CSCanada
Studies in Mathematical Sciences
Vol. 4, No. 2, 2012, pp. 48-53
DOI: 10.3968/j.sms.1923845220120402.2001

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
www.cscanada.org

On the Set of Indices of Convergence for Reducible Tournament Matrices

CHEN Xiaogen[a],*

[a]School of Information Science and Technology , Zhanjiang Normal University Zhanjiang 524048, People's Republic of China.
*Corresponding author.
Address: School of Information Science and Technology , Zhanjiang Normal University Zhanjiang 524048, People's Republic of China.

Supported by Guangdong Provinial Natural Science Foundation of China(No:9151027501000039)

Received 20 October, 2011; accepted 10 January, 2012

Abstract
We obtain that the set of indices of convergence for n by n reducible tournament matrices.

Key words
Boolean matrix; Reducible tournament matrix; Primitive exponent; Convergent index

CHEN Xiaogen (2012). On the Set of Indices of Convergence for Reducible Tournament Matrices. *Studies in Mathematical Sciences, 4*(2), 48-53. Available from: URL: http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220120402.2001 DOI: http://dx.doi.org/10.3968/j.sms.1923845220120402.2001

1. INTRODUCTION

A Boolean matrix is a matrix over the binary Boolean algebra $\{0, 1\}$, where the (Boolean)addition and (Boolean) multiplication in $\{0, 1\}$ are defined as $a + b = max\{a, b\}$, $ab = min\{a, b\}$(we assume $0 < 1$).

Let $\mathfrak{B}_n$ denote the set of all n by n matrices over the Boolean algebra $\{0, 1\}$.Then $\mathfrak{B}_n$ forms a finite multiplicative semigroup of order 2^{n^2}. Let $B \in \mathfrak{B}_n$. The sequence of powers $B^0 = I, B^1, B^2, \cdots$, clearly forms a finite sub-semigroup of $\mathfrak{B}_n$, and then there exists a least nonnegative integer $k = k(B)$ such that $B^k = B^{k+t}$ for some $t \geq 1$, and there exists a least positive integer $p = p(B)$ such that $B^k = B^{k+p}$. The integer $k = k(B)$ and $p = p(B)$ are called the index of convergence of B and the period of convergence of B respectively or simply the "index"and "period" of B.

For $B \in \mathfrak{B}_n$, if there is a permutation matrix P such that $PBP^T = A$, then we say B is permutation similar to a matrix A (written $B \sim A$).

A matrix $B \in \mathfrak{B}_n$ is reducible if $B \sim \begin{pmatrix} B_1 & 0 \\ C & B_2 \end{pmatrix}$, where B_1 and B_2 are square(non-vacuous), and B is irreducible if it is not reducible.

A Boolean matrix $B \in \mathfrak{B}_n$ is primitive if there is a nonnegative integer k such that $B^k = J$, the all-ones matrix. The least such k is called the exponent of B, denoted by $\gamma(B)$. It is easy to verify that if B is a primitive matrix, then $k(B) = \gamma(B)$. Hence, the concept of index of a Boolean matrix is in fact a generalization of the concept of the primitive exponent of a primitive matrix.

It is well known that B is primitive if and only if B is irreducible and $p(B) = 1$.

A matrix $A = [a_{ij}] \in \mathfrak{B}_n$ is called tournament matrix if $a_{ii} = 0(i = 1, 2, \ldots, n)$ and $a_{ij} + a_{ji} = 1(1 \leq i < j \leq n)$. Let $\mathfrak{T}_n$ denote the set of all $n \times n$ tournament matrices. Notice that a matrix $T_n \in \mathfrak{T}_n$ satisfies the equation

$$A_n + A_n^T = J_n - I_n$$

where J_n is the matrix of all $1'$s and I_n is the identity matrix.

Our main interests are in the study of the index $k(B)$. In particular, we are interested in the study of the index set (set of the indices) for various classes of n by n Boolean matrices. The index (or exponent)set problem of primitive Boolean matrices is already settled in [1]. In this paper we give the index set of reducible tournament matrices.

2. PRELIMINARIES

The notation and terminology used in this paper will basically follow those in [1]. For convenience of the reader, we will include here the necessary definitions and basic results in [3,5,6].

We use the following notations.

$$\bar{T}_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & 0 & 1 \\ 1 & \cdots & \cdots & 1 & 0 & 0 \end{pmatrix}_{n\times n} \quad (n \geq 3), \quad \mathbb{T}_l = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \end{pmatrix}_{l\times l},$$

$$\mathcal{T}_{3m} = \begin{pmatrix} \bar{T}_3 & 0 & \cdots & 0 \\ J & \bar{T}_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & \bar{T}_3 \end{pmatrix}, \quad \mathcal{I}_{3m} = \begin{pmatrix} I_3 & 0 & \cdots & 0 \\ J & I_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & I_3 \end{pmatrix},$$

where J is the matrix of all $1'$s, I_3 is the identity matrix of order 3.

Lemma 2.1 ([2]): Let $T_n \in \mathfrak{T}_n$. Then

$$T_n \sim \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ J & A_2 & 0 & \cdots & 0 \\ J & J & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \cdots & A_k \end{pmatrix},$$

where all the blocks J below the diagonal are matrices of 1's, and the diagonal blocks $A_1, \cdots, A_k$ are irreducible components of T_n. Let A_i be n_i by n_i matrix, $1 \leq i \leq k, 1 \leq n_i \leq n$. Then k and n_i are uniquely determined by T_n.

It is obvious that the irreducible tournament matrix of order 1 is zero matrix of order 1, the irreducible tournament matrix of order 2 is not exists, and the irreducible tournament matrix of order 3 is isomorphic to $\bar{T}_3$. Hence, in Lemma2.1, the diagonal blocks A_i is zero matrix of order 1, or $\bar{T}_3$, or irreducible tournament matrix of order $m_i(4 \leq m_i \leq n)$. Let $A_i \neq (0)_{1\times 1}$ (if there exists), $A_{i+1} = A_{i+2} = \ldots = A_{i+l_i} = (0)_{1\times 1}, A_{i+l_i+1} \neq (0)_{1\times 1}$(if there exists). Then

$$\begin{pmatrix} A_{i+1} & 0 & \cdots & 0 \\ J & A_{i+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & A_{i+l_i} \end{pmatrix} = \mathbb{T}_{l_i} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \end{pmatrix}_{l_i\times l_i}$$

Let $A_j \neq \bar{T}_3$ (if there exists), $A_{j+1} = A_{j+2} = \ldots = A_{j+q_i} = \bar{T}_3, A_{j+q_i+1} \neq \bar{T}_3$(if there exists). Then

$$\begin{pmatrix} A_{j+1} & 0 & \cdots & 0 \\ J & A_{j+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & A_{j+q_i} \end{pmatrix} = \mathcal{T}_{3q_i} = \begin{pmatrix} \bar{T}_3 & 0 & \cdots & 0 \\ J & \bar{T}_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \\ J & \cdots & J & \bar{T}_3 \end{pmatrix}_{3q_i \times 3q_i}.$$

We have

Lemma 2.2: Let $T_n \in \mathfrak{T}_n$. Then

$$T_n \sim \begin{pmatrix} B_1 & 0 & 0 & \cdots & 0 \\ J & B_2 & 0 & \cdots & 0 \\ J & J & B_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \cdots & B_g \end{pmatrix},$$

where all the blocks J below the diagonal are matrices of 1's, and the diagonal blocks B_i is $\mathcal{T}_{3q_i}$, or $\mathbb{T}_{l_i}$, or irreducible $T_{m_i} \in \mathfrak{T}_{m_i}$, $4 \leq m_i \leq n$, $1 \leq i \leq g$, $0 \leq 3q_i, l_i \leq n$, and q_i, l_i, m_i, g are uniquely determined by T_n.

Clearly, $p(\mathbb{T}_{l_i}) = p(T_{m_i}) = 1$, $p(\mathcal{T}_{3q_i}) = 3$ in Lemma2.2. Hence we have

Lemma 2.3: Let $T_n \in \mathfrak{T}_n$. Then $p(T_n) = 1$ or 3.

Lemma 2.4 ([3]): If $T_n \in \mathfrak{T}_n$ and $n \geq 4$. Then T_n is primitive if and only if T_n is irreducible.

It is obvious that 3×3 tournament matrix is not primitive, the primitive exponent of 4×4 irreducible tournament matrix is 9. For $n > 4$, we have

Lemma 2.5 ([3]): If $T_n \in \mathfrak{T}_n$ and $n \geq 5$, then $\gamma(T_n) \leq n + 2$.

Lemma 2.6 ([5]): Let $n \geq 5$,then $\gamma(\bar{T}_n) = n + 2$.

Lemma 2.7([5]): If $n \geq 5, T_n \in \mathfrak{T}_n$ is irreducible. Then $\gamma(T_n) = n + 2$ if and only if T_n is isomorphic to $\bar{T}_n$.

Lemma 2.8 ([3]): If $3 \leq e \leq n+2$ and $n \geq 6$, then there exists an irreducible $T_n \in \mathfrak{T}_n$ such that $\gamma(T_n) = e$.

For n by n tournament matrices, the set of primitive exponents is $\{3, 4, \cdots, n+2\}(n \geq 6)$ in [3]. We have that the set of indices of convergence for $n \times n$ reducible tournament matrices with period p.

3. THE SET OF INDICES OF CONVERGENCE FOR REDUCIBLE TOURNAMENT MATRICES

We use the following notations.

$\mathfrak{T}\mathfrak{I}_n$: irreducible matrices in $\mathfrak{T}_n$,

$\mathfrak{T}\mathfrak{R}_n$: reducible matrices in $\mathfrak{T}_n$,

$\mathfrak{T}\mathfrak{I}(n, p)$: matrices with period p in $\mathfrak{T}\mathfrak{I}_n$,

$\mathfrak{T}\mathfrak{R}(n, p)$: matrices with period p in $\mathfrak{T}\mathfrak{R}_n$,

$ITI(n, p)$: indices of convergence of matrices in $\mathfrak{T}\mathfrak{I}(n, p)$,

$ITR(n, p)$: indices of convergence of matrices in $\mathfrak{T}\mathfrak{R}(n, p)$.

$ITR(n)$: indices of convergence of matrices in $\mathfrak{T}\mathfrak{R}_n$.

Theorem 3.1: Let $T_n \in \mathfrak{T}\mathfrak{R}(n, p)$ and $n \geq 10$. Then $k(T_n) \leq n - p + 2$.

Proof: Let $T_n \in \mathfrak{T}\mathfrak{R}(n, p)$. It is obvious that $k(T_n) = \max_{1 \leq i \leq g}\{k(B_i)\}$ $=\max_{1 \leq i \leq g}\{l_i, k(B_{mi})\}=\max_{1 \leq i \leq g}\{l_i, m_i + 2\}$ in Lemma 2.2, where $n \geq 10$ and $4 \leq m_i < n$. By Lemma 2.3, $p(T_n) = 1$ or 3.

If $p(T_n) = 1$. There does not exist B_i that is $\mathcal{T}_{3q_i}$, $1 \leq i \leq g, 1 \leq 3q_i$, in Lemma 2.2. Hence $k(T_n) = \max_{1 \leq i \leq g}\{l_i, m_i + 2\} \leq n - 1 + 2 = n - p + 2$.

If $p(T_n) = 3$. There exists B_i that is $\mathcal{T}_{3q_i}$, $1 \leq i \leq g, 1 \leq 3q_i$, in Lemma 2.2. Hence $k(T_n) = \max_{1\leq i\leq g}\{l_i, m_i + 2\} \leq n - 3 + 2 = n - p + 2$.

Let $T_n^{(1)} = \begin{pmatrix} 0 & 0 \\ J & \bar{T}_{n-1} \end{pmatrix}$ and $T_n^{(2)} = \begin{pmatrix} \bar{T}_3 & 0 \\ J & \bar{T}_{n-3} \end{pmatrix}$. Then $T_n^{(1)} \in \mathfrak{TR}(n, 1)$ and $T_n^{(2)} \in \mathfrak{TR}(n, 3)$. By Lemma2.6, $k(T_n^{(1)}) = k(\bar{T}_{n-1}) = n - 1 + 2 = n + 1$ and $k(T_n^{(2)}) = k(\bar{T}_{n-3}) = n - 3 + 2 = n - 1$. where $n \geq 10$. We complete the proof.

Theorem 3.2: Let $n \geq 2$. Then

$$ITR(n,1) = \begin{cases} \{n\} & n = 2,3,4, \\ \{5,9\} & n = 5, \\ \{4,6,7,9\} & n = 6, \\ [3,9] & n = 7, \\ [3,n+1] & n \geq 8. \end{cases}$$

where $[3, n + 1] = \{3, 4, \cdots, n + 1\}$.

Proof: Note that $\mathbb{T}_l \in \mathfrak{TR}(n, 1)$ and $k(\mathbb{T}_l) = l(l \geq 1)$, hence $n(\geq 2) \in ITR(n, 1)$. It is easily verified that $ITI(5, 1) = \{4, 6, 7\}$. Now let $\tilde{T}_5 = \begin{pmatrix} 0 & 0 \\ J & T_4 \end{pmatrix}$, where $T_4 \in \mathfrak{TI}(4, 1)$, then $\tilde{T}_5 \in \mathfrak{TR}(5, 1)$ and $k(\tilde{T}_5) = k(T_4) = 9$.

Let $\tilde{T}_6 = \begin{pmatrix} 0 & 0 \\ J & T_5 \end{pmatrix}$, where $T_5 \in \mathfrak{TI}(5, 1)$, then $\tilde{T}_6 \in \mathfrak{TR}(6, 1)$ and $k(\tilde{T}_6) = k(T_5)$. Let $\hat{T}_6 = \begin{pmatrix} \mathbb{T}_2 & 0 \\ J & T_4 \end{pmatrix}$, where $T_4 \in \mathfrak{TI}(4, 1)$, then $\hat{T}_6 \in \mathfrak{TR}(6, 1)$ and $k(\hat{T}_6) = k(T_4)$. Hence $ITR(6, 1) = \{4, 6, 7, 9\}$. By Lemma 2.8, it is obvious that $ITR(7, 1) = [3, 9]$.

For $n \geq 8$, let $\tilde{T}_n = \begin{pmatrix} 0 & 0 \\ J & \bar{T}_{n-1} \end{pmatrix}$, where $\bar{T}_{n-1} \in \mathfrak{TI}(n - 1, 1)$, then $\tilde{T}_n \in \mathfrak{TR}(n, 1)$ and $k(\tilde{T}_n) = k(\bar{T}_{n-1})$. By Lemma 2.8, $ITR(n, 1) = [3, n + 1]$. We complete the proof.

Theorem 3.3: Let $n \geq 4$. Then

$$ITR(n,3) = \begin{cases} \{1\} & n = 4, \\ \{1,2\} & n = 5, \\ \{0,2,3\} & n = 6, \\ \{1,2,4,9\} & n = 7, \\ \{2,3,4,5,6,7,9\} & n = 8, \\ [0,9] & n = 9, \\ [0,n-1] & n > 10 \ \ and \ \ 3 \mid n, \\ [1,n-1] & n \geq 10 \ \ and \ \ 3 \nmid n. \end{cases}$$

where $[0, n - 1] = \{0, 1, 2, 3, \cdots, n - 1\}$.

Proof: Note that $k(\mathcal{T}_{3q_i}) = 0(3q_i > 0)$, hence $0 \in ITR(n, 3)$, where $n \geq 3$ and $3 \mid n$.

Let $\tilde{T}_4 = \begin{pmatrix} 0 & 0 \\ J & \bar{T}_3 \end{pmatrix}$, then $k(\tilde{T}_4) = 1 \in ITR(4, 3)$.

Let $\tilde{T}_5 = \begin{pmatrix} \mathbb{T}_2 & 0 \\ J & \bar{T}_3 \end{pmatrix}$, then $k(\tilde{T}_5) = 2 \in ITR(5, 3)$ and let $\hat{T}_5 = \begin{pmatrix} 0 & 0 & 0 \\ J & \bar{T}_3 & 0 \\ J & J & 0 \end{pmatrix}$, then $k(\hat{T}_5) = 1 \in ITR(5, 3)$.

Let $\tilde{T}_6 = \begin{pmatrix} \mathbb{T}_3 & 0 \\ J & \bar{T}_3 \end{pmatrix}$, then $k(\tilde{T}_6) = 3 \in ITR(6, 3)$ and let $\hat{T}_6 = \begin{pmatrix} \mathbb{T}_2 & 0 & 0 \\ J & \bar{T}_3 & 0 \\ J & J & 0 \end{pmatrix}$, then $k(\hat{T}_6) = 2 \in ITR(6, 3)$. It is easy to see that $ITR(7, 3) = \{1, 2, 4, 9\}$, $ITR(8, 3) = \{2, 3, 4, 5, 6, 7, 9\}$ and $ITR(9, 3) = [0, 9]$.

Suppose $n \geq 10$. Let $\tilde{T}_n = \begin{pmatrix} \mathbb{T}_3 & 0 \\ J & T_{n-3} \end{pmatrix}$, where $T_{n-3} \in \mathfrak{T}\mathfrak{I}(n-3, 1)$, by Lemma 2.8, $\tilde{T}_n \in \mathfrak{T}\mathfrak{R}(n, 3)$ and $k(\tilde{T}_n) = k(\tilde{T}_{n-3}) \in [3, n-1]$.

If $n = 3m$. Let

$$\tilde{T}_n = \begin{pmatrix} \mathbb{T}_1 & 0 & 0 & 0 & 0 & 0 \\ J & \bar{T}_3 & 0 & 0 & 0 & 0 \\ J & J & \mathbb{T}_1 & 0 & 0 & 0 \\ J & J & J & \bar{T}_3 & 0 & 0 \\ J & J & J & J & \mathbb{T}_1 & 0 \\ J & J & J & J & J & \mathcal{T}_{3(m-3)} \end{pmatrix},$$

and

$$\hat{T}_n = \begin{pmatrix} \mathbb{T}_1 & 0 & 0 & 0 \\ J & \bar{T}_3 & 0 & 0 \\ J & J & \mathbb{T}_2 & 0 \\ J & J & J & \mathcal{T}_{3(m-2)} \end{pmatrix},$$

then $k(\tilde{T}_n) = 1$, and $k(\hat{T}_n) = 2$.

If $n = 3m + 1$. Let $\tilde{T}_n = \begin{pmatrix} \mathbb{T}_1 & 0 \\ J & \mathcal{T}_{3m} \end{pmatrix}$, and

$$\hat{T}_n = \begin{pmatrix} \mathbb{T}_2 & 0 & 0 & 0 \\ J & \bar{T}_3 & 0 & 0 \\ J & J & \mathbb{T}_2 & 0 \\ J & J & J & \mathcal{T}_{3(m-2)} \end{pmatrix},$$

then $k(\tilde{T}_n) = 1$, and $k(\hat{T}_n) = 2$.

If $n = 3m + 2$. Let $\tilde{T}_n = \begin{pmatrix} \mathbb{T}_1 & 0 & 0 & 0 \\ J & \bar{T}_3 & 0 & 0 \\ J & J & \mathbb{T}_1 & 0 \\ J & J & J & \mathcal{T}_{3(m-1)} \end{pmatrix}$, and $\hat{T}_n = \begin{pmatrix} \mathbb{T}_2 & 0 \\ J & \mathcal{T}_{3m} \end{pmatrix}$, then $k(\tilde{T}_n) = 1$, and $k(\hat{T}_n) = 2$. Hence If $n \geq 10$, then

$$ITR(n, 3) = \begin{cases} [0, n-1] & n > 10 \ \ and \ \ 3 \mid n, \\ [1, n-1] & n \geq 10 \ \ and \ \ 3 \nmid n. \end{cases}$$

We complete the proof.

By Theorem 3.2 and Theorem 3.3, we have

Corollary 3.4 Let $n \geq 8$. Then

$$ITR(n) = \begin{cases} [0, n+1] & n > 10 \ \ and \ \ 3 \mid n, \\ [1, n+1] & n \geq 10 \ \ and \ \ 3 \nmid n. \end{cases}$$

REFERENCES

[1] Bolian Liu (2006). *Combinatorial Matrix Theory* (Second Edition). Bejing: Science Press.

[2] Richard A.Brualdi (2006). *Combinatorial Matrix Classes* (First Edition). Cambridge University Press. New York.

[3] J.W., Moon & N.J., Pullman (1967). *On the Power of Tournament Matrices*. *Comb. Theory, 3*, 1–9.

[4] Jiang ZM & Shao JY (1991). On the Set of Indices of Convergence for Reducible Matrices *Linear Algebra and Its Application, 148*, 265–278.

[5] Xue-mei Ye (2007). Characterization of the Tournament with Primitive Exponent Reaching Its Secondary Value J*ournal of Mathematical Research and Exposition, 4*, 715–718.

[6] Bo Zhou & Jian Shen (2002). On Generalized Exponents of Tournaments *Taiwanese J. Math., 6*, 565–572.