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Halpern-type Iterations for Strong Relatively Nonexpansive Multi-valued Mappings in Banach Spaces

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Abstract
In this paper, an iterative sequence for strong relatively nonexpansive multi-valued mapping by modifying Halpern's iterations is introduced, and then some strong convergence theorems are proved. At the end of the paper some applications are given also.

Key words
Multi-valued mapping; Strong relatively nonexpansive; Fixed point; Iterative sequence; Normalized duality mapping

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1. INTRODUCTION

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let D be a nonempty closed subset of a real Banach space E. A single-valued mapping $T : D \to D$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in D$. Let $N(D)$ and $CB(D)$ denote the family of nonempty subsets and nonempty closed bounded subsets of D, respectively. The Hausdorff metric on $CB(D)$ is defined by

$$H(A_1, A_2) = max\{\sup_{x\in A_1} d(x, A_2), \sup_{y\in A_2} d(y, A_1)\}, \tag{1.1}$$

for $A_1, A_2 \in CB(D)$, where $d(x, A_1) = inf\{\|x - y\|, y \in A_1\}$. The multi-valued mapping $T : D \to CB(D)$ is called nonexpansive if $H(T(x), T(y)) \le \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to N(D)$ if $p \in T(p)$. The set of fixed points of T is represented by $F(T)$.

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in E. \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be uniformly convex if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$. E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.3}$$

exists for all $x, y \in U$. E is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

Remark 1.1 The following basic properties for Banach space E and for the normalized duality mapping J can be found in Cioranescu [1].

(i) If E is an arbitrary Banach space, then J is monotone and bounded;

(ii) If E is a strictly convex Banach space, then J is strictly monotone;

(iii) If E is a a smooth Banach space, then J is single-valued, and hemi-continuous, i.e., J is continuous from the strong topology of E to the weak star topology of E;

(iv) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E;

(v) If E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*, JJ^* = I_E^*$ and $J^*J = I_E$;

(vi) If E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping J is single-valued, one-to-one and onto;

(vii) A Banach space E is uniformly smooth if and only if E^* is uniformly convex. If E is uniformly smooth, then it is smooth and reflexive.

Let E be a smooth Banach space. In the sequel, we always use $\phi : E \times E \to \mathbb{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.4}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{1.5}$$

In addition, the function ϕ has the following property:

$$\phi(y, x) = \phi(z, x) + \phi(y, z) + 2\langle z - y, Jx - Jz\rangle, \quad \forall x, y, z \in E \tag{1.6}$$

and

$$\phi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz) \leq \lambda\phi(x, y) + (1 - \lambda)\phi(x, z), \tag{1.7}$$

for all $\lambda \in [0, 1]$ and $x, y, z \in E$.

Let C is a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E. Following Alber [2], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = arg \inf_{y\in C} \phi(y, x), \quad \forall x \in E.$$

Let D be a nonempty subset of a smooth Banach space. A mapping $T : D \to E$ is relatively nonexpansive [3–5], if the following properties are satisfied:

(R1) $F(T) \neq \emptyset$;

(R2) $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in D$;

(R3) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in D converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0, it follows that $p \in F(T)$.

If T satisfies (R1) and (R2), then T is called quasi-ϕ-nonexpansive[6].

Recently, Weerayuth Nilsrakoo[7] introduced the following iterative sequence for finding a fixed point of strongly relatively nonexpansive mapping $T : D \to E$. Given $x_1 \in D$,

$$x_{n+1} = \Pi_D J^{-1}(\alpha_n Ju + (1 - \alpha_n)JTx_n)$$

where D is nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, Π_D is the generalized projection of E onto D and $\{\alpha_n\}$ is a sequences in (0,1).

Iterative methods for approximating fixed points of multi-valued mappings in Banach spaces have been studied by some authors, see for instance [8−12].

Let D be a nonempty closed convex subset of a smooth Banach space E. A mapping $T : D \to N(D)$ is relatively nonexpansive multi-valued mapping [12], if the following properties are satisfied:

(S1) $F(T) \neq \emptyset$;

(S2) $\phi(p, z) \leq \phi(p, x), \forall x \in D, z \in T(x), p \in F(T)$;

(S3) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in D which weakly to p and $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$, it follows that $p \in F(T)$.

Let D be a nonempty closed convex subset of a smooth Banach space E. We define a strongly relatively nonexpansive multi-valued mapping as follows.

Defnition 1.2 A multi-valued mapping $T : D \to N(D)$ is called strongly relatively nonexpansive, if T satisfies (S1), (S2), (S3)and

(S4) If whenever $\{x_n\}$ is a bounded sequence in D such that $\phi(p, x_n) - \phi(p, z_n) \to 0$, for some $p \in F(T), z_n \in T(x_n)$, it follows that $\phi(z_n, x_n) \to 0$.

In this article, inspired by Weerayuth Nilsrakoo [7], we introduce the following iterative sequence for finding a fixed point of strongly reatively nonexpansive multi-valued mapping $T : D \to N(D)$. Given $u \in E, x_1 \in D$,

$$x_{n+1} = \Pi_D J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jw_n) \tag{1.8}$$

where $w_n \in Tx_n$ for all $n \in N$, D is a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, Π_D is the generalized projection of E onto D and $\{\alpha_n\}$ is sequences in (0,1). We proved the strong convergence theorems in uniformly convex and uniformly smooth Banach space E.

2. PRELIMINARIES

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

First, we recall some conclusions.

Lemma 2.1 (Cf. [13, Proposition 2]). Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$.

Remark 2.2 For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space E, we have

$$\phi(x_n, y_n) \to 0 \iff x_n - y_n \to 0 \iff Jx_n - Jy_n \to 0.$$

Lemma 2.3 (Cf. [13, Propositions 4 and 5]). Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in E$;

(b) If $x \in E$ and $z \in C$, then $z = \Pi_C x \iff \langle z - y, Jx - Jz\rangle \geq 0, \forall y \in C$;

(c) For $x, y \in E, \phi(x, y) = 0$ if and only $x = y$.

Remark 2.4. The generalized projection mapping Π_C above is relatively nonexpansive and $F(\Pi_C) = C$.

Let E be a reflexive, strictly convex and smooth Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*, that is, $J^* = J^{-1}$. We will use the following mapping $V : E \times E^* \to R$ studied in [2]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.3}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.5 (Cf. [2] and [14, Lemma 3.2]). Let E be a reflexive, strictly convex and smooth Banach space. Then

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 (Cf. [15, Lemma 2.1]). Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n$$

for all $n \in \mathbb{N}$, where the sequences $\{\gamma_n\}$ in (0,1) and $\{\delta_n\}$ in $\mathbb{R}$ satisfy the following conditions: $\lim\limits_{n\to\infty} \gamma_n = 0$, $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup\limits_{n\to\infty} \delta_n \leq 0$. Then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.7 (Cf. [16, Lemma 3.1]). Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \in \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_{k+1}} \text{ and } a_k \leq a_{m_{k+1}}.$$

Infact, $m_k = max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.8 (Cf. [12, Proposition 2.1]). Let E be a strictly convex and smooth Banach space, and D a nonempty closed convex subset of E. Suppose $T : D \to N(D)$ is a relatively nonexpansive multi-valued mapping. Then, $F(T)$ is closed and convex.

3. MAIN RESULTS

In this section, we use Halpern's idea [17] for finding fixed point of strongly relatively nonexpansive multi-valued mappings in a uniformly convex and smooth Banach space. In the sequel, we shall need the following lemma.

Lemma 3.1 Let D be a nonempty closed convex subset of a uniformly convex and smooth Banach space E, $T : D \to N(D)$ be a relatively nonexpansive multi-valued mapping, $x \in E$ and $x^* = \Pi_{F(T)}x$. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences such that $\phi(z_n, x_n) \to 0$ and $\phi(z_n, y_n) \to 0, z_n \in Tx_n$. Then

$$\limsup_{n\to\infty} \langle y_n - x^*, Jx - Jx^* \rangle \leq 0.$$

Proof. From the uniform convexity of E and Lemma 2.1,

$$z_n - x_n \to 0 \text{ and } y_n - x_n \to 0.$$

From property (R3) of the mapping T, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup y \in F(T)$ and

$$\begin{aligned}\limsup_{n\to\infty} \langle y_n - x^*, Jx - Jx^* \rangle &= \limsup_{n\to\infty} \langle x_n - x^*, Jx - Jx^* \rangle \\ &= \limsup_{i\to\infty} \langle x_{n_i} - x^*, Jx - Jx^* \rangle\end{aligned}$$

From Lemma 2.3(b), we immediately obtain that

$$\limsup_{n\to\infty} \langle y_n - x^*, Jx - Jx^* \rangle = \langle y - x^*, Jx - Jx^* \rangle \leq 0$$

Theorem 3.2 Let D be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E and let $T : D \to N(D)$ be a strongly relatively nonexpansive multi-valued mapping. Let $\{x_n\}$ be the iterative sequence defined by (1.8), $\{\alpha_n\}$ is sequence in (0,1) satisfying

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(C2) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$

Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}u$.

Proof. Let $y_n \equiv J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n)$. Then $x_{n+1} \equiv \Pi_D y_n$. By Lemma 2.8, $F(T)$ is nonempty, closed and convex, so, we can define the generalized projection $\Pi_{F(T)}$ onto $F(T)$. Putting $u^* = \Pi_{F(T)}u$, we first show that $\{x_n\}$ is bounded. From Remark 2.4 and (1.7), we have

$$\begin{aligned}\phi(u^*, x_{n+1}) &\le \phi(u^*, y_n) = \phi(u^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n))\\ &\le \alpha_n\phi(u^*, u) + (1-\alpha_n)\phi(u^*, w_n)\\ &\le \alpha_n\phi(u^*, u) + (1-\alpha_n)\phi(u^*, x_n)\\ &\le max\{\phi(u^*, u), \phi(u^*, x_n)\}.\end{aligned}$$

By induction, we have

$$\phi(u^*, x_{n+1}) \le max\{\phi(u^*, u), \phi(u^*, x_1)\},$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and so is the sequence $\{Tx_n\}$. From Condition (C1) and (1.7), we obtain

$$\begin{aligned}\phi(w_n, y_n) &= \phi(w_n, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n))\\ &\le \alpha_n\phi(w_n, u) + (1-\alpha_n)\phi(w_n, w_n)\\ &= \alpha_n\phi(w_n, u) \to 0, \quad (n\to\infty).\end{aligned} \tag{3.1}$$

From Remark 2.4, Lemma 2.5 and (1.7), we have

$$\begin{aligned}\phi(u^*, x_{n+1}) &\le \phi(u^*, y_n) = v(u^*, Jy_n)\\ &\le v(u^*, Jy_n - \alpha_n(Ju - Ju^*)) - 2\langle y_n - u^*, -\alpha_n(Ju - Ju^*)\rangle\\ &= v(u^*, \alpha_n Ju^* + (1-\alpha_n)Jw_n) + 2\alpha_n\langle y_n - u^*, Ju - Ju^*\rangle\\ &= \phi(u^*, J^{-1}(\alpha_n Ju^* + (1-\alpha_n)Jw_n)) + 2\alpha_n\langle y_n - u^*, Ju - Ju^*\rangle\\ &\le \alpha_n\phi(u^*, u^*) + (1-\alpha_n)\phi(u^*, w_n) + 2\alpha_n\langle y_n - u^*, Ju - Ju^*\rangle\\ &\le (1-\alpha_n)\phi(u^*, x_n) + 2\alpha_n\langle y_n - u^*, Ju - Ju^*\rangle,\end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$.

The rest of the proof will be divided into two parts.

Case1. Suppose that there exists $n_0 \in N$ such that $\{\phi(u^*, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\phi(u^*, x_n)\}$ is then convergent. Then

$$\lim_{n\to\infty}(\phi(u^*, x_n) - \phi(u^*, x_{n+1})) = 0. \tag{3.3}$$

Notice that

$$\phi(u^*, x_{n+1}) \le \alpha_n\phi(u^*, u) + (1-\alpha_n)\phi(u^*, w_n).$$

It followsfrom (3.3) and Condition (C1) that

$$\begin{aligned}\phi(u^*, x_n) - \phi(u^*, w_n) &= \phi(u^*, x_n) - \phi(u^*, x_{n+1}) + \phi(u^*, x_{n+1}) - \phi(u^*, w_n)\\ &\le \phi(u^*, x_n) - \phi(u^*, x_{n+1}) + \alpha_n(\phi(u^*, u) - \phi(u^*, w_n)) \to 0.\end{aligned}$$

Since T is strongly relatively nonexpansive multi-valued mapping,

$$\phi(w_n, x_n) \to 0.$$

It follows from (3.1) and Lemma 3.1 that

$$\limsup_{n\to\infty}\langle y_n - u^*, Ju - Ju^*\rangle \le 0. \tag{3.4}$$

From (3.2), we have

$$\phi(u^*, x_{n+1}) \le (1-\alpha_n)\phi(u^*, x_n) + 2\alpha_n\langle y_n - u^*, Ju - Ju^*\rangle. \tag{3.5}$$

It follows from Lemma 2.6, (3.4) and (3.5) that

$$\lim_{n\to\infty} \phi(u^*, x_n) = 0.$$

Hence the conclusion follows from Lemmas 2.1.

Case2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(u^*, x_{n_i}) \leq \phi(u^*, x_{n_i+1}),$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.7, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\phi(u^*, x_{m_k}) \leq \phi(u^*, x_{m_k+1}) \text{ and } \phi(u^*, x_k) \leq \phi(u^*, x_{m_k+1}),$$

for all $k \in \mathbb{N}$. This together with Condition (C1) gives

$$\begin{aligned}\phi(u^*, x_{m_k}) - \phi(u^*, w_{m_k}) &= \phi(u^*, x_{m_k}) - \phi(u^*, x_{m_k+1}) + \phi(u^*, x_{m_k+1}) - \phi(u^*, w_{m_k})\\ &\leq \alpha_{m_k}(\phi(u^*, u) - \phi(u^*, w_{m_k})) \to 0.\end{aligned}$$

This implies that

$$\phi(w_{m_k}, x_{m_k}) \to 0.$$

It now follows from (3.1) and Lemma 3.1 that

$$\limsup_{n\to\infty} \langle y_{m_k} - u^*, Ju - Ju^* \rangle \leq 0. \tag{3.6}$$

From (3.2), we have

$$\phi(u^*, x_{m_k+1}) \leq (1 - \alpha_{m_k})\phi(u^*, x_{m_k}) + 2\alpha_{m_k}\langle y_{m_k} - u^*, Ju - Ju^* \rangle. \tag{3.7}$$

Since $\phi(u^*, x_{m_k}) \leq \phi(u^*, x_{m_k+1})$, we have

$$\begin{aligned}\alpha_{m_k}\phi(u^*, x_{m_k}) &\leq \phi(u^*, x_{m_k}) - \phi(u^*, x_{m_k+1}) + 2\alpha_{m_k}\langle y_{m_k} - u^*, Ju - Ju^* \rangle\\ &\leq 2\alpha_{m_k}\langle y_{m_k} - u^*, Ju - Ju^* \rangle.\end{aligned}$$

In particular, since $\alpha_{m_k} > 0$, we get

$$\phi(u^*, x_{m_k}) \leq 2\langle y_{m_k} - u^*, Ju - Ju^* \rangle.$$

It follows from (3.6) that $\phi(u^*, x_{m_k}) \to 0$. This together with (3.7) gives

$$\phi(u^*, x_{m_k+1}) \to 0.$$

But $\phi(u^*, x_k) \leq \phi(u^*, x_{m_k+1})$ for all $k \in \mathbb{N}$. We conclude that $x_k \to u^*$.

This implies that $\lim_{n\to\infty} x_n = u^*$ and the proof is finished.

Remark 3.3 The result [12, Theorem 3.3] and [18, Corollary 8] is a special case of our result.

Lemma 3.4 Let D be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : D \to N(D)$ be a relatively nonexpansive multi-valued mapping. Let U be the mapping defined by

$$U = J^{-1}(\lambda J + (1 - \lambda)JT),$$

where $\lambda \in (0, 1)$, then $U : D \to N(D)$ is strongly relatively nonexpansive multi-valued mapping and $F(U) = F(T)$.

The proof is similar to the proof of [19, Lemmas 3.1 and 3.2].

Applying Theorem 3.2 and Lemma 3.4, we have the following result.

Theorem 3.5 Let D be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and let $T : D \to N(D)$ be a relatively nonexpansive multi-valued mapping. Let $\{x_n\}$ be a sequence in D defined by $u \in E$, $x_1 \in D$ and

$$x_{n+1} = \Pi_D J^{-1}(\alpha_n Ju + (1-\alpha_n)(\lambda Jx_n + (1-\lambda)Jz_n))$$

where $z_n \in Tx_n$ for all $n \in \mathbb{N}$, $\{\alpha_n\}$ is a sequence in (0,1) satisfying Conditions (C1) and (C2), and $\lambda \in (0,1)$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}u$.
Remark 3.6 In Theorems 3.2 and 3.5, the condition of the nonempty interior of fixed point set of T is not needed.

4. APPLICATION TO ZERO POINT PROBLEM OF MAXIMAL MONOTONE MAPPINGS

Let E be a smooth, strictly convex and reflexive Banach space. An operator $A : E \to 2^{E^*}$ is said to be monotone, if $\langle x-y, x^*-y^*\rangle \geq 0$ whenever $x, y \in E$, $x^* \in Ax$, $y^* \in Ay$. We denote the zero point set $\{x \in E : 0 \in Ax\}$ of A by $A^{-1}0$. A monotone operator A is said to be maximal, if its graph $G(A) := \{(x,y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then $A^{-1}0$ is closed and convex. Let A be a maximal monotone operator, then for each $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that $J(x) \in J(x_r) + rA(x_r)$ (see, for example, [2]). We define the *resolvent* of A by $J_r x = x_r$. In other words $J_r = (J + rA)^{-1}J$, $\forall r > 0$. We know that J_r is a single-valued relatively nonexpansive mapping and $A^{-1}0 = F(J_r)$, $\forall r > 0$, where $F(J_r)$ is the set of fixed points of J_r. We have the following
Theorem 4.1 Let E, $\{\alpha_n\}$ be the same as in Theorem 3.2. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $J_r = (J + rA)^{-1}J$ for all $r > 0$ such that $A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $u, x_1 \in E$ and

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)JJ_r x_n),$$

then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}u$.
Proof. In Theorem 3.2 taking $D = E$, $T = J_r$, $r > 0$, then $T : E \to E$ is a single-valued relatively nonexpansive mapping and $A^{-1}0 = F(T) = F(J_r)$, $\forall r > 0$ is a nonempty closed convex subset of E. Therefore all the conditions in Theorem 3.2 are satisfied. The conclusion of Theorem 4.1 can be obtained from Theorem 3.2 immediately.

REFERENCES

[1] Cioranescu, I. (1990). Geometry of Banach Spaces, Duality Mappings and nonlinear Problems. *Kluwer Academic, Dordrecht*.

[2] Alber, Y. I. (1996). Metric and Generalized Projection Operators in Banach Spaces. *Marcel Dekker, 178*, 15-50. New York.

[3] Matsushita, S. and Takahashi, W. (2004). Weak and Strong Convergence Theorems for Relatively Nonexpansive Mappings in a Banach Spaces. *Fixed Point Theory Appl.,* 37-47.

[4] Matsushita, S. and Takahashi, W. (2004). An Iterative Algorithm for Relatively Nonexpansive Mappings by Hybrid Method and Applications. In *Proceedings of the Third International Conference on Nonlinear Analysis and Convex Analysis* (pp. 305-313).

[5] Matsushita, S. and Takahashi, W. (2005). A Strong Convergence Theorem for Relatively Nonexpansive Mappings in a Banach Spaces. *J. Approx. Theory., 134*, 257-266.

[6] Nilsrakoo, W. and Saejung, S. (2008). Strong Convergence to Common fixed Points of Countable Relatively Quasi-Nonexpansive Mappings. *Fixed Point Theory Appl.*, DOI:10.1155/2008/312454.

[7] Nilsrakoo, W. (2011). Halpern-Type Iterations for Strongly Relatively Nonexpansive Mappings in Banach Spaces. *Comput. Math. Appl., 62*, 4656-4666.

[8] Jung, J. S. (2007). Strong Convergence Theorems for Multivalued Nonexpansive Nonself-Mappings in Banach Spaces. *Nonlinear Anal., 66*, 2345-2354.

[9] Shahzad, N. & Zegeye, H. (2008). Strong Convergence Results for Nonself Multimaps in Banach Spaces. *Proc. Am. Soc., 136*, 539-548.

[10] Shahzad, N. & Zegeye, H. (2009). On Mann and Ishikawa Iteration Schems for Multi-Valued Maps in Banach Spaces. *Nonlinear Anal., 71*, 838-844.

[11] Song, Y. & Wang, H. (2009). Convergence of Iterative Algorithms for Multivalued Mappings in Banach Spaces. *Nonlinear Anal., 70*, 1547-1556.

[12] Homaeipour, S. & Razani, A. (2011). Weak and Strong Convergence Theorems for Relatively Nonexpansive Multi-Valued Mappings in Banach Spaces. *Fixed Point Theory Appl., 73*, DOI:10.1186/1687-1812-2011-73.

[13] Kamimura, S. & Takahashi, W. (2002). Strong Convergence of a Proximal-Type Algorithm in a Banach Space. *SIAMJ. Optim. 13*, 938-945.

[14] Kohsaka, F. & Takahashi, W. (2004). Strong Convergence of an Iterative Sequence for Maximal Monotone Operators in a Banach Space. *Abstr. Appl. Anal.,* 239-249.

[15] Xu, H. K. (2002). Another Control Condition in an Iterative Method for Nonexpansive Mappings. *Bull. Aust. Math. Soc., 65*, 109-113.

[16] Maing, P. E. (2008). Strong Convergence of Projected Subgradient Methods for Nonsmooth and Nonstrictly Convex Minimization. *Set-Valued Anal., 16*, 899-912.

[17] Halpren, B. (1967). Fixed Points of Nonexpansive Maps. *Bull. Amer. Math. Soc., 73*, 957-961.

[18] Saejung, S. (2010). Halperns Iteration in Banach Spaces. *Nonlinear Anal., 73*, 3431-3439.

[19] Kohsaka, F. & Takahashi, W. (2007). Approximating Common fixed Points of Countable Families of Strongly Nonexpansive Mappings. *Nonlinear Stud., 14*, 219-234.