD, the equivalent proposition of SPD is deduced from PGF and it helps us to consider the generalized stuttering Poisson distribution.

Theorem 2.1 For a discrete random variable X with $P(s) = \sum_{i=0}^{r} p_i s^i (|s| \leq 1)$. Then, taking logarithm of PGF and expand it to a power series

$$g_X(s) \triangleq \ln(\sum_{i=0}^{r} p_i s^i) = \sum_{i=0}^{r} b_i s^i, |s| \leq 1$$

with $\sum_{i=1}^{r} b_i = \lambda < \infty, (b_i \geq 0)$, where $g_X(s)$ is cumulants generation function of a discrete random variable. Then, if and only if the discrete random variable obeys SPD.

Proof. Sufficiency. With $\sum_{i=0}^{r} b_i s^i = \ln(\sum_{i=0}^{r} p_i s^i)$ and $\sum_{i=1}^{r} b_i = \lambda < \infty$, we know that $g_X(s)$ is absolutely convergent in $|s| \leq 1$. Hence $\sum_{i=0}^{r} b_i = \ln(\sum_{i=0}^{r} p_i) = 0$. Let $b_i = a_i\lambda$, it yield to

$$P(s) = e^{\sum\limits_{i=1}^{r} b_i s^i - \sum\limits_{i=1}^{r} b_i} = e^{\sum\limits_{i=1}^{r} a_i \lambda (s^i - 1)}.$$

Set $\lambda' t = \lambda$, then $X \sim SP(\lambda\alpha_1, \lambda\alpha_2, \cdots)$.

Necessity. The parameters of SPD satisfy $\sum_{i=1}^{r} b_i = \lambda < \infty$. □

Example 2.1 Negative binomial distribution (NBD):

$$P_k = \begin{pmatrix} -r \\ k \end{pmatrix} p^r (p-1)^k, \ (p \in (0,1), \ k = 1, 2, \cdots).$$

The PGF of NBD is $[\frac{p}{1-(1-p)s}]^r$. The logarithm of PGF is $r \ln p + \sum\limits_{i=1}^{\infty} \frac{r(1-p)^i}{i} s^i, (|s| \leq 1)$, then we have

$$\sum_{i=1}^{r} \frac{rq^i}{i} < r \sum_{i=1}^{r} q^i = rq \cdot \frac{1-q^r}{1-q} < \frac{rq}{1-q}, (q = 1-p).$$

So the NBD is equivalent to SPD with parameters $(rq, \frac{rq^2}{2}, \cdots, \frac{rq^i}{i}, \cdots)$. This reveals the essential properties of NBD. NBD, which is a mixture distribution of Poisson distribution and logarithmic distribution [18,21], is an important distribution in automobile insurance.

2.2. Compound Poisson Sums

W. Feller [8] named that the family of independent and identically distributed (i.i.d.) non-negative random variables $\{X_i, i \geq 1\}$ whose sums $Y_N = \sum_{i=1}^{N} X_i$ with $N \sim P(\lambda t)$ (X_i is independent of N) is compound Poisson distribution (the accurate name should be non-negative discrete compound Poisson distribution or SPD). The PGF of is $G_X(s) = \sum_{i=0}^{r} b_i s^i$, ($|s| \leq 1$), by using double conditional expectation of s^{Y_N}, we have

$$E(E(s^{Y_N} \,|\, N = n)) = E([G_X(s)]^n) = e^{\lambda[G_X(s)-1]} = e^{\lambda \sum\limits_{i=1}^{r} b_i(s^i - 1)}.$$

SPD have infinitely divisible property via its PGF. A non-negative discrete distribution is called infinitely divisible for any $n > 1$, its PGF can be represented as the nth power of some other PGF. Thus a SPD with PGF $e^{\lambda t \Sigma_{i=1}^{r} \alpha_i (s^i - 1)}$ can be represented as the nth power of the other PGF $e^{\frac{\lambda t}{n} \Sigma_{i=1}^{r} \alpha_i (s^i - 1)}$. The expression $e^{\frac{\lambda t}{n} \Sigma_{i=1}^{r} \alpha_i (s^i - 1)}$ is SPD with parameters $(\frac{\alpha_1 \lambda t}{n}, \cdots, \frac{\alpha_r \lambda t}{n})$.

Next, we obtain "compound compound Poisson" sums is also SPD in Theorem 2.2.

Theorem 2.2 The family of non-negative random variables $\{X_i, i \geq 1\}$ is i.i.d.. When $N \sim SP(\alpha_1 \lambda t, \cdots, \alpha_r \lambda t)$ and X_i is independent of N, "compound compound Poisson" sums $Y_N = \sum_{i=1}^{N} X_i$ is SPD.

Proof. $G_X(s) = \sum_{i=0}^{\infty} b_i s^i$, ($|s| \leq 1$), using double conditional expectation of s^{Y_N}, we have

$$\begin{aligned} P_{Y_N}(s) &= E(s^{Y_N}) = E_N(E(s^{Y_N} \,|\, N = n)) \\ &= E_N([G_X(s)]^n) = e^{\lambda \sum\limits_{i=1}^{r} \alpha_i [G(s)^i - 1]}. \end{aligned}$$

Noticed that $|G_X(s)| \leq \sum_{i=0}^{\infty} b_i = 1$, hence

$$\lambda \sum_{i=1}^{r} \left| \alpha_i [G_X(s)^i - 1] \right| \leq \lambda \sum_{i=1}^{r} \alpha_i |G_X(s)|^i + \lambda \leq 2\lambda.$$

So $P_{Y_N}(s)$ is absolutely convergent in $|s| \leq 1$,

$$\begin{aligned} \lambda \sum_{i=1}^{r} \alpha_i [G(s)^i - 1] &= \lambda \sum_{i=1}^{r} \alpha_i (b_0 + b_1 s + b_2 s^2 + \cdots)^i - \lambda \\ &= \sum_{i=1}^{\infty} \lambda c_i s^i - \sum_{j=1}^{r} \lambda \alpha_j (1 - b_0^j) \end{aligned}$$

where c_i are derived from multinomial expand. We don't need to have the accurate expression of c_i. Let $\Sigma_{j=1}^{r} \alpha_j (1 - b_0^j) = c$, then

$$P_{Y_N}(s) = e^{\lambda \sum\limits_{i=1}^{\infty} c_i (s^i - 1)}, \sum_{i=1}^{\infty} \lambda c_i = c.$$

Noticed that $c_i > 0$ and $\sum_{i=1}^{r} \lambda c_i = c < \infty$, Y_N is SPD by using Theorem 2.1. $\square$

In Theorem 2.2, a discrete compound Poisson distribution (or SPD) is a special case of discrete compound SPD (compound compound Poisson distribution) when $N \sim P(\lambda t)$. Theorem 2.2 concludes that "compound $\cdots$ compound Poisson distribution" is SPD. In practice, many claims may be from superimposed events. That explains why some distributions in non-life insurance are equivalent to SPD. Besides SPD, other generalized Poisson model have wide applications in non-life insurance actuarial model and risk model such as mixed Poisson process [2,18], and doubly stochastic Poisson processes [2].

3. GENERALIZED STUTTERING POISSON DISTRIBUTION

In this chapter, general stuttering Poisson is defined by weakening conditions of Theorem 2.1. Furthermore, we show that any non-negative discrete random variable X with $P\{X = 0\} > 0.5$ obeys generalized SPD.

3.1. Generalized Stuttering Poisson Model

L. Janossy [7] used independent increments, stationary increments and Chapman-Kolmogoroff equations to construct the system of functional equation

$$P_i(t + \Delta t) = \sum_{k=0}^{i} P_k(\Delta t)P_{i-k}(t),\ (i = 0, 1, \cdots) \tag{2}$$

Solving (2) from one to one will deduce to (1).

When $i = 0$, we have $P_0(t + \Delta t) = P_0(t)P_0(\Delta t)$, the solution of $P_0(t)$ is $P_0(t)$=$\mathrm{e}^{-\lambda t}$.

When $i = 1$, we have $P_1(t + \Delta t) = P_0(t)P_1(\Delta t) + P_1(t)P_0(\Delta t)$, the solution is $P_1(t) = \alpha_1 t e^{-\lambda t}$.

When $i = 2, \cdots$, by the system of functional equations (2) we have

$$P_n(t) =$$
$$[\alpha_n \lambda t + \cdots + \sum_{\substack{k_1+\cdots+k_u+\cdots k_n=i, k_u\in\mathbb{N} \\ 1\cdot k_1+\cdots+uk_u+\cdots+nk_n=n}} \frac{\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_n^{k_n}}{k_1!k_2!\cdots k_n!}(\lambda t)^i + \cdots + \frac{\alpha_1^n(\lambda t)^n}{n!}]e^{-\lambda t}, \tag{3}$$

where $\lambda \geq 0,\ \alpha_i \in \mathbb{R},\ n$=$0, 1, 2, \cdots, r$. L. Janossy [7] proved (3) by mathematical induction.

There are no nonnegative restriction in (3), implies

$$P_i(\Delta t) = \lambda\alpha_i\Delta t + o(\Delta t),\ (i = 1, 2, \cdots)$$

then we obtain $\sum_{i=1}^{\infty} \alpha_i = 1$.

Since α_i are not necessarily to be nonnegative, we suppose that α_i may take negative value and satisfy $\sum_{i=1}^{\infty} \alpha_i = 1$ to have a new distribution family.

Definition 3.1 Generalized stuttering Poisson distribution (GSPD): For a discrete random variable $X(P\{X = i\} = p_i,\ i = 0, 1, 2, \cdots)$, the form of PGF is

$$e^{\lambda t \sum\limits_{i=1}^{\infty} \alpha_i (s^i - 1)} = \sum_{i=0}^{\infty} p_i s^i,\ (p_i \geq 0, |s| \leq 1)$$

and satisfies $\lambda > 0$, $\sum_{i=1}^{\infty} \alpha_i = 1$.

When $\alpha_i \equiv 0$, if $\alpha_i \geq r+1$, we name it rth generalized stuttering Poisson distribution (GSPD). Denote

$$\xi(t) \sim GSP(\alpha_1 \lambda t, \alpha_2 \lambda t, \cdots)$$

with parameters $(\alpha_1\lambda t, \alpha_2\lambda t, \cdots) \in \mathbb{R}^\infty$ $(\sum_{i=1}^{\infty} \alpha_i = 1)$. It is obvious that SPD is a subfamily of GSPD.

Theorem 3.1 Any non-negative discrete random variable $X(P\{X = i\} = p_i,\ i = 0, 1, 2, \cdots)$ with $P\{X = 0\} > 0.5$ obeys GSPD.

Proof. The PGF of X is $P(s) = \sum\limits_{i=0}^{\infty} p_i s^i$ $(|s| \leq 1)$. Let $Q(s) = \dfrac{p_1}{p_0}s + \dfrac{p_2}{p_0}s^2 + \cdots$, expand $\ln[P(s)]$ with Taylor series at zero, that is

$$\ln[P(s)] = \ln[1 + Q(s)] + \ln p_0 = \ln p_0 + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} Q^i(s) \triangleq \sum_{i=0}^{\infty} b_i s^i.$$

Noticed that $|Q(s)| \leq p_0^{-1}(p_1 + p_2 + \cdots)$. If $p_0^{-1}(p_1 + p_2 + \cdots) < 1$, then we have $\sum_{i=1}^{\infty} b_i s^i$, which is absolutely convergent in $|s| \leq 1$. Solving the inequality, we get $p_0 > 0.5$. On the other hand

$$\sum_{i=1}^{\infty} \frac{b_i}{-\ln p_0} = \frac{\ln[P(1)] - \ln p_0}{-\ln p_0} = 1$$

judging from Definition 3.1, X obeys GSPD. □

3.2. Zero-Inflated Model

The zero-inflated model is a random event containing excess zeros count data in the unit time [19]. For example, the number of claims to an insurance company is almost zero, otherwise substantial losses will lead insurance company to bankrupt. Zero-inflated model has too much count data that Poisson distribution is disable to forecast. It is obvious that condition in Theorem 3.1 satisfies zero-inflated model for $P\{X = 0\} > 0.5$.

Example 3.1 and Theorem 3.1 show that the probability of zero occurrence is more than positive occurrences. For example, the cumulant of Bernoulli distribution $P(s) = p + (1-p)s$, $(p > 0.5)$ is

$$\ln[p + (1-p)s] = -\ln p + \frac{1-p}{p}s - \frac{1}{2}(\frac{1-p}{p})^2 s^2 + \frac{1}{3}(\frac{1-p}{p})^3 s^3 + \cdots.$$

Let t=1, thus the other form of PDF is $P(s) = e^{-\ln p \Sigma_{i=1}^{\infty}[-\frac{1}{i \ln p}\left(\frac{p-1}{p}\right)^i (s^i - 1)]}$. The possibility of counted occurrences at a given instant is

$$P_i(\Delta t) = \frac{-1}{i}\left(\frac{p-1}{p}\right)^i \Delta t + o(\Delta t),\ (i = 1, 2, \cdots)$$

Intuitive explanation: The possibility of two or more counted occurrences at a given instant is permitted with negative and positive probability in turn. Have a "negative probability" in an instant can be considered that demands leave and arrive alternatively in group or superposition. Leaving and arriving with same probability can be seen as zero probability.

3.3. Waiting Time Distributions of Generalized Stuttering Poisson Process

Akin to the properties and statement of the waiting time distributions of Poisson process [3], we obtain distribution of the waiting time untill the nth event by two methods.

Definition 3.2 Generalized stuttering Poisson process $\xi(t)$ satisfies the following properties:

Property 1: $\xi(0) = 0$.

Property 2: $\xi(t)$ has independent increments and stationary increments .

Property 3: The number of events $\xi(t)$ in any interval of length t is generalised stuttering Poisson distributed (SPD is a special case of it).

In Example 3.1, we may consider the number of events $\xi(t)$ in any interval of length t with $P\{\xi(t)=0\}=p$, $P\{\xi(t)=0\}=1-p$ $(p > 0)$. The others number of events do not occur. Or we just consider the number of events $\xi(t)$ obeys SPD. We denote the time of the first event by T_1, and T_n denote the waiting time between the $(n-1)$th event and the nth event. We shall not directly determine the distribution of the T_n. Another quantity of interest is $W_n=\Sigma_{i=1}^{n} T_n$ $(n \geq 1)$, the waiting time until the nth event (the arrival time of the nth event).

It is easily seen that the nth event will occur prior to or at time t if and only if the number of events occurring by time t is at least n. Hence,

$$F_{W_n} = P\{W_n \leq t\} = P\{N(t) \geq n\} = \sum_{i=n}^{\infty} P_i(t) = 1 - \sum_{i=0}^{n-1} P_i(t)$$

$$f_{W_n} = F'_{W_n} = -\sum_{i=0}^{n-1} {P_i}'(t) \tag{4}$$

Next, we deduce ${P_i}'(t)$ from (2). Substitute $P_k(\Delta t)=\lambda\alpha_k \Delta t+o(\Delta t)$ $(k=1,2,\cdots)$ and $P_0(\Delta t)=1-\lambda\Delta t+o(\Delta t)$ to (2), then

$$P_i(t+\Delta t) = [1-\lambda\Delta t+o(\Delta t)]P_i(t)+[\lambda\alpha_1\Delta t+o(\Delta t)]P_{i-1}(t)+\cdots+[\lambda\alpha_i\Delta t+o(\Delta t)]P_0(t)$$

Rewrite this relation in the form

$$\frac{P_i(t+\Delta t)-P_i(t)}{\Delta t} = \lambda[-P_i(t)+\alpha_1 P_{i-1}(t)+\cdots+\alpha_{i-1}P_1(t)+\alpha_i P_0(t)]+o(\Delta t)$$

As $\Delta t \to 0$, the limit of the left side exsists, yields

$${P_i}'(t) = \lambda[-P_i(t)+\alpha_1 P_{i-1}(t)+\cdots+\alpha_{i-1}P_1(t)+\alpha_i P_0(t)].$$

Set $S_n=\Sigma_{i=0}^{n} P_i(t)$. Substituting ${P_i}'(t)$ to (4), then we have

$$\begin{aligned} f_{W_n} &= \sum_{i=0}^{n-1} \lambda P_i(t) - \sum_{i=1}^{n-1} \lambda[\alpha_1 P_{i-1}(t)+\alpha_2 P_{i-2}(t)+\cdots+\alpha_i P_0(t)] \\ &= \lambda[S_{n-1} - \sum_{i=1}^{n-1} \alpha_i S_{n-1-i}] \end{aligned}$$

A special case: When $\alpha_1 = 1$, W_n obeys Erlang distribution, that is

$$f_{W_n} = \lambda(S_{n-1} - \alpha_1 S_{n-2}) = \lambda P_{n-1}(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

In the following theorem, f_{W_n} is obtained by PGF of the GSDP's tailed distribution. It yields to distribution of the waiting time untill the nth event by higher-order partial and mixed derivatives.

Theorem 3.2 Set the tailed distribution of GSPD $Q_n(t)=\Sigma_{i=n}^{\infty}P_i(t)$ $(n \geq 1)$. Let $P(s,t)=\Sigma_{n=0}^{\infty}P_n(t)s^i=e^{\lambda t\sum\limits_{i=1}^{\infty}a_i(s^i-1)}$ $|s| \leq 1$. Write $F(s,t) = \sum\limits_{n=1}^{\infty} Q_n(t)s^n$ $|s| \leq 1$, the tailed probability generating function of $Q_n(t)$. Then

$$F(s,t) = \frac{s}{1-s}[1 - e^{\lambda t\sum\limits_{i=1}^{\infty}a_i(s^i-1)}],$$

hence

$$f_{W_n} = \frac{1}{n!}\cdot\frac{\partial^{n+1}}{\partial s^n \partial t}\left(\frac{s}{1-s}[1 - e^{\lambda t\sum\limits_{i=1}^{\infty}a_i(s^i-1)}]\right)\Bigg|_{s=0}.$$

Proof. The tailed probatility generating function of $Q_n(t)$ is

$$\begin{aligned} F(s,t) &= \sum_{n=1}^{\infty} Q_n(t)s^n \\ &= [P_1(t) + P_2(t) + \cdots]s + [P_2(t) + P_3(t) + \cdots]s^2 + \cdots \end{aligned}$$

$$sF(s,t) = [P_1(t) + P_2(t) + \cdots]s^2 + [P_2(t) + P_3(t) + \cdots]s^3 + \cdots.$$

Then, we have

$$F(s,t) - sF(s,t) = [1 - P_0(t)]s - P_1(t)s^2 - P_2(t)s^3 - \cdots = s[1 - P(s,t)]$$

$$\Rightarrow F(s,t) = \frac{s}{1-s}[1 - P(s,t)] = \frac{s}{1-s}[1 - e^{\lambda t\sum\limits_{i=1}^{\infty}a_i(s^i-1)}].$$

Since $\dfrac{\partial F(s,t)}{\partial t} = \sum\limits_{n=1}^{\infty} {Q_n}'(t)s^n$ $|s| \leq 1$, hence

$$\begin{aligned} f_{W_n} = {Q_i}'(t) &= \frac{1}{n!}\cdot\frac{\partial^n}{\partial s^n}\left(\frac{\partial F(s,t)}{\partial t}\right)\Bigg|_{s=0} \\ &= \frac{1}{n!}\cdot\frac{\partial^{n+1}}{\partial s^n \partial t}\left(\frac{s}{1-s}[1 - e^{\lambda t\sum\limits_{i=1}^{\infty}a_i(s^i-1)}]\right)\Bigg|_{s=0} \end{aligned}$$

□

4. STATISTIC OF GENERALIZED STUTTERING POISSON DISTRIBUTION

Cumulants κ_n, moments m_n and central moments c_n of GSPD are deduced from probability generating function and moment generating function. It can be used in SPD too.

A discrete random variable X has PGF $P(s)=\sum_{i=0}^{\infty} b_i s^i$ ($|s| < 1$), where moment generating function is $M_X(s)=P(e^s)$. Expanding e^{sX} with Taylor series at zero, we have

$$M_X(s) = E(\sum_{n=0}^{\infty} \frac{(sX)^n}{n!}) = \sum_{n=0}^{\infty} \frac{EX^n}{n!} s^n \triangleq \sum_{n=0}^{\infty} \frac{m_k}{n!} s^n, \ (|s| \leq 1),$$

where m_k is kth moment (k=0, 1, 2, $\cdots$).

4.1. Cumulants of Generalized Stuttering Poisson Distribution

Definition 4.1 Cumulants generating function of a random variable is

$$g_X(s) \triangleq \ln(M_X(s)) = \sum_{n=0}^{\infty} \kappa_n \frac{s^n}{n!}, \ (|s| \leq 1)$$

where coefficients κ_n ($n = 0, 1, 2, \cdots$) is nth cumulants. It is explicit that $\kappa_0 = 0$.

Theorem 4.1 $\xi(t) \sim GSP(\alpha_1\lambda t, \cdots, \alpha_r\lambda t)$, then the nth cumulant of $\xi(t)$ is

$$\kappa_n = \sum_{i=1}^{r} \alpha_i \lambda t i^n. \tag{5}$$

Proof. Cumulants generating function of GSPD is $g_X(s) = \sum_{i=1}^{r} \alpha_i \lambda t(e^{is} - 1)$. Expanding e^{sX} with Taylor series at zero, that is

$$\begin{aligned}&\sum_{i=1}^{r} \alpha_i \lambda t(e^{is} - 1)\\ =& - \lambda t + \lambda t[\alpha_1(1 + \sum_{j=1}^{\infty} \frac{s^j}{j!}) + \alpha_2(1 + \sum_{j=1}^{\infty} \frac{(2s)^j}{j!}) + \cdots + \alpha_r(1 + \sum_{j=1}^{\infty} \frac{(rs)^j}{j!})]\\ =&(\sum_{i=1}^{r} \alpha_i \lambda t i)s + (\sum_{i=1}^{r} \alpha_i \lambda t i^2)\frac{s^2}{2!} + \cdots + (\sum_{i=1}^{r} \alpha_i \lambda t i^l)\frac{s^l}{l!} + \cdots .\end{aligned}$$

Comparing coefficients of t^n, we have $\kappa_n = \sum_{i=1}^{r} \alpha_i \lambda t i^n$. When $r \to \infty$, κ_n may divergence. □

4.2. Moments of Generalized Stuttering Poisson Distribution

The moments of generalized stuttering Poisson distribution are deduced from cumulants.

Theorem 4.2 If $\xi(t) \sim GSP$ $(\alpha_1\lambda t, \cdots, \alpha_r\lambda t)$, then the recursion formula of nth moments m_n is

$$m_{n+1} = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) \kappa_{n+1-j} m_j, (\kappa_1 = m_1 = \sum_{i=1}^{r} \alpha_i \lambda t i). \tag{6}$$

Proof. Expanding $\ln[M_X(s)]$ with Taylor series at zero, that is

$$\begin{aligned}
\ln[M_X(s)] &= \ln(1+\sum_{n=1}^{\infty}\frac{m_n}{n!}s^n) \\
&= \sum_{n=1}^{\infty}\frac{m_n}{n!}s^n - \frac{1}{2}(\sum_{n=1}^{\infty}\frac{m_n}{n!}s^n)^2 + \cdots + \frac{(-1)^{i-1}}{i}(\sum_{n=1}^{\infty}\frac{m_n}{n!}s^n)^i + \cdots \\
&= m_1 s + \frac{m_2 - m_1^2}{2!}s^2 + \frac{m_3 - 3m_1m_2 + 2m_1^3}{3!}s^3 \\
&\quad + \frac{m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4}{4!}s^4 + \cdots .
\end{aligned}$$

By the definition of cumulant, we have

$$\kappa_1 = m_1 = EX = \sum_{i=1}^{r}\alpha_i\lambda t i \Rightarrow m_1 = \sum_{i=1}^{r}\alpha_i\lambda t i,$$

$$\kappa_2 = m_2 - m_1^2 = E(X-EX)^2 = \sum_{i=1}^{r}\alpha_i\lambda t i^2 \Rightarrow m_2 = \sum_{i=1}^{r}\alpha_i\lambda t i^2 + (\sum_{i=1}^{r}\alpha_i\lambda t i)^2,$$

$$\kappa_3 = m_3 - 3m_1m_2 + 2m_1^3 = E(X-EX)^3 = \sum_{i=1}^{r}\alpha_i\lambda t i^3 \Rightarrow m_3 = \kappa_3 + 3m_1m_2 - 2m_1^3,$$

$$\kappa_4 = m_4 - 4m_1m_3 - 3m_2^2 + 12m_1^2m_2 - 6m_1^4 = E(X-EX)^4 - 3[E(X-EX)^2]^2, \cdots$$

By taking the derivative of both side of $M_X(s)=\mathrm{e}^{g_X(s)}$ again and again with respect to s, using Leibniz formula we obtain

$$M_X^{(n+1)}(s) = \sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix}\left[g_X^{(1)}(s)\right]^{(n-i)}M_X^{(i)}(s). \tag{7}$$

Substitute $s=0$ to the nth derivatives of $g_X(s)$ and $M_X(s)$, hence

$$M_X^{(n)}(s)\Big|_{s=0} = m_n, \quad g_X^{(n)}(t)\Big|_{s=0} = \kappa_n.$$

Substituting cumulants and moments into (6), we have

$$m_{n+1} = \sum_{j=1}^{n}\begin{pmatrix} n \\ j \end{pmatrix}\kappa_{n+1-j}m_j = \sum_{j=1}^{n}\begin{pmatrix} n \\ i \end{pmatrix}(\sum_{i=1}^{r}\alpha_i\lambda t i^{n+1-j})m_j. \tag{8}$$

The relationship between cumulants and moment is

$$\kappa_n = m_n - \sum_{j=1}^{n-1}\begin{pmatrix} n-1 \\ i \end{pmatrix}\kappa_{n-j}m_j, (\kappa_1 = m_1) \tag{9}$$

$$\triangleq f(m_1, m_2, \cdots, m_n). \tag{10}$$

□

Remark Higher cumulants ($n \geq 4$) are different to central moment. Arguing from $\kappa_n = \frac{d^n}{dt^n}\ln[M_X(s)]\Big|_{s=0}$ and Faà Di Bruno formula, it can deduce to κ_n too.

Example 4.2 An alternative approach to compute the mean and variance of Bernoulli distribution

$$\kappa_n = \sum_{i=1}^{\infty} \alpha_i \lambda i^n = \sum_{i=1}^{\infty} \frac{(1-p)^i}{-ip^i \ln p}(-\ln p) i^n = \sum_{i=1}^{\infty} (\frac{1-p}{p})^i i^{n-1}$$

$$\Rightarrow EX = \sum_{i=1}^{\infty} (\frac{1-p}{p})^i = 1-p,\ DX = \sum_{i=1}^{\infty} i(\frac{1-p}{p})^i = p - p^2.$$

4.3. Central Moments of Generalized Stuttering Poisson Distribution

The moments of generalized stuttering Poisson distribution are deduced from cumulants. Expanding e^{sX} with Taylor series at zero, we have

$$M_{X-EX}(s) = E(\sum_{n=0}^{\infty} \frac{(s(X-EX))^n}{n!}) = \sum_{n=0}^{\infty} \frac{E(X-EX)^n}{n!} s^n \triangleq \sum_{n=0}^{\infty} \frac{c_k}{n!} s^n, (|s| \leq 1),$$

where c_k is kth central moment ($k = 0, 1, 2, \cdots$).

Theorem 4.3 If $\xi(t) \sim GSP(\alpha_1\lambda t, \cdots, \alpha_r\lambda t)$, then the recursion formula of nth moments c_n is

$$c_{n+1} = \sum_{j=0}^{n} \binom{n}{i} \kappa^*_{n+1-j} c_j,\ \left(c_0 = 1, c_1 = 1, \kappa^*_n = \begin{cases} \kappa_n, n \neq 1 \\ 0, n = 1 \end{cases}\right) \tag{11}$$

Proof. Expanding $\ln[M_{X-EX}(s)]$ with Taylor series at zero and comparing the coefficients of cumulant generating function, we obtain

$$\kappa^*_1 = 0,\ \kappa^*_2 = \sum_{i=1}^{r} \alpha_i \lambda t i^2,$$
$$\kappa^*_3 = \sum_{i=1}^{r} \alpha_i \lambda t i^3,\ \kappa^*_4 = \sum_{i=1}^{r} \alpha_i \lambda t i^4 + 3(\sum_{i=1}^{r} \alpha_i \lambda t i^2)^2, \cdots$$

Since

$$M^{(n+1)}_{X-EX}(s) = \sum_{i=0}^{n} \binom{n}{i} [g^{(1)}_{X-EX}(s)]^{(n-i)} M^{(i)}_{X-EX}(s),$$

Substitute $s = 0$ to the nth derivatives of $g_{X-EX}(s)$ and $M_{X-EX}(s)$, hence

$$M^{(n)}_{X-EX}(s)\Big|_{s=0} = c_n,\ g^{(n)}_{X-EX}(s)\Big|_{s=0} = \kappa^*_n.$$

Similarly to the proof in Theorem 4.2, displacing κ_n with κ^*_n, we have (11). □

5. CUMULANT ESTIMATION OF GENERALIZED STUTTERING POISSON DISTRIBUTION

5.1. Cumulant Estimation

Y.C. Patel [20] gave moment estimator of the parameters of Hermite distribution. Y.C. Patel [4] estimated the parameters of the triple and quadruple stuttering Poisson distributions with maximum likelihood estimation moment estimation, and mixed moment estimation of the parameter. Use sample moments $\hat{m}_1$ and central moments $\hat{c}_2$ and $\hat{c}_3$, let λt=1, then

$$\begin{gathered}\begin{pmatrix}\hat{m}_1\\ \hat{c}_2\\ \hat{c}_3\end{pmatrix} \triangleq \begin{pmatrix}1 & 2 & 3\\ 1 & 4 & 9\\ 1 & 8 & 27\end{pmatrix}\begin{pmatrix}\hat{\alpha}_1\\ \hat{\alpha}_2\\ \hat{\alpha}_3\end{pmatrix}\\ \Rightarrow \begin{pmatrix}\hat{\alpha}_1\\ \hat{\alpha}_2\\ \hat{\alpha}_3\end{pmatrix} = \begin{pmatrix}3 & \frac{-5}{2} & \frac{1}{2}\\ \frac{-3}{2} & 2 & \frac{-1}{2}\\ \frac{1}{3} & \frac{-1}{2} & \frac{1}{6}\end{pmatrix}\begin{pmatrix}\hat{m}_1\\ \hat{c}_2\\ \hat{c}_3\end{pmatrix},\end{gathered} \tag{12}$$

where $E\hat{\alpha}_i = \alpha_i + O(\frac{1}{n})\,, (i = 1, 2, 3)$.

When $n \geq 4$, from the computing in the proof of Theorem 4.2, higher cumulants are different to central moment or moment. Central moment or moment are nonlinear combination of $\alpha_i(i = 1, 2, \cdots)$. Thus it is difficult to estimate the parameters by using central moment or moment estimation.

Theorem 4.1 implies that κ_n is linear combination of $\alpha_i(i = 1, 2, \cdots)$. Therefore, firstly, we use sample moment κ_n to calculate in (5). From (5), when $r < \infty$, κ_n is convergence. Secondly, by solving the following system of linear equations of $\hat{\alpha}_i(i = 1, 2, \cdots, n)$ by means of 0th to nth cumulants formula

$$\begin{pmatrix}\hat{\kappa}_0\\ \hat{\kappa}_1\\ \vdots\\ \hat{\kappa}_n\end{pmatrix} \triangleq \begin{pmatrix}1 & 1 & \cdots & 1\\ 0 & 1 & \cdots & n\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 1^n & \cdots & n^n\end{pmatrix}\begin{pmatrix}-1\\ \hat{\alpha}_1\\ \vdots\\ \hat{\alpha}_n\end{pmatrix}, \tag{13}$$

where the coefficients matrix in (13) is invertible Vandermonde matrix. By solving the linear system equation we have

$$\hat{\alpha}_i = \sum_{j=1}^{n} b_{ij}\hat{\kappa}_j \triangleq T_i(\xi_1, \xi_2, \cdots, \xi_l), (i = 1, 2, \cdots, n) \tag{14}$$

Thus (14) is the cumulant estimation of GSPD, $T_i(\xi_1, \xi_2, \cdots, \xi_l)$ is a statistic of the samples ξ_i (i=1, 2, $\cdots$, l).

5.2. Consistent Estimation

Assuming the samples to the power of n, ξ_i^n (i=1, 2, $\cdots$, l) are i.i.d.. Let samples nth moment $A_{ln}=\frac{1}{l}\Sigma_{i=1}^{l}\xi_i^n$, arguing from Khintchine's law of large numbers, for all $\epsilon > 0, n$=0, 1, 2 $\cdots$, we obtain

$$\lim_{l\to\infty} P\{|A_{ln} - m_n| \geq \varepsilon\} = 0$$

From the relationship between cumulants and moment (4), $f(x_1, x_2, \cdots, x_n)$ is continuous of several variables from (9),(10), then

$$\kappa_n = f(m_1, m_2 \cdots, m_n), \hat{\kappa}_{ln} = f(A_{l1}, A_{l2} \cdots, A_{ln}).$$

It implies

$$\lim_{l \to \infty} P\{|\kappa_n - \hat{\kappa}_{ln}| \geq \varepsilon\} = 0 \tag{15}$$

by the convergence in probability properties of transformation [21]. Using the linear relation in (14), we have

$$\{|\alpha_i - \hat{\alpha}_i| \geq \varepsilon\} = \{\sum_{i=1}^{n} b_{ij} |\kappa_i - \hat{\kappa}_{li}| \geq \varepsilon\} \subset \bigcup_{i=1}^{n} \{b_{ij} |\kappa_i - \hat{\kappa}_{li}| \geq \frac{\varepsilon}{n}\}$$

then

$$\lim_{l \to \infty} P\{|\alpha_i - \hat{\alpha}_i| \geq \varepsilon\} \leq \sum_{i=1}^{n} \lim_{l \to \infty} P\{|\kappa_i - \hat{\kappa}_{li}| \geq \frac{\varepsilon}{b_{ij} n} = \varepsilon'\} = 0.$$

Thus we prove cumulant estimation of α_i $(i = 1, 2, \cdots)$ is consistent estimator.

6. APPLICATIONS

R.M. Adelson [1] put forward the recursion formula of SPD's probability density function by using Leibniz formula

$$\begin{aligned} P_{j+1}(t) &= \frac{1}{j+1}[\alpha_1 \lambda P_j(t) + 2\alpha_2 \lambda P_{j-1}(t) + \cdots + (j+1)\alpha_{j+1} \lambda P_0(t)], \\ P_0(t) &= e^{-\lambda t}, \end{aligned} \tag{16}$$

(16) avoids tediously computing the sum of much index in (1) by recursion relation. There are no nonnegative restriction to $\alpha_i (i = 1, 2, \cdots)$, so (16) can be used in GSPD.

Now we use SPD to fit auto insurance claims data [22] (the car insurance claims data from the following Table 1), and then we compare the goodness of fit with some other distributions.

From data in Table 1, total insurance policies are n=106974. The probability of zero claim policies is far greater than 0.5. Obviously it is zero-inflated data. The number of the insurance policy of ith is x_j (j=1, 2, $\cdots$, 106974), so the mean value and the 2th and the 3th central moment of the insurance policy claims rate is

$$m_1 = 0.1010806364, \ c_2 = 0.1074468102, \ c_3 = 0.1216468798.$$

According to (9) and (12), we have

$$\hat{\alpha}_1 = 0.97255, \ \hat{\alpha}_2 = 0.02496, \ \hat{\alpha}_3 = 0.00249.$$

Thus we can infer that the probability of claims of customer who buy two copies of the same insurance is only 2.496%, and three copies of the same insurance is only 0.249%. Employing recursion relation (16) and (9), we obtain $\hat{p}_i$. And then we figure out $n\hat{p}_i$. Analogously, consider quadruple SPD fitting, in this case we have

$$\tilde{\alpha}_1 = 0.97151, \ \tilde{\alpha}_2 = 0.02703, \ \tilde{\alpha}_3 = 0.00112, \ \tilde{\alpha}_4 = 0.00034.$$

In Table 1, we assume the data come from Poisson distribution (PD), triple SPD (compute by recursion formula (16)), quadruple SPD (compute by recursion formula (16)), negative binomial distribution, respectively, and then estimate the probability of different numbers of claims.

Table 1
The Comparison of Auto Insurance Claims Data of Different Distributions Fitting Effect (Moment Estimation or Cumulant Estimation)

i	Methods	0	1	2	3	4	i > 4
v_i	**Observed frequency**	**96978**	**9240**	**704**	**43**	**9**	**0**
$n\hat{p}_i$	Estimate by PD	96689.5	9773.5	494.5	16.6	0.4	0
$n\hat{p}_i$	Estimate by triple SPD	96974.1	9256.0	679.2	60.4	4.0	0.3
$n\hat{p}_i$	Estimate by quadruple SPD	96977.3	9243.2	697.6	49.1	6.1	0.7
$n\hat{p}_i$	Estimate by NPD	96985.4	9222.5	711.7	50.7	3.5	0.2

Constructing test statistical: $\eta = \sum_{i=0}^{4} \frac{v_i^2}{n\hat{p}_i} - n$, by calculating, we get

$$\eta_{PD} = 345.1250,\ \eta_{3-SPD} = 12.5786,$$
$$\eta_{4-SPD} = 2.8963,\ \eta_{NBD} = 10.1294.$$

From Pearson's chi-squared test theory, in one hand, $\chi_4^2(0.01) = 12.277$, given significant level of 0.1 we accept that claims data obey quadruple SPD or negative binomial distribution. In the other hand, $\chi_4^2(0.5) = 3.357$, given significant level of 0.5 we accept that claims data obey quadruple SPD fitting. It is thus clear that quadruple SPD fitting effect is better than that of negative binomial distribution fitting effect, and NBD is better than triple SPD. The goodness of Poisson distribution model is worst in those four distributions.

ACKNOWLEDGEMENTS

The authors want to thank Sun Chenmin, Wang Jian, Prof. Xie Minyu, Prof. Zhao Hui, Prof. Chen Yingbao, Lec. Yin Xiaobo, Prof. C.S. Kumar for their support and helpful comments. This paper is supported by "National University Student Innovation Test Plan".

REFERENCES

[1] Adelson, R.M. (1966). Compound Poisson distributions. *Operational Research, 17*(1), 73-75.
[2] Bening , V.E., & Korolev, V.Y. (2002). *Generalized Poisson models and their applications in insurance and finance*. Utrecht: VSP B.V..
[3] Ross, S.M. (2009). *Introduction to probability models* (10th ed.). Los Angeles: Academic Press.
[4] Patel, Y.C. (1976). Estimation of the parameters of the triple and quadruple stuttering-Poisson distributions. *Technometrics, 32*(4), 865-873.

[5] Roman, S. (1980). The formula of FAA Di Bruno. *The American Mathematical Monthly, 87*(10), 805-809.
[6] Zhang, H., Chu, L., & Diao, Y. (2012). Using system of differential equations to solve probability density function of stuttering Poisson distribution. *Journal of Higher Correspondence Education (Natural Sciences)(In Chinese), 25*(4), 3-5.
[7] Jnossy, L., Rnyi, A., & Aczl, J. (1950). On composed Poisson distributions I. *Acta Mathematica Hungarica, 1*(2-4), 209-224.
[8] Feller, W. (1950). *An Introduction to Probability Theory and Its Applications Vol.1* . New York: Wiley.
[9] Adelson, R.M. (1965). Some properties of the Hermite distribution. *Biometrika, 156*(2), 381-394.
[10] Galliher, H.P., Morse, P.M., & Simond, M.(1959). Dynamics of two classes of continuous-review inventory systems. *Operations Research, 7*(3), 362-383.
[11] Moothathu, T.S., & Kumar, C.S. (1995). Some properties of the stuttering Poisson distribution. *Calcutta Statistical Association Bulletin,* (45), 125-130.
[12] Gallaher, D.J. (2007). Two periodic review inventory models with backorders and stuttering Poisson demands. *AIIE transaction, 1*(2), 164-171.
[13] Feeney, G.J., & Sherbrooke, C.C. (1966). The $(s-1, s)$ inventory policy under compound Poisson demand. *Management Science, 12*(5), 391-411.
[14] Chen, J., Jackson, P.L., & Muckstadt, J.A. (2011). Exact analysis of a lost sales model under stuttering Poisson demand. *Operational Research, 59*(1), 249-253.
[15] Kuczura, A. (1973). Batch input to a multiserver queue with constant service times. *The Bell System Technical Journal, 52*(1), 83-99.
[16] Mitchell, R., Rappold, R.A., & Faulkner, W.B. (1983). On an analysis of air force EOQ data with an application to reorder point calculation. *Management Science, 29*(4), 440-446.
[17] Giles, D.E. (2010). Hermite regression analysis of multi-modal count data. *Economics Bulletin, 30*(4), 2936-2945.
[18] Klugman, S.A., Panjer, H.H., & Willmot, G.E. (2008). *Loss models: from data to decisions* (3rd ed.). New York: Wiley.
[19] Lambert, D. (1992). Zero-Inated Poisson Regression, with an Application to Defects in Manufacturing. *Technometrics, 34*(1), 1-14.
[20] Patel, Y.C. (1976). Even point estimation and moment estimation in Hermite distribution. *International Biometric Society, 32*(4), 865-873.
[21] Shao, J. (2010). *Mathematical statistics* (2nd ed.). New York: Springer.
[22] Yan, J.P. (2006). *Non-life insurance actuarial science (In Chinese).* Beijing: Peking University Press.