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Generalized Ridges and Ravines on an Equiform Motion

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Abstract
In this paper we investigate a new type of ridges and ravines of the configuration space corresponding to an equiform motion in the Euclidean space R^3. Necessary and sufficient conditions for the existence of generalized ridges and ravines are expressed as a partial differential inequalities involving the principal curvatures. For special case we obtain the solution of the differential equations which characterize some type of singularities. The singularities are displayed through figures [1, 2, 3].

Key words
Equiform motion; Configuration space; Generalized ridges and ravines

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1. INTRODUCTION

Recent advances in pattern recognition, computer vision, medical imaging, and free-from shape design inspired a fresh interest in surface features associated with singularities of the intrinsic geometric quantities on the surface. Intrinsic geometry has been proposed and studied for smoothing surfaces or getting a hierarchical description of surfaces[4,5]. Therefore, in order to describe a shape (think of wrinkles on a face or think of the nose as a feature of facial shape) we use a characterization of a certain types of singularities of a shape. The simplest example of singularities is given by the smoothing of a plane curve by its curvature. The main features of a plane curve are its points of inflections where the curvature is zero and the vertices where the curvature has a local maximum or minimum. For surfaces there are two principal curvatures and the features will be interested depend on the parabolic curves where one of these curvature is zero (the Gaussian curvature vanishes), the ridge or ravine curves where are them have a maximum or minimum on its corresponding line of curvature and umbilic points where they are equal. Parabolic points are associated with inflections on object contours. Ridge and ravine curves are very important for shape recognition. In particular, the principal curvatures are non differentiable functions at umbilic points, hence umbilics will become singular points depending on the variation of the principal curvatures. At parabolic points the Gaussian curvature of a surface vanishes. They are the boundaries between elliptic and hyperbolic regions. Alternatively, they are the points where the tangent planes have a specially higher order contact with the surface [6]. Parabolic points can be classified into two regions D_1 and D_2 as follows: a point is called a D_1 - parabolic point if the larger principal curvature, say k_1, is zero; likewise and a D_2 - parabolic point is where the smaller principal curvature k_2 equal zero. A more degenerate type is the flat umbilic, where $k_1 = k_2 = 0$. If the surface is closed and oriented so that the curvature is positive at convex regions, then the D_2 - parabolics are the boundary between convex elliptic regions and hyperbolic regions and the D_1 - parabolics are the boundaries of the concave elliptic regions.

2. CNFIGURATION SPACE

It's well-known that the similar (equiform) motion is defined as Rigid motion with scaling. This motion can be represented by a translation vector T and a rotation matrix A as the following:

$$\bar{x} = \rho A x + T \tag{1}$$

where $A^t A = AA^t = I$ and ρ is the scaling factor [3] and [10]. Also the space of all possible rigid transformations of an object constitute the configuration space of the motion.

Thus the configuration space is defined as the space of all directions of any system. This space has the structure of a manifold which is called configuration manifold of the motion. From (1), it is easy to see that the similar motion can be defined through a linear mapping as in the following:

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} \rho A & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{2}$$

The linear map (2) in question may be defined explicitly in the 3-dimensional Euclidean space as

$$\bar{x}(u^\alpha) = \rho(u^\beta) A(u^\beta) + T(u^\alpha) \tag{3}$$

for some fixed parameter u^β and $\alpha = 1, 2$.

Without loss of generality, we consider the following representation of (3) as follows:

$$\bar{x}(u^1, u^2) = \rho(u^2) R_z(u^2) x(u^1) + T(u^1, u^2) \tag{4}$$

where $x = x(u^1)$ is a regular representation of a curve **C** (the profile curve) in the plane xz $(y = 0)$ in R^3, $R_z(u^2)$ is the rotation matrix around z - axis, $\rho(u^2)$ is the equiform factor and T is the translation vector. In this construction the matrix $R_z(u^2)$ is given as

$$R_z(u^2) = \begin{bmatrix} \cos u^2 & -\sin u^2 \\ \sin u^2 & \cos u^2 \end{bmatrix}$$

The configuration space (4) has several forms as the following:

(i) Natural rigid motion, $\rho = 1$, $T = const.$.

(ii) Natural rotation $\rho = 1$, $T = 0$

(iii) Generalized rigid motion along rotation axis, $\rho = 1$, $T = T(u^2)$.

(iv) Helical motion $\rho = 1$, $T = au^1$, $a = const.$.

Consider the motion for which the displacement along z - axis, *i.e.*

$$T\left(u^1,u^2\right)=\left(0,0,t\left(u^2\right)\right)=\tilde{T}\left(u^2\right) \tag{5}$$

Thus, we have the parametric representation of the motion under consideration as follows:

$$X\left(u^1,u^2\right)=\rho\left(u^2\right)\left(f\left(u^1\right)\cos u^2\,,\ f\left(u^1\right)\sin u^2\,, h\left(u^1\right)+\hat{T}\left(u^2\right)\right) \tag{6}$$

where

$$\rho \neq 0,\ \hat{T}=\frac{\tilde{T}\left(u^2\right)}{\rho\left(u^2\right)} \text{ and } X\left(u^1\right)=\left(f\left(u^1\right),\ 0,\ h\left(u^1\right)\right).$$

3. INTRINSIC GEOMETRY OF THE CONFIGURATION SPACE

The representation (6) characterizes a surface in the Euclideon space R^3. The metric properties on the surface is given from the metric tensior $g_{\alpha\beta}$ where

$$\begin{aligned} g_{11} &= \rho^2\lambda\ ,\ \lambda = f'^2 + h'^2 \\ g_{12} &= \rho\left(f\, f'\,\rho' + \mu\, h'\right), \quad \mu = \rho'\, h + t' \\ g_{22} &= f^2\left(\rho'^2 + \rho^2\right) + \mu^2, \qquad ' = \frac{d}{du^2} \end{aligned} \tag{7}$$

The function $\lambda = \lambda\ (u^1,\ u^2) > 0$ represents the arc length on the profile curve. Thus the metric on the configuration space of the generalized equiform motion is given by

$$g = Det\left(g_{\alpha\beta}\right) = \rho^2\left(h'^2 f^2\rho'^2 + \lambda f^2\rho^2 + f'^2\mu^2 - 2f\, f'\rho'\,\mu h'\right) \tag{8}$$

The normal vector field on the configuration space (6) is defined as

$$N = \frac{\rho}{\sqrt{g}}\left(\mu f'\sin u^2 - f\,h'\left(\rho'\sin u^2 + \rho\cos u^2\right),\ f\,h'\left(\rho'\cos u^2 - \rho\sin u^2\right) - \mu f'\cos u^2\,,\ \rho f f'\right) \tag{9}$$

The singularities on the configuration space (6) of the generalized equiform motion is given from $g = 0$ or equivalently

$$h'^2 f^2\rho'^2 + \lambda f^2\rho^2 + f'^2\mu'^2 - 2f\, f'\rho'\,\mu h' = 0 \tag{10}$$

Remark 1.
There is no singularity on the natural rotation ($\rho = 1,\ t = 0$) because of the condition (10) does not satisfied in this case.

Remark 2.
The singularities occur in the case of the generalized rigid motion ($\rho = 1$) if $t'' < 0$.

The singularities on the configuration space (6) are displayed in Fig. 1, 2 in the case $\rho \neq$ const. In these figures, the singularities are illustrated through the configuration space and its Gauss map $G : N \to S^2$ (1) in Fig. 3, 4 respectively for all the points of the configuration space. The components of the curvature tensor (the 2nd fundamental quantities) are given as $<x_{\alpha\beta}, N> = L_{\alpha\beta}$ or explicitly

$$L_{11} = \frac{f\rho^3 \lambda^{\frac{3}{2}} k_c}{\sqrt{g}}$$

$$L_{12} = \frac{\rho^2 f'}{\sqrt{g}}(f\rho' h' - f'\mu) \tag{11}$$

$$L_{22} = \frac{\rho f}{\sqrt{g}}(f h' \gamma + f'\alpha)$$

where

$$k_c = \frac{h'' f' - f'' h'}{\lambda^{\frac{3}{2}}} \tag{12}$$

is the curvature of the profile curve $c : x(u^1) = (f(u^1), 0, h(u^1))$ and

$$v = \rho'' h + t'', \quad \gamma = \rho^2 + 2\rho'^2 - \rho''\rho, \quad \alpha = \rho v - 2\rho'\mu \tag{13}$$

Thus the determinant of the 2nd fundamental tensor $L_{\alpha\beta}$ is given from

$$L = Det(L_{\alpha\beta}) = \frac{\rho^4 \wp}{g} \tag{14}$$

where

$$\wp = f^2 \lambda^{\frac{3}{2}} k_c (f h' \gamma + f'\alpha) - f'^2 (f\rho' h' - f'\mu)^2 \tag{15}$$

The Gaussian curvature of the surface (6) corresponding to the configuration space of the generalized equiform motion is given from

$$G = \frac{L}{g} = \frac{\rho^4 \wp}{g^2} = k_1 k_2 \tag{16}$$

where k_1, k_2 are the principal curvatures of the surface at an arbitrary point (u^1, u^2).

4. RIDGES AND RAVINES

A ridge point is a point where the surface has a higher order contact with one of the osculating spheres, or equivalently, where the principal curvature has an extreme value along the corresponding line of curvature. Ridge points can also be classified into two types according to the maximum (Ridge) or minimum (ravine) values of the principal curvature [7,8,9].

The Gaussian curvature G is a function of two variables, so the singularities (critical or extremes) are defined from

$$\frac{\partial G}{\partial u^1}=0,\ \frac{\partial G}{\partial u^2}=0 \tag{17}$$

We call these singularities the generalized ridge or Ravines of the equiform motion depending on the extremes of the principal curvatures as we shall show in the following.

From (17) we have

$$\begin{aligned}&\frac{\rho^4(\wp_1 g-2\wp g_1)}{g^3}=0\ ,\\ &\frac{\rho^3(g(4\rho'\wp+\wp_2\rho)-2\rho\wp g_2)}{g^3}=0,\end{aligned} \tag{18}$$

$$where\qquad \rho\neq 0,\ \wp_\alpha=\frac{\partial\wp}{\partial u^\alpha},\ g_\alpha=\frac{\partial g}{\partial u^1}\ \ ,\ \ \alpha=1,2.$$

Or equivalently ($g\neq 0,\rho\neq 0$)

$$\begin{aligned}&\frac{\wp_1}{\wp}-\frac{2g_1}{g}=0,\\ &2\left(2\frac{\rho'}{\rho}-\frac{g_2}{g}\right)+\frac{\wp_2}{\wp}=0.\end{aligned} \tag{19}$$

By integration we have

$$\frac{\wp}{g^2}=\eta(u^2) \tag{20}$$

$$\frac{\rho^2\sqrt{\wp}}{g}=\zeta(u^1) \tag{21}$$

Thus, we have the proof of the following:

Theorem 1. The relations (20) and (21) are the necessary conditions for the existence of singularities of the

Gaussian curvature.

In the natural rotation ($t = 0$, $\rho = 1$) it is easy to see $\wp = \wp(u^1)$, $g = g(u^1)$ but from (20) we have $\frac{\wp}{g^2} = \eta(u^2)$ which gives a contradiction because of, in this case, the singularities are depend on the profile curve only, i.e. on the parameter u^2. Thus we have:

Theorem 2. The necessary condition for singularities of the configuration space corresponding to the natural rotation ($\rho = 1$) is given from the condition $\frac{\sqrt{\wp}}{g} = \zeta(u^1)$ only.

To determine the type of singularities, *i.e.* the points at which occur the generalized ridges or ravines, we use the necessary conditions as in the form

$$\begin{aligned} &G = k_1 k_2, \\ &G_1 = k_1 k_{2;1} + k_{1;1} k_2 = 0, \\ &G_2 = k_1 k_{2;2} + k_{1;2} k_2 = 0, \end{aligned} \tag{22}$$

where $G_\alpha = \frac{\partial G}{\partial u^\alpha}$, $k_{\alpha;\beta} = \frac{\partial k_\alpha}{\partial u^\beta}$, $\alpha, \beta = 1,2$.

These conditions determine the points (u^1, u^2)on the configuration space at which occur the generalized ridges or ravines.

For non planer point $[k_1\ k_2]^t \neq 0$, the conditions (22) are equivalent to

$$k_{2;1} k_{1;2} + k_{1;1} k_{2;2} = 0 \tag{23}$$

The second derivates are given as

$$\begin{aligned} &G_{11} = 2k_{1;1} k_{2;1} + k_1 k_{2;11} + k_{1;11} k_2, \\ &G_{22} = 2k_{1;2} k_{2;2} + k_1 k_{2;22} + k_{1;22} k_2, \\ &G_{12} = k_{1;2} k_{2;1} + k_1 k_{2;12} + k_{1;12} k_2 + k_{1;12} k_{2;2}, \end{aligned} \tag{24}$$

where $G_{\alpha\beta} = \frac{\partial G}{\partial u^\alpha \partial u \beta}$, $k_{\alpha;\beta\gamma} = \frac{\partial k_\alpha}{\partial u^\beta \partial u^\gamma}$, $\alpha, \beta, \gamma = 1, 2$.

The sufficient conditions for the existence of a maxima or minima, i.e. the generalized ridges or ravines, are given from the well known conditions on the Hessian matrix ($G_{\alpha\beta}$). These conditions are given from the following theorems [10,11,12].

Theorem 3. The sufficient condition for the existence of a generalized ridge near elliptic point ($K > 0$) is one of the inequalities

$$\begin{aligned} &k_1, k_2 > 0\ , k_{1,ij\alpha\beta}, k_{2,ij\alpha\beta} < 0 \\ &k_1, k_2 < 0\ , k_{1,ij\alpha\beta}, k_{2,ij\alpha\beta} > 0 \end{aligned} \tag{25}$$

must be satisfied.

Theorem 4. The sufficient condition for the existence of a generalized ravine near elliptic point ($K > 0$) is one the inequalities

$$k_1, k_2 > 0 \, , k_{1,ij\alpha\beta}, k_{2,ij\alpha\beta} > 0$$
$$k_1, k_2 < 0 \, , k_{1,ij\alpha\beta}, k_{2,ij\alpha\beta} < 0 \qquad (26)$$

must be satisfied.

Theorem 5. The sufficient condition for the existence of a generalized elliptic near hyperbolic point ($K < 0$) is one of the inequalities

$$k_1 < 0 \, , k_2 > 0 \, , k_{1,ij\alpha\beta} < 0, k_{2,ij\alpha\beta} > 0$$
$$k_1 > 0 \, , k_2 < 0 \, , k_{1,ij\alpha\beta} > 0, k_{2,ij\alpha\beta} < 0 \qquad (27)$$

must be satisfied.

Theorem 6. The sufficient condition for the existence a of generalized ravine near hyperbolic point ($K < 0$) is one of the inequalities

$$k_1 > 0 \, , k_2 < 0 \, , k_{1,ij\alpha\beta} < 0, k_{2,ij\alpha\beta} > 0$$
$$k_1 < 0 \, , k_2 > 0 \, , k_{1,ij\alpha\beta} > 0, k_{2,ij\alpha\beta} < 0 \qquad (28)$$

must be satisfied.

4. UMBILICAL POINTS

From (7) and (11) one can show the condition for an umbilici is

$$\text{Rank} \begin{pmatrix} g_{11} & g_{12} & g_{22} \\ L_{11} & L_{12} & L_{22} \end{pmatrix} \leq 1 \qquad (29)$$

which is equivalent to simultaneously requiring that

$$g_{11} L_{22} - g_{22} L_{11} = 0 \, ,$$
$$g_{22} L_{12} - g_{12} L_{22} = 0 \, . \qquad (30)$$

If the configuration space patched by principal patch (orthogonal lines of curvatures as the parametric curves) we have

$$g_{12} = L_{12} = 0 \, , \; g_{11} L_{22} - g_{22} L_{11} = 0 \qquad (31)$$

Since the principal curvatures are not C^{∞} functions at the umbilici points, we can not define the ridges or ravines. From (31) , (7) and (11), it is easy to see that the umbilics are given from the differential equations

$$f f' \rho' + \mu h' = 0, \quad f \rho' h' - f' \mu = 0, \; f' \neq 0 \; ,$$
$$(f h' \gamma + f' \alpha) - \sqrt{\lambda} \, k_c \left(f^2 (\rho'^2 + \rho^2) + \mu^2 \right) = 0 \, . \qquad (32)$$

In the natural rotation ($\rho = 1$, $t = 0$) we have

$$\lambda\, h' - (h''f' - f''\,h') = 0 \tag{33}$$

If the profile curve parameterized by the arc length (natural parameter) we have

$$\lambda = 1 + h'^2,\ f = u^2, f' = 1, f'' = 0. \tag{34}$$

Thus the umbilics in this case are given from

$$(1 + h'^{\,2})h' - h'' = 0 \tag{35}$$

Thus the point (u^1, u^2) on the configuration space of the natural rotation is an umbilic point if it satisfied the differential equation (35). The differential equation (35) has a general solution in the form

$$h(u^1) = \sin^{-1}\left(c_1 e^{u^1} + c_2\right), \tag{36}$$

where c_1 and c_2 are arbitrary constants.

From (34) , (36) and (6) we have a configuration space corresponding to the natural rotation as in the following form

$$x(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, \sin^{-1}(c_1 e^{u^1} + c_2)). \tag{37}$$

The normal vector field on this surface is given as

$$\mathrm{N} = \frac{1}{1+\xi^2}\left(-\xi \cos u^2, -\xi \sin u^2, 1\right) \tag{38}$$

where $\xi = c_1 e^{u^1} / \sqrt{1 - \left(c_1 e^{u^1} + c_2\right)^2}$

Remark 3. All the points on the surface (37) are of type umbilical as we see in figures 5 and 6 for special values of the constants c_1 and c_2. Also the Gass image of this surface are given through figures 7 and 8 respectively.

Remark 4. The construction of the configuration space (37) and its Gauss image (38) depends on the constants c_1 and c_2 as the following:
(*i*) $c_1 > 0$ and $c_2 < -1$, as in Fig. 5, 7.
(*ii*) $c_1 < 0$ and $-1 < c_2 < 1$, as in Fig. 6, 8.

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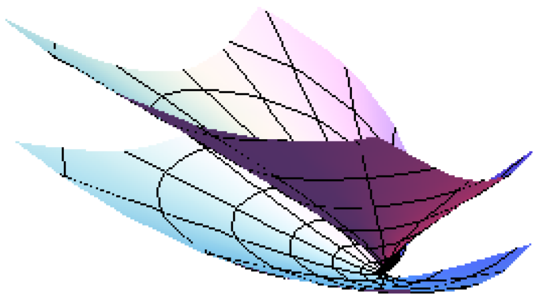

Figure 1
Configuration Space, $\rho = v^2, f = u^2, h = u^2 - 3u+1, t = \sin v$

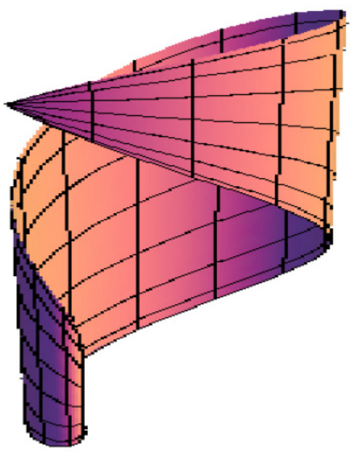

Figure 2
Configuration Space

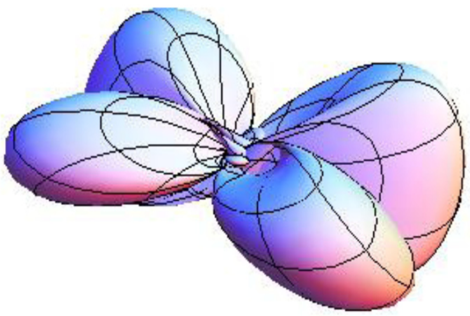

Figure 3
Gauss Image G_1

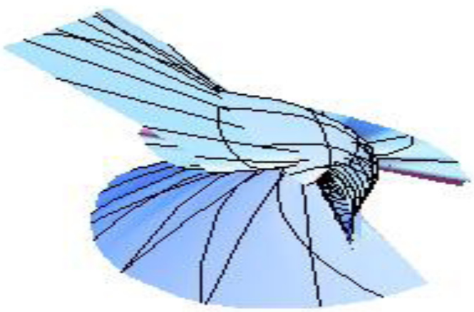

Figure 4
Gauss Image G_2

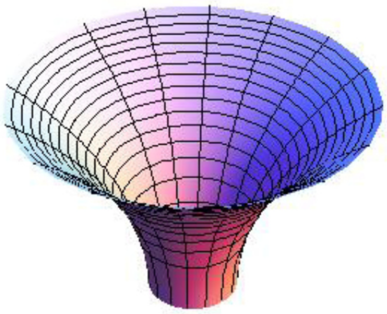

Figure 5
Umbilical Configuration Space $c_1 \prec 0$, $-1 \leq c_2 \prec 1$

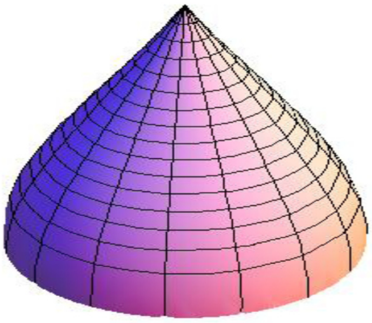

Figure 6
Umbilical Configuration Space $c_1 \succ 0$, $c_2 \prec -1$

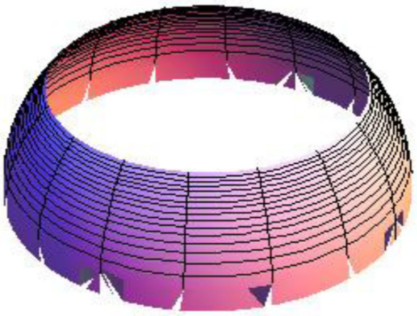

Figure 7
Gauss Image G_3

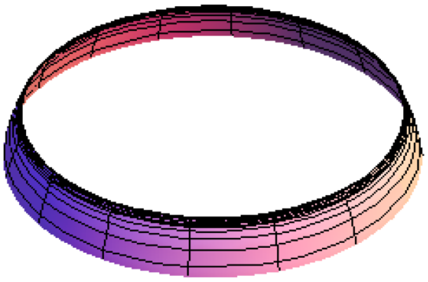

Figure 8
Gauss Image G_4