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Analytical and Numerical Investigation of Soliton Solutions of Coupled Equations in Quadratic Media: A Novel Approach

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Abstract

Solitons have been hitherto studied in Kerr media. However, we show that the optical solitons are not only limited to non-linear Kerr media. In this paper, a new method is devised to achieve the analytical approximate solutions to the soliton packages in one and two dimensions. Inputting $\chi^{(1)}$ and $\chi^{(2)}$ to Maxwell equations and using second-harmonic generation of type I, the coupled motion equations are obtained. Some states of spatial bright solitons are studied for large values of α. The variational method is briefly reviewed and analytical approximate solution to quadratic-soliton coupled motion equations is investigated in one and two dimensions and also numerically solved. The comparison of analytical and numerical results indicates that our method is more accurate than the variational method.

Key words

Bright and dark solitons; Spatial optical soliton; Quadratic Kerr media; Variational method; Second-harmonic generation

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1. INTRODUCTION

Nowadays, thanks to advancements in the telecommunication industry and also the growing demand for high-speed data transfer, the use of optical communication technology is inevitable. Most of optical fibres and information processing systems have been hitherto designed based on the linear effect of optical materials. However, the speed and capacity limitations of such systems led to the use of non-linear optics as a new solution.

For instance, the third-order non-linear Kerr effect can compensate for the broadening resulting from dispersion in the fibre. Therefore, the balance between the non-linear effect and dispersion causes the pulse to propagate along the fibre without distortion. These pulses are called temporal solitons. The utilization of these solitons can increase the data transfer rate of optical fibres hundreds of times.

The interesting point is that the non-linear properties of optical materials have been widely applied not only to information-transfer technologies but also to control and process high-speed signals. Hence, the idea of light guiding light was developed, which brought about the employment of spatial solitons, the guided beams, to process optical signals. The spatial solitons are self-guided self-trapped optical beams resulting from the balance between the non-linear effect and diffraction and they do not broaden during propagation in non-linear media[1].

The first observations of spatial solitons were performed by Ashkin and Bjorkholm in the early 1970s[2]. Scientists then observed some beams whose spatial dimensions did not broaden along any tangent direction. But unfortunately, these investigations did not attract a lot of interest until 1990s that a revolutionary approach to studying the spatial optical, or more physical, solitons of solitary waves was mooted[3]. They concluded that solitons are not limited only to non-linear Kerr media. There is a special kind of spatial solitons whose formation depends on the second-order non-linear polarization term, $\chi^{(2)}$. Although the investigation of wave propagation in second-order non-linear media attracted a great deal of interest in the late 1960s and early 1970s, scientists' attention was shifted onto materials with third-order non-linear effects ($\chi^{(3)}$) after the discovery of the possibility that there may be optical solitons in this type of materials. However, these materials have some limitations that were completely known at that time[4]. In fact, only the ultrafast optical Kerr effect is important for optical telecommunication devices. In terms of quantum mechanics, the value range of the non-linear Kerr effect of optical materials is lower than the level required to manufacture desirably efficient optical devices. The search for appropriate non-linear optical materials caused scientists to switch to optical materials of $\chi^{(2)}$ again. In these materials, the self-trapping phenomenon results from the energy exchange between two or more different-frequency waves. The solitons formed in these materials are called quadratic solitons[5] in which a relatively strong and saturable non-linear effect having a tendency to trap the wave packet in the space results from the interaction of the input beam with the waveguide and its second harmonic. Since the obtained solitons include the frequency field of fundamental and second-harmonic radiations, they are inherently multi-coloured[6].

The quadratic solitons were predicted by Sukhorukov and Karamzin in the early 1970s[7]. However, from then onwards their investigation did not receive much attention until 1995 that they were empirically observed by Stegeman, Schiek, and Torruellas. In this experiment, the crystals of potassium titanyl phosphate (KTP) and lithium niobate ($LiNbO_3$) cut for phase matching were irradiated using a pulsed Nd:YAG laser. At low-energy inputs, the input infrared signal produced a green beam and they both were diffracted. But when the input power was near to the threshold value, the obtained beam was not diffracted and bright solitons were formed[6].

The quadratic solitons are formed only in waveguides with inversion asymmetry (the media in which the polarization direction does not change with changing the electric-field direction) such as KTP and $LiNbO_3$, which are used in switching and optical processing systems, owing to their unique properties[8]. The spatial and temporal quadratic solitons are similarly defined. However, the empirical observation of temporal quadratic solitons is difficult because their group-velocity dispersion is extremely smaller than the quadratic non-linear effects. Hence the spatial quadratic solitons have received more attention from researchers. Despite these solitons' stability, the equations governing them are not integrable and not solved using general analytic methods. The variational method is one of the methods for solving non-linear Schrödinger (NLS) equations. In 1983, Anderson employed this method to solve third-order non-linear Schrödinger equations[9] and showed that his results are approximately consistent with the results obtained by Hasegawa and Tappert[10]. In this method, the parameters of trial functions can be achieved through Lagrangian minimization. However, explicitly obtaining parameters imposes some limitations on the type of trial functions and consequently poses obstacles for the cases for which there is not any appropriate trial function. In 1995, Steblina and Kivshar utilized a variational method to solve the coupled equations governing quadratic solitons and provided approximate results for the soliton shape[11]. The Gaussian approximation they employed precisely predicted power distribution between first (fundamental) and second harmonics. However, their results did not appropriately correspond with the real shape of the wave. The comparison between the Gaussian results obtained by the variational method and the exact results achieved through numerical solution suggests that these results coincide with each other only at some specific points and not at asymptotic points($x \to \pm\infty$).

Here a new method is devised to obtain the analytical approximate solutions to the soliton packages in one and two dimensions. First, we briefly review the previously conducted variational method and then investigate the analytical approximate solution to the quadratic-soliton coupled equations in one and two dimensions. These equations are also numerically solved. Finally, through comparing the analytical and numerical results, we show that our method is more accurate than the variational method[12-14].

2. VARIATIONAL METHOD

As we know[15-20]:

$$\mathrm{i}\frac{\partial v}{\partial z}+r\frac{\partial^2 v}{\partial x^2}-\beta v+wv^*=0 \tag{1.1}$$

$$\mathrm{i}\sigma\frac{\partial w}{\partial z}-i\delta\frac{\partial w}{\partial x}+s\frac{\partial^2 w}{\partial x^2}-\sigma(2\beta+\Delta)w+\frac{v^2}{2}=0 \tag{1.2}$$

where $\Delta = Z_d\Delta k$, $\delta \equiv \delta_2 r_0/|\gamma_2|$, $r\equiv \mathrm{sign}(\gamma_1)$, $s\equiv\mathrm{sign}(\gamma_2)$, and $\sigma = |\gamma_1/\gamma_2|$ The dimensionless parameter β is proportional to the non-linearity-induced phase velocity shift[21-28].

Eqs. (1.1) and (1.2) are simplified as follows[29]:

$$\mathrm{i}\frac{\partial \tilde{v}}{\partial \tilde{z}}+\tilde{r}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}-\tilde{v}+\tilde{w}\tilde{v}^*=0 \tag{2.1}$$

$$\mathrm{i}\sigma\frac{\partial \tilde{w}}{\partial \tilde{z}}-i\tilde{\delta}+\tilde{s}\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}-\alpha\tilde{w}+\frac{\tilde{v}^2}{2}=0 \tag{2.2}$$

w h e r e $\alpha=\sigma(2\beta+\Delta)/\beta$, $\tilde{r}=r\,\mathrm{sign}(\beta)$, $\tilde{s}=s\,\mathrm{sign}(\beta)$, and $\tilde{\delta}=\delta\,\mathrm{sign}(\beta)\sqrt{|\beta|}$. and . Regarding the spatial states, (2.1) and (2.2) are only possible in $\tilde{r}=\tilde{s}=1$ (Bright soliton) or or $\tilde{r}=\tilde{s}=-1$ (Dark soliton)[15]. Hereafter, we neglect δ and $\bar{\delta}$ in the above equations to simplify. From this equivalent assumption, the weak spatial walk-off angle is for the spatial state or weak unmatching of group velocity is for the temporal state. After eliminating walk-off, (2.1) and (2.2) are transformed into the coupled equations below:

$$\mathrm{i}\frac{\partial v}{\partial z}+r\frac{\partial^2 v}{\partial x^2}-v+wv^*=0 \tag{3.1}$$

$$\mathrm{i}\sigma\frac{\partial w}{\partial z}+s\frac{\partial^2 w}{\partial x^2}-\alpha w+\frac{v^2}{2}=0 \tag{3.2}$$

Considering the static state, in which the value of self-trapped solutions does not depend on z, we have the following ordinary differential equations for the bright spatial solitons (r=s=+1) in one dimension:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}-v+wv=0 \tag{4.1}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2}-\alpha w+\frac{1}{2}v^2=0 \tag{4.2}$$

where $v(x)$ and $w(x)$ are real and $\mapsto$ is positive. We will obtain these equations from

$$\gamma_2\equiv(1/4k_0),[\gamma_2\equiv(-1/2)(\partial^2k_2/\partial\omega^2)\,|\,\omega=2\omega_0]$$

in the next section.

Eqs. (4.1) and (4.2) can be considered as an equation of the two-dimensional motion of a particle on the surface (v,w) that is defined by the Hamiltonian below:

$$H=\frac{1}{2}(Pv^2+Pw^2)+U(v,w) \tag{5}$$

where $Pv=dv/dx$, $Pw=dw/dx$, and $U(v,w)=0.5(v^2w-\alpha w^2-v^2)$ is the two-dimensional potential of the system.

When a=1, the analytical solutions to (4) will be as follows[5, 7, 32]:

$$v_s(x)=\pm\frac{3\sqrt{2}}{2}\mathrm{sech}^2(\frac{x}{2}) \tag{6.1}$$

$$w_s(x)=\frac{3}{2}\mathrm{sech}^2(\frac{x}{2}) \tag{6.2}$$

Therefore, there are analytical solutions for a=1 and a>>1. Steblina used a variational method to achieve the analytical approximate solutions for a<1.

By employing this method, the static solutions to (4) are the solutions to the variational problem dL=0 where L is given by:

$$L=\int_{-\infty}^{\infty}\frac{1}{2}\left[(\frac{\mathrm{d}v}{\mathrm{d}x})^2+(\frac{\mathrm{d}w}{\mathrm{d}x})^2\right]-U(v,w)\mathrm{d}x \tag{7}$$

The trial functions of v and w are considered the solutions to Gaussian profile:

$$v(x)=A\mathrm{e}^{-\rho x^2} \tag{8.1}$$

$$w(x)=B\mathrm{e}^{-\gamma x^2} \tag{8.2}$$

where A, B, ρ, and γ are positive and real. By inserting (8) into (7) and achieving $(\partial L/\partial a_j)=0$ where $a_j=\{A,B,\rho,\gamma\}$, we obtain:

$$A^2=\frac{(\rho+1)(2\rho+\gamma)(\gamma+\alpha)}{\sqrt{\rho\gamma}} \tag{9}$$

$$B=\frac{(\rho+1)\sqrt{2\rho+\gamma}}{\sqrt{2\rho}} \tag{10}$$

$$\gamma=\frac{4\rho^2}{1-\rho} \tag{11}$$

where *r* is a positive and real root of the third-order equation below:

$$20\rho^3 + (4 - 3\alpha)\rho^2 + 4\alpha\rho = \alpha \tag{12}$$

All the parameters and consequently the shape of the spatial soliton (Eqs. (8.1) and (8.2)) in one dimension are obtained through selecting the value of *a*.

Whenever we consider *a*=1 in (12), we will have *r*=0.2. Accordingly:

$$\gamma = 0.2, \qquad A = 6\sqrt{3}\,/\,5 \simeq 2.08 = v(0), \qquad B = 3\sqrt{6}\,/\,5 \simeq 1.47 = w\left(0\right) \tag{13}$$

In comparison with the results of (6.1) and (6.2), $v(0)=(3/(2)^{1/2})=2.12$ and $w(0)=(3/2)=1.5$, an approximate 2% offset can be observed.

Figure 1 compares the numerical and analytical solutions to Eqs. (8) to (12) for *a*=10. As is evident, there is not a good coincidence between the variational and numerical solutions, particularly in the wave packet tail[31, 32, 34].

3. ANALYTICAL SOLUTIONS IN ONE DIMENSION

One of the methods for finding analytical solutions is to input ansatz solutions to the basic equations[33]. We start from (3.1) and (3.2) again. By considering bright spatial solitons (r=s=+1), the most general solutions to (3.1) and (3.2) are given by[33]:

$$v = v_0(x,z)\mathrm{e}^{\mathrm{i}\theta_1(x,z)} \tag{14.1}$$

$$w = w_0(x,z)\mathrm{e}^{\mathrm{i}\theta_2(x,z)} \tag{14.2}$$

where v_0, w_0, and θ_i are real.

Using the static condition of solutions for the waves whose wave packet does not change through progress, it means:

$$\frac{\partial v_0}{\partial z} = \frac{\partial w_0}{\partial z} = 0 \tag{15}$$

and inserting (14.1) and (14.2) into (3.1) and (3.2), we obtain:

$$\begin{aligned} &-v_0\frac{\partial\theta_1}{\partial z} + \frac{\partial^2 v_0}{\partial x^2} + \mathrm{i}\frac{\partial\theta_1}{\partial x}\frac{\partial v_0}{\partial x} + \mathrm{i}\frac{\partial v_0}{\partial x}\frac{\partial\theta_1}{\partial x} \\ &+\mathrm{i}v_0\frac{\partial^2\theta_1}{\partial x^2} - v_0\frac{\partial^2\theta_1}{\partial x^2} - v_0 + w_0 v_0 \mathrm{e}^{\mathrm{i}(\theta_2 - 2\theta_1)} = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} &-\sigma w_0\frac{\partial\theta_2}{\partial z} + \frac{\partial^2 w_0}{\partial x^2} + \mathrm{i}\frac{\partial\theta_2}{\partial x}\frac{\partial w_0}{\partial x} + \mathrm{i}\frac{\partial w_0}{\partial x}\frac{\partial\theta_2}{\partial x^2} \\ &+\mathrm{i}w_0\frac{\partial^2\theta_2}{\partial x^2} - w_0\frac{\partial^2\theta_2}{\partial x^2} - \alpha w_0 + \frac{v_0^2}{2}\mathrm{e}^{\mathrm{i}(2\theta_1 - \theta_2)} = 0 \end{aligned} \tag{17}$$

If the imaginary and real parts of (16) and (17) are separated, we will have:

$$-v_0 \frac{\partial \theta_1}{\partial z} + \frac{\partial^2 v_0}{dx^2} - v_0 \frac{\partial^2 \theta_1}{\partial x^2} - v_0 + w_0 v_0 \cos(\theta_2 - 2\theta_1) = 0 \tag{18.1}$$

$$+\frac{1}{v_0} \frac{\partial}{\partial x}(v_0^2 \frac{\partial \theta_1}{\partial x}) + w_0 v_0 \sin(\theta_2 - 2\theta_1) = 0 \tag{18.2}$$

$$-\sigma w_0 \frac{\partial \theta_2}{\partial z} + \frac{\partial^2 w_0}{\partial x^2} - w_0 \frac{\partial^2 \theta_2}{\partial x_2} - \alpha w_0 + \frac{v_0^2}{2} \cos(2\theta_1 - \theta_2) = 0 \tag{19.1}$$

$$\frac{1}{w_0} \frac{\partial}{\partial x}(w_0^2 \frac{\partial \theta_2}{\partial x}) + \frac{v_0^2}{2} \sin(2\theta_1 - \theta_2) = 0 \tag{19.2}$$

If it is assumed that $\theta_1 = k_1 z$ and $\theta_2 = k_2 z$ where k_1 and k_2 are constant, the dependence of phase on z will be linear and the above equations will be simplified. Consequently, for (18.2) and (19.2) we have:

$$\sin(\theta_2 - 2\theta_1) = 0 \Rightarrow \theta_2 = 2\theta_1 \pm n\pi \quad (n \in N) \tag{20}$$

The above result contains an important physical concept that in order to obtain static solutions, the average value of the second harmonic must be twice as large as that of the first (fundamental) harmonic. To simplify the calculations and not to eliminate the generality of the problem, q_i can be considered zero[33].

Therefore, (18.1) and (19.1) are simplified as follows:

$$\frac{\mathrm{d}^2 v_0}{\mathrm{d}x^2} - v_0 + w_0 v_0 = 0 \tag{21.1}$$

$$\frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2} - \alpha w_0 + \frac{v_0^2}{2} = 0 \tag{21.2}$$

For a>>1, we know that $w_0 \simeq v_0^2 / 2\alpha$ where v_0 is the solution to the NLS equation below:

$$\frac{\mathrm{d}^2 v_0}{\mathrm{d}x^2} - v_0 + \frac{1}{2\alpha} \left| v_0 \right|^2 v_0 = 0 \tag{22}$$

Accordingly, they will be[30]:

$$v_0(x) \simeq 2\sqrt{\alpha}\,\mathrm{sech}(x) \tag{23.1}$$

$$w_0(x) \simeq 2\,\mathrm{sech}^2(x) \tag{23.2}$$

We already have the solutions to (6.1) and (6.2) for a=1. By utilizing the above equations and the solution to (6.2), it is observed that the form of $w_0(x)$ remains the same. Hence the following equation is considered for the shape of the second harmonic:

$$w_0(x) = w_{\mathrm{m}}\,\mathrm{sech}^2(\frac{x}{p}) \tag{24}$$

where w_m, the amplitude maximum, and p are unknown parameters. By inserting (24) into (21.1), the following linear eigenvalue equation is achieved:

$$\left[\frac{d^2}{dx^2} + w_m \operatorname{sech}^2\left(\frac{x}{p}\right)\right] v_0 = v_0 \tag{25}$$

We consider the equation below for the second harmonic:

$$v_0(x) = v_m \operatorname{sech}^p\left(\frac{x}{p}\right) \tag{26}$$

By inputting (26) to (25) and considering a bright single soliton ($v_0(x) \neq 0$), we will have:

$$\operatorname{sech}^2\left(\frac{x}{p}\right)\left[w_m - 1 - \frac{1}{p}\right] = 0 \tag{27}$$

that immediately results in:

$$w_m = 1 + \frac{1}{p} \tag{28}$$

Now, a relatively simple analysis is considered to obtain the unknown parameters (w_m, v_m, and p). First, we assume the solutions to $v_0(x)$ and $w_0(x)$ coincide with the real shape of the soliton solutions to (21.1) and (21.2) at the soliton peak, x=0. Therefore, by inserting these solutions into (21.2) for x=0, we obtain:

$$2w_m\left(\alpha + \frac{2}{p^2}\right) = v_m^2 \tag{29}$$

At the next step, we benefit from the fact that (21.1) and (21.2) are similar to the equations of a dynamical system of the motion of a particle with extended velocities (dv_0/dx, dw_0/dx) and potential $U_d(v_0, w_0) = (v_0^2 w_0 - v_0^2 - \alpha w_0^2)/2$ This is a conservative system whose Hamiltonian is given by:

$$H_d = \frac{1}{2}\left[\left(\frac{dv_0}{dx}\right)^2 + \left(\frac{dw_0}{dx}\right)^2\right] + U_d(v_0, w_0) \tag{30}$$

Regarding bright solitons, the field fades at infinity and consequently $H_d \equiv 0$. When $v_0(x)$ and $w_0(x)$ reach their maximum at x=0 ((dv_0/dx)=(dw_0/dx)=0), the zero potential ($U_d|_{x=0}=0$) is a prerequisite for asymptotic satisfaction ($H_d=0$). This means that:

$$v_m^2 w_m - v_m^2 - \alpha w_m^2 = 0 \tag{31}$$

Therefore, employing (28) to (31), the unknown parameters are obtained as follows:

$$v_m^2 = \frac{\alpha w_m^2}{(w_m - 1)}, \qquad p = \frac{1}{(w_m - 1)}, \qquad \alpha = \frac{4(w_m - 1)^3}{(2 - w_m)} \tag{32}$$

v_m, p, and w_m can be obtained through selecting a. If $0 < \alpha < +\infty$, other parameters will be:

$$0 < v_{\mathrm{m}} < +\infty, \qquad 1 < w_{\mathrm{m}} < 2, \qquad +\infty > p > 1; \tag{33}$$

and if a=1:

$$v_{\mathrm{m}} = \frac{3}{\sqrt{2}}, \qquad w_{\mathrm{m}} = \frac{3}{2}, \qquad p = 2 \tag{34}$$

that are the same as the results of (6.1) and (6.2). For $\alpha \longrightarrow +\infty$, the results are identical to the results obtained from (23.1) and (23.2).

4. APPROXIMATE ANALYTICAL SOLUTIONS IN TWO DIMENSIONS

Here we employ a similar method to investigate the shape of quadratic solitons in bulk waveguides. We start with the following two-dimensional coupled equations:

$$\mathrm{i}\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - v + wv^* = 0 \tag{35.1}$$

$$\mathrm{i}\sigma\frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \alpha w + \frac{1}{2}v^2 = 0 \tag{35.2}$$

We seek to obtain static solutions with circular symmetry. The same as the one-dimensional method, they are considered as follows:

$$v = v_0\left(x, y, z\right)\mathrm{e}^{\mathrm{i}\theta_1\left(x,y,z\right)} = v_0\left(r\right)\mathrm{e}^{\mathrm{i}\theta_1\left(r,z\right)} \tag{36.1}$$

$$w = w_0\left(x, y, z\right)\mathrm{e}^{\mathrm{i}\theta_2\left(x,y,z\right)} = w_0\left(r\right)\mathrm{e}^{\mathrm{i}\theta_2\left(r,z\right)} \tag{36.2}$$

By inserting (36.1) and (36.2) into (35.1) and (35.2) and considering the assumptions we made for the one-dimensional method, we obtain:

$$\frac{\mathrm{d}^2 v_0}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}v_0}{\mathrm{d}r} - v_0 + v_0 w_0 = 0 \tag{37.1}$$

$$\frac{\mathrm{d}^2 w_0}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w_0}{\mathrm{d}r} - \alpha w_0 + \frac{1}{2}v_0^2 = 0 \tag{37.2}$$

where v_0 and w_0 are real, and $r = \sqrt{x^2 + y^2}$.

Since Eqs. (37.1) and (37.2) are not integrable, they are just solved through approximate methods. Thus we consider the second harmonic as:

$$w_0\left(r\right)=w_sF\left(b_sr\right) \tag{38}$$

where the function F determines the shape of the second harmonic. w_s and b_s represent relative amplitude and the inverse width of wave packet, respectively. Also, the shape of the first harmonic is considered as:

$$v_0\left(r\right)=v_sF^p\left(b_sr\right) \tag{39}$$

We consider a=1 to determine the shape of the second harmonic. The analysis of this state is further simplified by assuming $w_0(r)=v_0(r)/(2)^{1/2}=F(r)$. The function $F(r)$ fulfils the equation below:

$$\frac{\mathrm{d}^2F}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r}-F+F^2=0 \tag{40}$$

Here we obtain the solutions to (40) through the variational method. First it should be rewritten according to the variational method:

$$\frac{\delta L}{\delta F}=0 \tag{41}$$

where d denotes the variational derivative and L represents the Lagrangian of (40) that is given by:

$$L\left(F\right)=\int_0^{+\infty}\left[(\frac{\mathrm{d}F}{\mathrm{d}r})^2+F^2-\frac{2}{3}F^3\right]rdr \tag{42}$$

Next, we consider F as follows:

$$F_0(r)=F_{\mathrm{m}}\operatorname{sech}^2(b_0r) \tag{43}$$

Based on (41), when F is the exact solution to (40), the Lagrangian reaches its minimum. Accordingly, the extremum points of the integral $L(F_0)$ must be calculated to obtain the amplitude maximum (F_{m}) and inverse width (b_0) in (43). It means:

$$\frac{\partial L(F_0)}{\partial F_{\mathrm{m}}}=\frac{\partial L(F_0)}{\partial b_0}=0 \tag{44}$$

Through solving them, we have[36]:

$$F_{\mathrm{m}}=\frac{15(4\log 2-1)}{32\log 2-11}\approx 2.3781 \tag{45}$$

$$b_0=\frac{1}{2}\sqrt{\frac{5(4\log 2-1)}{8\log 2+1}}\approx 0.5818 \tag{46}$$

Eqs. (43), (45), and (46) will be useful for the following calculations. Also, the following equation can be substituted for dF/dr:

$$(\frac{\mathrm{d}F}{\mathrm{d}r})^2 \simeq 4b_0^2F^2(1-\frac{F}{F_\mathrm{m}}) \tag{47}$$

To obtain the parameters of (38) and (39), we first input them to (37.1) and (37.2), and then remove the coefficients of the exponents zero and one ($b_\mathrm{s}^2 \approx b_\mathrm{s}$):

$$pb_\mathrm{s}^2\left[1+4b_0^2(p-1)\right]=0 \tag{48}$$

$$pb_\mathrm{s}^2\left[1+4b_0^2(p-1)F_\mathrm{m}^{-1}\right]=w_\mathrm{s} \tag{49}$$

Afterwards, we consider (37.2) when r goes to zero:

$$w_\mathrm{s}b_\mathrm{s}^2(1-F_\mathrm{m})-\alpha w_\mathrm{s}+\frac{1}{2}v_\mathrm{s}^2F_\mathrm{m}^{2p-1}=0 \tag{50}$$

in which (40) is rewritten as a simpler form. Because r approaches zero ($r\rightarrow 0$), we substitute F_m, the amplitude maximum, for F.

To more appropriately investigate the asymptotic state of far fields and calculate all the parameters, we need another equation that is preferably obtained from a conservation law. The dynamical system defined by (37.1) and (37.2) is not Hamiltonian due to the loss term $(1/r)/(\mathrm{d}F/\mathrm{d}r)$, but we can still approximately calculate the parameters in two dimensions as well as in one dimension. We insert (38) and (39) into (37.2), multiply the product by $\mathrm{d}F(b_\mathrm{s}r)/\mathrm{d}r$, integrate it over the interval $0<r<+\infty$, apply the approximation $b_\mathrm{s}^2 \approx b_\mathrm{s}$, and employ the following equation:

$$\int_0^{+\infty}(\frac{dF}{dr})^2\frac{\mathrm{d}r}{r}=\frac{1}{3}F_\mathrm{m}^3-\frac{1}{2}F_\mathrm{m}^2 \tag{51}$$

to obtain the equation below:

$$w_\mathrm{s}b_\mathrm{s}^2(\frac{1}{2}-\frac{1}{3}F_\mathrm{m})-\frac{1}{2}\alpha w_\mathrm{s}+\frac{1}{4p+2}v_\mathrm{s}^2F_\mathrm{m}^{2p-1}=0 \tag{52}$$

Eq. (52) must be rewritten with respect to F_m and b_0 (the known parameters) to calculate p. Hence, we obtain the following equations from (48) and (49):

$$(b_\mathrm{s})^{-2}=p\left[1+4b_0^2(p-1)\right] \tag{53}$$

$$\frac{u_\mathrm{s}^2}{b_\mathrm{s}^2}F_\mathrm{m}^{2p-1}=\frac{2\alpha w_\mathrm{s}}{b_\mathrm{s}^2}-2w_\mathrm{s}(1-F_\mathrm{m}) \tag{54}$$

By inputting (53) and (54) to (52) and simplifying it to a third-order equation:

$$8\alpha b_0^2p^3+2\alpha(1-6b_0^2)p^2+\left[(4/3)F_\mathrm{m}-2\right]+\alpha(4b_0^2-1)p+\left[1-(4/3)F_\mathrm{m}\right]=0 \tag{55}$$

Now p can be obtained through selecting a. Moreover, b_s, w_s, and v_s can be calculated using (48) and (49). Next, by employing (43), (45), and (46) at a=1, and also (38) and (39), the final solutions will be as follows:

$$v_0(r) = v_m \operatorname{sech}^{2p}(rb) \tag{56}$$

$$w_0(r) = w_m \operatorname{sech}^{2}(rb) \tag{57}$$

where $v_m = v_s F_m^p$, $w_m = w_s F_m$, and $b = b_s b_0$ .Eqs. (56) and (57) determine the shape of the soliton in bulk media. When α=1, F_m=1.5, and b_0=0.5, these equations will be identical to the one-dimensional equations obtained at the previous section.

5. RESULTS AND DISCUSSION

To numerically solve (4.1) and (4.2), we consider them as a system of four ordinary first-order differential equations:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = p_v; \quad \frac{\mathrm{d}p_v}{\mathrm{d}x} = v - wv; \quad \frac{\mathrm{d}w}{\mathrm{d}x} = p_w; \quad \frac{\mathrm{d}p_w}{\mathrm{d}x} = \alpha w - \frac{v^2}{2} \tag{58}$$

where $v(x)$ and $w(x)$ are the coordinates of a mechanical problem with the Hamiltonian below:

$$H = \frac{1}{2}p_v^2 + \frac{1}{2}p_w^2 + \frac{1}{2}(v^2 w - \alpha w^2 - v^2) \tag{59}$$

The self-trapped solutions to (58) are discrete trajectories in the four-dimensional space (v, w, p_v, p_w). These trajectories start from (0, 0, 0, 0) at $x \to -\infty$ and asymptotically end with (0, 0, 0, 0) $x \to +\infty$ at. A two-point boundary problem must be solved to trace these trajectories[35]. The boundary conditions should satisfy at two points (starting and ending points of the integration interval). Of course, by applying symmetry, we can sometimes consider the interval from zero to infinity.

The shooting method is one of the methods for solving boundary value problems. In this method, values corresponding to the boundary conditions are considered for boundary-dependent variables. Then, the ordinary differential equations are integrated using a Runge–Kutta method from the initial point to the next boundary point[35]. In this method, the integration of differential equations is the same as tracing the trajectory of a bullet fired toward a specific target and selecting precise initial conditions is the same as accurately targeting, which is performed by selecting $1<w_m<2$ that is obtained using (22) for $\alpha>0$. This condition can also be achieved using the fact that the Hamiltonian is conservative at x=0. By selecting an interval for v_m and w_m and using an iteration loop and Runge–Kutta method, we can obtain precise solutions. These solutions are compared with the analytical solutions (see Figures 2-9). The circles and squares denote the numerical solutions and the curves represent the analytical solutions to Eqs. (24) and (26), which are termed first harmonic and second harmonic. The comparison indicates that the numerical and analytical solutions exactly coincide with each other for many values of α, especially for $\alpha>1$. Furthermore, in comparison with the Gaussian curves (Figure 1), there is a better coincidence between the hyperbolic curves and numerical solutions. Table 1 provides the numerical and analytical results for α=1.

Table 1
Analytical and Numerical Solutions for First and Second Harmonic Amplitudes (*v*,*w*) at *a*=1.

X	V (numerical)	V (analytical)	W (numerical)	W (analytical)
–7.1784	0.00653	0.00646	0.00451	0.00457
–6.93912	0.00827	0.00821	0.00577	0.0058
–6.69984	0.01049	0.01042	0.00735	0.00737
–6.46056	0.01331	0.01323	0.00936	0.00935
–6.22128	0.0169	0.01679	0.01191	0.01187
–5.982	0.02145	0.02131	0.01514	0.01507
–5.74272	0.0269	0.02703	0.019	0.01911
–5.50344	0.03413	0.03428	0.02412	0.02424
–5.26416	0.04329	0.04345	0.0306	0.03072
–5.02488	0.05488	0.05504	0.03879	0.03892
–4.7856	0.06952	0.06968	0.04915	0.04927
–4.54632	0.08798	0.08812	0.0622	0.06231
–4.30704	0.1112	0.11131	0.07863	0.07871
–4.06776	0.14035	0.14038	0.09924	0.09927
–3.82848	0.17681	0.17672	0.12502	0.12496
–3.5892	0.22219	0.22194	0.15711	0.15693
–3.34992	0.27838	0.27788	0.19684	0.19649
–3.11064	0.34746	0.34662	0.24569	0.2451
–2.87136	0.43163	0.43034	0.30521	0.3043
–2.63208	0.53305	0.53117	0.37692	0.3756
–2.3928	0.65354	0.65094	0.46212	0.46028
–2.15352	0.79417	0.79071	0.56156	0.55912
–1.91424	0.94622	0.95028	0.66908	0.67195
–1.67496	1.12359	1.12748	0.7945	0.79725
–1.43568	1.31399	1.3175	0.92913	0.93161
–1.1964	1.50948	1.5124	1.06737	1.06943
–0.95712	1.69889	1.70108	1.2013	1.20284
–0.71784	1.86852	1.86991	1.32124	1.32223
–0.47856	2.00368	2.00435	1.41681	1.41729
–0.23928	2.09106	2.09124	1.47861	1.47873
0	2.12132	2.12132	1.5	1.5
0.23928	2.09106	2.09124	1.47861	1.47873
0.47856	2.00368	2.00435	1.41681	1.41729
0.71784	1.86852	1.86991	1.32124	1.32223
0.95712	1.69889	1.70108	1.2013	1.20284
1.1964	1.50948	1.5124	1.06737	1.06943
1.43568	1.31399	1.3175	0.92913	0.93161
1.67496	1.12359	1.12748	0.7945	0.79725
1.91424	0.94622	0.95028	0.66908	0.67195
2.15352	0.79417	0.79071	0.56156	0.55912
2.3928	0.65354	0.65094	0.46212	0.46028
2.63208	0.53305	0.53117	0.37692	0.3756
2.87136	0.43163	0.43034	0.30521	0.3043
3.11064	0.34746	0.34662	0.24569	0.2451
3.34992	0.27838	0.27788	0.19684	0.19649
3.5892	0.22219	0.22194	0.15711	0.15693
3.82848	0.17681	0.17672	0.12502	0.12496
4.06776	0.14035	0.14038	0.09924	0.09927
4.30704	0.1112	0.11131	0.07863	0.07871
4.54632	0.08798	0.08812	0.0622	0.06231
4.7856	0.06952	0.06968	0.04915	0.04927
5.02488	0.05488	0.05504	0.03879	0.03892
5.26416	0.04329	0.04345	0.0306	0.03072
5.50344	0.03413	0.03428	0.02412	0.02424
5.74272	0.0269	0.02703	0.019	0.01911
5.982	0.02145	0.02131	0.01514	0.01507
6.22128	0.0169	0.01679	0.01191	0.01187
6.46056	0.01331	0.01323	0.00936	0.00935
6.69984	0.01049	0.01042	0.00735	0.00737
6.93912	0.00827	0.00821	0.00577	0.0058
7.1784	0.00653	0.00646	0.00451	0.00457

6. CONCLUSION

Heretofore, most investigations of solitons have been carried out in non-linear Kerr media. In such media, the odd orders are considered as a non-linear effect in polarization. The solitons produced in such media are often utilized in optical fibres. However, we showed that the optical solitons are not limited only to non-linear Kerr media. The coupled motion equations were achieved by considering the linear term $x^{(1)}$ and nonlinear quadratic polarization ($x^{(2)}$) in the Maxwell equations and employing second-harmonic generation (type I). Some specific states of spatial bright solitons were studied at large values of *a* and their analytical solutions were obtained. It was shown that the coupled equations' parameters *r* and *s* must equal positive one for stable static bright solitons to appear. We obtained the analytical solutions to the equations in the static state and compared them with the numerical solutions. It was demonstrated that in comparison with the analytical solutions obtained through stationary Gaussian process (SGP), the analytical solutions to our coupled equations coincide with the numerical solutions more appropriately. The solitons investigated in this paper are being widely used in switching and all-optical processing systems and they have a major role in increasing the speed of such systems.

As future proposals:

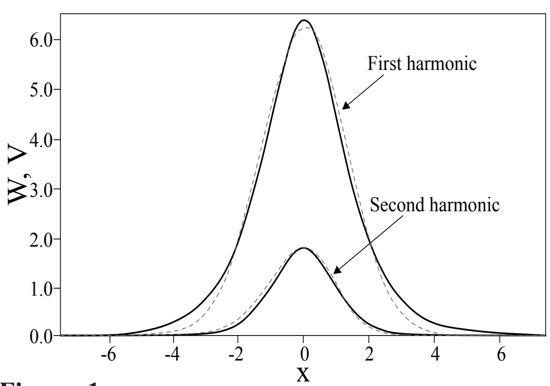

Figure. 1
Gaussian Solutions to Eqs. (8) to (12) and Numerical Solutions at α =10

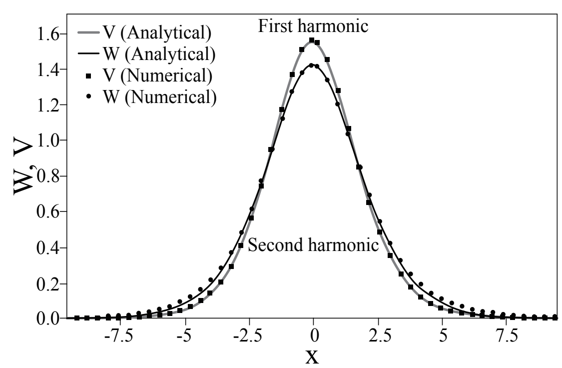

Figure. 2
Analytical Solutions to (24), (26), and (32) and Numerical Solutions at α =0.5

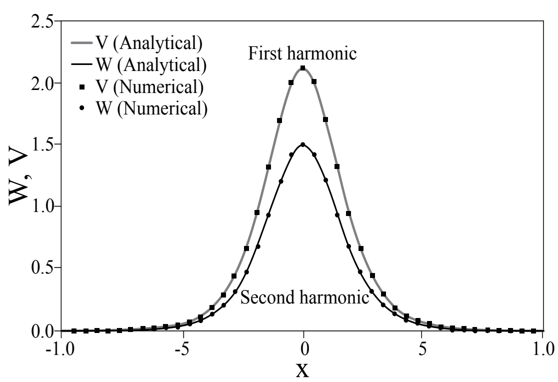

Figure. 3
Analytical Solutions to (24), (26), and (32) and Numerical Solutions at α =1

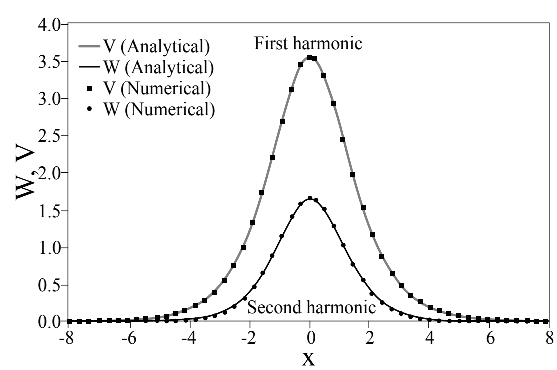

Figure. 4
Analytical Solutions to (24), (26), and (32) and Numerical Solutions at α =3

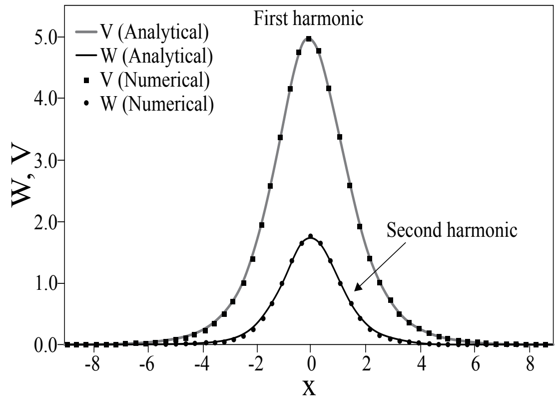

Figure. 5
Analytical Solutions to (24), (26), and (32) and Numerical Solutions at α =6

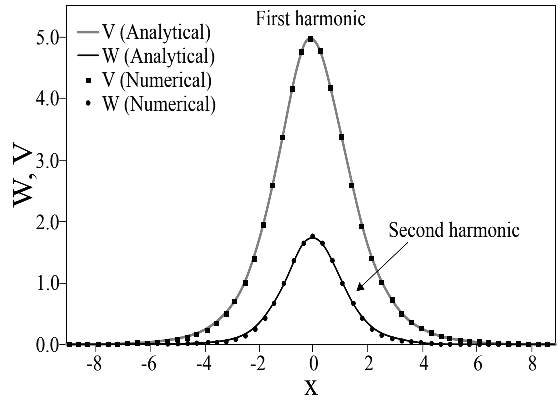

Figure. 6
Analytical Solutions to (24), (26), and (32) and Numerical Solutions at α =10

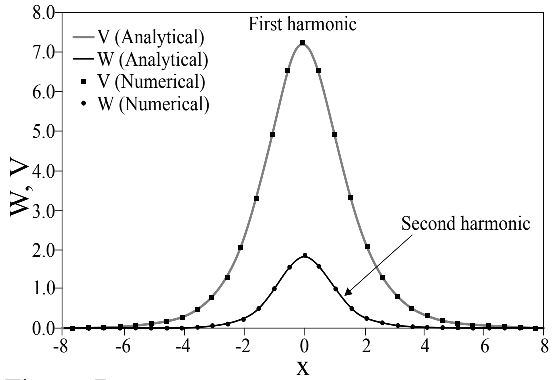

Figure. 7
Analytical Solutions to (24), (26), and (32) and Numerical Solutions at α =13

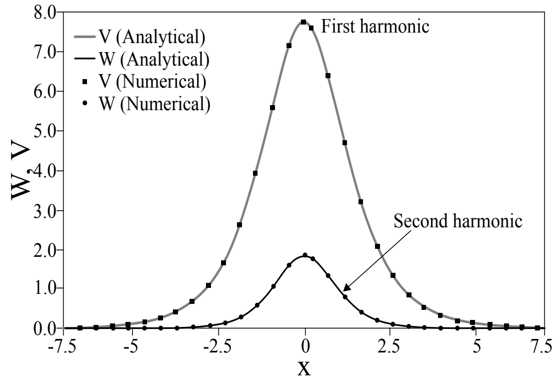

Figure. 8
Analytical Solutions to (24), (26), and (32) and Numerical Solutions at α =15

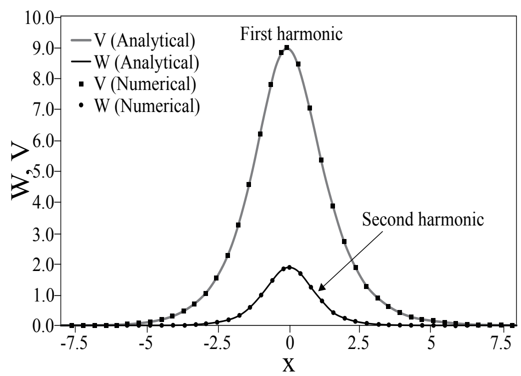

Figure. 9
Analytical Solutions to (24), (26), and (32) and Numerical Solutions at α =20

The numerical investigation of these solitons: Eqs. (3.1) and (3.2) can be solved through the split-step beam propagation method (BPM). The integration of the partial differential equations (3) with respect to the propagation parameter z is similar to the initial-input propagation along the coordinate z. This input can be considered as the static $w(x)$ and $v(x)$ solutions to (4.1) and (4.2), which was calculated for different values of a in this paper. By little perturbation, we consider these solutions as the initial conditions of (3) and investigate their progress along the axis z. The studies suggest that the wave amplitudes of $w(x)$ and $v(x)$ tend to their initial non-perturbational values by inputting perturbation, which results in soliton stability.

Both the cubic ($\chi^{(3)}$) and ($\chi^{(2)}$) quadratic non-linear effects can be present in the medium. This matter arises from the situation that there is always a little $\chi^{(3)}$ in a quadratic nonlinear medium. These effects affect the dynamics of wave nonlinear propagation.

The spatial bright solitons ($r = s = +1$) were studied in this paper. The dark ($r = s = -1$) or double (($r = +1, s = -1$), ($r = -1, s = +1$)) solitons can be studied in the future.

Given that considering SHG of type I led to two coupled motion equations, SHG of type II can be employed to investigate three coupled equations.

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