CSCanada
Studies in Mathematical Sciences
Vol. 5, No. 1, 2012, pp. [49–57]
DOI: 10.3968/j.sms.1923845220120501.1133

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
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Closed Form Solution of a Symmetric Competitive System of Rational Difference Equations

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Received: June 15, 2012/ Accept: August 25, 2012/ Published: August 31, 2012

Abstract: In this paper, we will study a symmetric competitive three-dimensional system of difference equations in the form:

$$x_{n+1} = \frac{x_n}{z_n y_n} \ \& \ y_{n+1} = \frac{y_n}{x_n z_n} \ \& \ z_{n+1} = \frac{z_n}{y_n x_n} \tag{1}$$

where the initial values x_0, y_0, and z_0 are nonzero real numbers. Moreover, we have studied periodicity of solutions for this system. Finally we will give some numerical examples as applications.

Key words: Difference equation; Solutions; Convergence; Periodicity; Competitive

Ibrahim, T.F. (2012). Closed Form Solution of a Symmetric Competitive System of Rational Difference Equations. *Studies in Mathematical Sciences, 5*(1), 49–57. Available from http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220120501.1133 DOI: 10.3968/j.sms.1923845220120501.1133

1. INTRODUCTION

Recently, there has been great interest in studying systems of difference equations. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models [11] describing real life situations in population biology [7], economic, probability theory, genetics, psychology, etc.

The study of properties of rational difference equations and systems of rational difference equations has been an area of interest in recent years, see book [10] and the references therein [6].

A first order system of difference equations

$$x_{n+1} = f(x_n, y_n) \ \& \ y_{n+1} = g(x_n, y_n) \tag{2}$$

where $n = 0, 1, ...,$ $(x_0, y_0) \in R, R \subset \mathbb{R}$, $(f, g) : R \to R$.

f, g are continuous function is competitive, if $f(x, y)$ is non-decreasing in x and non-increasing in y; and $g(x, y)$ is non-increasing in x and non-decreasing in y. System (2) where the functions f and g have monotonic character opposite of the monotonic character in competitive system will be called anti-competitive.

It is well known that the dynamical properties of competitive populations have received great attention from both theoretical and mathematical biologists [16] due to its universal prevalence and importance. Competitive and anti-competitive systems were studied by many authors [1–4,7,8,14–16].

In a modeling setting, the two-dimensional competitive system of nonlinear rational difference equations

$$x_{n+1} = \frac{x_n}{a + y_n} \ \& \ y_{n+1} = \frac{y_n}{b + x_n}$$

where the parameters a, b and c are positive, represents the rule by which two discrete, competitive populations reproduce from one generation to the next. The phase variables x_n and y_n denote population sizes during the n-th generation and sequence or orbit $\{(x_n, y_n) : n = 0, 1, ...\}$ describes how the populations evolve over time. Competitive between the populations is reflected by the fact that the transition function for each population is a decreasing function of the other population size. For instance, Hassell and Comins [11] studied a discrete (difference) single age-class model for two-species competition and its stability properties discussed.

There are many papers in which systems of difference equations have studied. For some other recent papers on systems of difference equations, see [5,9,12] and the related references therein.

Our goal, in this paper, is studying a symmetric competitive three-dimensional system of difference equations in the form:

$$x_{n+1} = \frac{x_n}{z_n y_n} \ \& \ y_{n+1} = \frac{y_n}{x_n z_n} \ \& \ z_{n+1} = \frac{z_n}{y_n x_n}$$

where the initial values x_0, y_0 and z_0 are nonzero real numbers. Moreover, we have studied the periodicity of solutions for this system of nonlinear rational difference equations in (1). Finally we will give some numerical examples as applications.

2. CLOSED FORM SOLUTION FOR THE THREE-DIMENSIONAL SYSTEM (1)

In this section, we try to deduce the closed form solution for the three-dimensional system of nonlinear rational difference equations (1) by two methods:

The first method depends on the following suggested notations:

Suppose that $x_0 = a,\ y_0 = b,\ z_0 = c$. Let $A_1 = bc,\ B_1 = ac$, and $C_1 = ab$.

Also consider the following notations: $A_2 = bc$, $B_2 = ac$, and $C_2 = ab$. Moreover, we can consider

$$A_3 = A_1^3,\ B_3 = B_1^3,\ C_3 = C_1^3,\ A_4 = A_1^5,\ B_4 = B_1^5,\ C_4 = C_1^5.$$

In general, we have

$$A_p = bc\prod_{i=1}^{p-2} A_i^2, \qquad B_p = ac\prod_{i=1}^{p-2} B_i^2$$

$$C_p = ab\prod_{i=1}^{p-2} C_i^2,\ where \quad p \geq 3. \tag{2.1}$$

Theorem 2.1 Suppose that $\{x_n,\ y_n,\ z_n\}$ are solutions of system (1). The solutions of the system (1), with the above notations (2.1), are given by:

$$x_1 = a/A_1,\ y_1 = b/B_1,\ z_1 = c/C_1 \tag{2.2}$$

where $n \geq 2$ and a, b, c are non-zero real numbers .

Proof. Firstly, $x_1 = \dfrac{x_0}{y_0 z_0} = \dfrac{a}{A_1}$. Similarly, $y_1 = B/b_1$ and $z_1 = C/c_1$.

Now, by mathematical induction, we will prove that equations (II) are true for $n \geq 2$. In the beginning we try to prove that equations (2.2) are true for $n = 2$.

$$x_2 = \frac{x_1}{y_1 z_1} = \frac{a/A_1}{(b/B_1)(c/C_1)} = \frac{aB_1C_1}{A_1(bc)} = \frac{aB_1C_1}{A_1A_2} = a\frac{\prod_{i=1}^{2-1} B_iC_i}{\prod_{j=1}^{2} A_j}$$

Similarly $y_2 = b\dfrac{\prod_{i=1}^{2-1} A_iC_i}{\prod_{j=1}^{2} B_j}$ and $z_2 = c\dfrac{\prod_{i=1}^{2-1} A_iB_i}{\prod_{j=1}^{2} C_j}$.

Now suppose that the equations (2.2) is true for $n = r$. This means that

$$x_r = a\frac{\prod_{i=1}^{r-1} B_iC_i}{\prod_{j=1}^{r} A_j} \ \&\ y_r = b\frac{\prod_{i=1}^{r-1} A_iC_i}{\prod_{j=1}^{r} B_j} \ \&\ z_r = c\frac{\prod_{i=1}^{r-1} A_iB_i}{\prod_{j=1}^{r} C_j}$$

Finally we prove that the equations (2.2) is true for $n = r+1$.

$$x_{r+1} = \frac{x_r}{y_r z_r} = \frac{a\dfrac{\prod\limits_{i=1}^{r-1} B_i C_i}{\prod\limits_{j=1}^{r} A_j}}{\left(b\dfrac{\prod\limits_{i=1}^{r-1} A_i C_i}{\prod\limits_{j=1}^{r} B_j}\right)\left(c\dfrac{\prod\limits_{i=1}^{r-1} A_i B_i}{\prod\limits_{j=1}^{r} C_j}\right)}$$

$$= \frac{a\prod\limits_{i=1}^{r-1} B_i C_i}{\prod\limits_{j=1}^{r} A_i\left(bc\dfrac{\prod\limits_{i=1}^{r-1} A_i^2 B_i C_i}{\prod\limits_{j=1}^{r} B_i C_i}\right)} = \frac{a\left(\prod\limits_{i=1}^{r-1} B_i C_i\right) B_r C_r}{\left(\prod\limits_{j=1}^{r} A_j\right) A_{r+1}} = a\frac{\prod\limits_{i=1}^{r} B_i C_i}{\prod\limits_{j=1}^{r+1} A_j}$$

Similarly $y_{r+1} = b\dfrac{\prod\limits_{i=1}^{r} A_i C_i}{\prod\limits_{j=1}^{r+1} B_j}$ and $z_{r+1} = c\dfrac{\prod\limits_{i=1}^{r} A_i B_i}{\prod\limits_{j=1}^{r+1} C_j}$, which complete the proof.

□

Remarks 2.2 We have the following properties:

i) $x_n y_n = \dfrac{ab}{A_n B_n}\left(\prod\limits_{k=1}^{n-1} C_k^2\right)$

ii) $y_n z_n = \dfrac{bc}{B_n C_n}\left(\prod\limits_{k=1}^{n-1} A_k^2\right)$

iii) $x_n z_n = \dfrac{ac}{A_n C_n}\left(\prod\limits_{k=1}^{n-1} B_k^2\right)$

As a direct result, we have the following property

Corollary 2.3

$$x_n y_n z_n = \frac{abc}{A_n B_n C_n}\left(\prod_{k=1}^{n-1} A_k B_k C_k\right)$$

Now we will give the solution by other method in the following theorem:

Theorem 2.4 Suppose that $\{x_n, y_n, z_n\}$ are solutions of system (1). Then, the solutions of the system (1) are given by:

$$x_{2n-1} = \frac{(x_0)^{\alpha_{n-1}}}{(y_0 z_0)^{\beta_{n-1}}} \;\&\; x_{2n} = \frac{(x_0)^{\beta_n}}{(y_0 z_0)^{\alpha_{n-1}}}$$

$$y_{2n-1} = \frac{(y_0)^{\alpha_{n-1}}}{(x_0 z_0)^{\beta_{n-1}}} \;\&\; y_{2n} = \frac{(y_0)^{\beta_n}}{(x_0 z_0)^{\alpha_{n-1}}}$$

$$z_{2n-1} = \frac{(z_0)^{\alpha_{n-1}}}{(x_0 y_0)^{\beta_{n-1}}} \;\&\; z_{2n} = \frac{(z_0)^{\beta_n}}{(x_0 y_0)^{\alpha_{n-1}}}$$

where

$$\alpha_{m+1} = 4\alpha_m + 1,\ \alpha_0 = 1,\ and\ m \geq 0$$

$$\beta_{m+1} = 4\beta_m - 1,\ \beta_0 = 1,\ and\ m \geq 0 \tag{3}$$

where the initial values x_0, y_0 and z_0 are non-zero real numbers and $n \geq 1$.

Remark 2.5 In fact (3) represent first order linear difference equations. It is easy to see that the solutions of linear difference equations (3) are given by:

$$\alpha_m = (1/3)(4^{m+1} - 1)$$
$$\beta_m = (1/3)(2 \times 4^{m+1} + 1)$$

So the solution of system (1) can be rewritten in the following theorem:

Theorem 2.6 Suppose that $\{x_n, y_n, z_n\}$ are solutions of system (1). Then, the solutions of the system (1) are given by:

$$x_{2n-1} = \frac{(x_0)^{(1/3)(4^n-1)}}{(y_0z_0)^{(1/3)(2\times4^{n-1}+1)}}\ \&\ x_{2n} = \frac{(x_0)^{(1/3)(2\times4^n+1)}}{(y_0z_0)^{(1/3)(4^n-1)}}$$
$$y_{2n-1} = \frac{(y_0)^{(1/3)(4^n-1)}}{(x_0z_0)^{(1/3)(2\times4^{n-1}+1)}}\ \&\ y_{2n} = \frac{(y_0)^{(1/3)(2\times4^n+1)}}{(x_0z_0)^{(1/3)(4^n-1)}}$$
$$z_{2n-1} = \frac{(z_0)^{(1/3)(4^n-1)}}{(y_0x_0)^{(1/3)(2\times4^{n-1}+1)}}\ \&\ z_{2n} = \frac{(z_0)^{(1/3)(2\times4^n+1)}}{(y_0x_0)^{(1/3)(4^n-1)}}$$

where the initial values x_0, y_0 and z_0 are non-zero real numbers and $n \geq 1$.

Corollary 2.7 If $x_0 = y_0 = z_0 = a$, $a \in \mathbb{R}$, such that $a \neq 0, 1, -1$, the solutions for the system (1) take the form:

$$x_n = \left\{\frac{1}{a}, a, \frac{1}{a}, a, \frac{1}{a}, a, \ldots\ldots\ldots\ldots\right\}$$
$$y_n = \left\{\frac{1}{a}, a, \frac{1}{a}, a, \frac{1}{a}, a, \ldots\ldots\ldots\ldots\right\}$$
$$z_n = \left\{\frac{1}{a}, a, \frac{1}{a}, a, \frac{1}{a}, a, \ldots\ldots\ldots\ldots\right\}$$

Remark 2.8 If $a = \pm 1$, then $x_n = y_n = z_n = \pm 1$.

Corollary 2.9 We have the following properties between the solutions of system (1):

i) $x_{2n-1}x_{2n} = \dfrac{(x_0)^{4^n}}{(y_0z_0)^{\frac{4^n}{2}}}$

ii) $y_{2n-1}y_{2n} = \dfrac{(y_0)^{4^n}}{(x_0z_0)^{\frac{4^n}{2}}}$

iii) $z_{2n-1}z_{2n} = \dfrac{(z_0)^{4^n}}{(x_0y_0)^{\frac{4^n}{2}}}$

Remark 2.10 If $x_0 = y_0 = z_0 = a$, $a \in \mathbb{R}$, then $x_{2n-1}x_{2n} = y_{2n-1}y_{2n} = z_{2n-1}z_{2n} = 1$.

3. PERIODICITY OF SOLUTIONS FOR SYSTEMS

In this section, we will determine the conditions for periodicity of system (1). (see [10,16]).

Definition 3.1 (Periodicity) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Theorem 3.2 The solution $\{x_n, y_n, z_n\}$ for system (1) is periodic of period k solution if the following conditions satisfy:

i) $\prod_{j=n+1}^{n+k-1} \frac{B_j C_j}{A_j} = \frac{A_{n+k}}{B_n C_n}$

ii) $\prod_{j=n+1}^{n+k-1} \frac{B_j A_j}{C_j} = \frac{C_{n+k}}{B_n A_n}$

iii) $\prod_{j=n+1}^{n+k-1} \frac{A_j C_j}{B_j} = \frac{B_{n+k}}{A_n C_n}$

Corollary 3.3 If the solution $\{x_n, y_n, z_n\}$ for system (1) is periodic solution of period k, then

$$\prod_{j=n+1}^{n+k-1} A_j B_j C_j = \frac{A_{n+k} B_{n+k} C_{n+k}}{(A_n B_n C_n)^2}$$

where A_j, B_j and C_j are defined in Section 2.

4. NUMERICAL RESULTS

In this section, we deal with some numerical examples.

Example 4.1 When the initial conditions are $x_0 = y_0 = z_0 \neq \pm 1$, then the system is periodic of period 2 (see Figure 1 and Corollary 2.7).

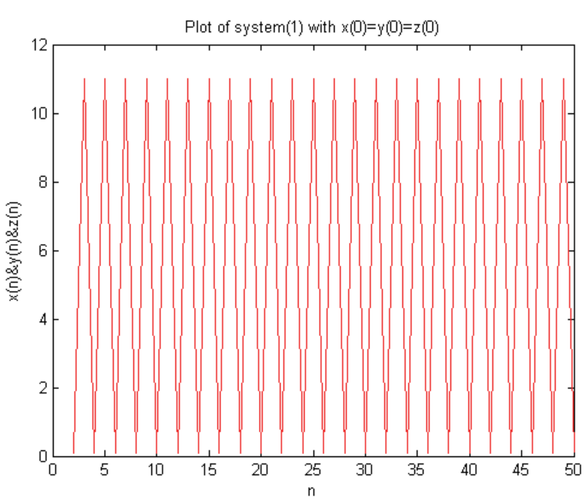

Figure 1
The Periodicity of System (1) with Period 2 in Case $\mathbf{x_0 = y_0 = z_0}$

Example 4.2 When the initial conditions are $x_0 = y_0 = z_0 = -1$, then the system (1) converge to -1 (see Figure 2, Corollary 2.7 and Remark 2.8).

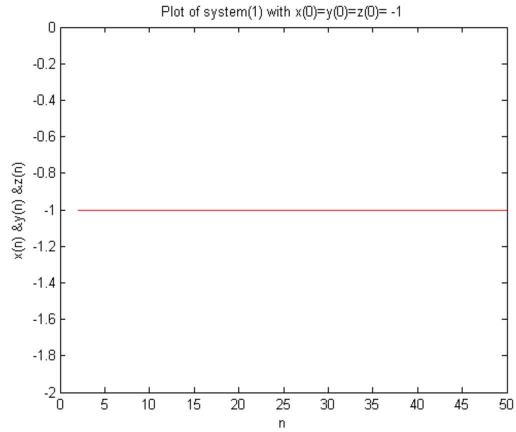

Figure 2
The Convergence of System (1) in Case $\mathbf{x_0 = y_0 = z_0 = -1}$

Example 4.3 When the initial conditions are $x_0 = y_0 = z_0 = 1$, then the system (1) converge to 1 (see Figure 3, Corollary 2.7 and Remark 2.8).

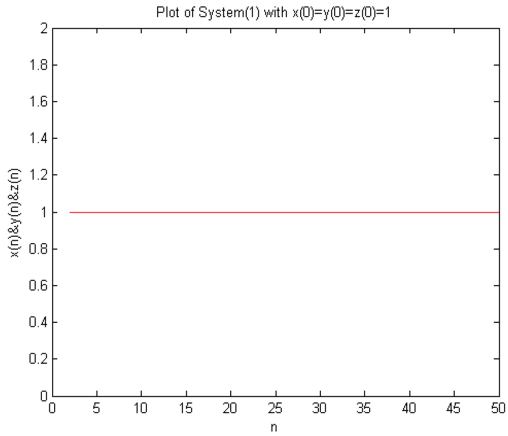

Figure 3
The Convergence of System (1) in Case $\mathbf{x_0 = y_0 = z_0 = 1}$

5. CONCLUSION

In this paper, we introduce a new technique or method which can be used it to find the solutions for some ordinary difference equations and some systems of ordinary difference equations We have already used this technique to give the solutions for a competitive system of nonlinear rational difference equations of three-dimension. We expect that this technique can be used to deduce the solutions for some partial difference equations and some systems of partial difference equations. We suggest to do algorithms for the solutions solved by this technique. We look forward to widely use this method in many difference equations.

ACKNOWLEDGEMENTS

This article was funded by the Deanship of Scientific Research (DSR), the King khalid University, Abha (under Grant No. KKU_S180_33). The author, therefore, acknowledge with thanks DSR technical and financial support. Moreover, the author is grateful anonymous referees for their valuable suggestions that improved the quality of this study.

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