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The Category of L-fuzzy Quantales

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Abstract
In this paper,the definition of the L-fuzzy quantales is given, we study the properties of L-fuzzy quantales from the categorical point of view, the product,equalizers,co-equalizers and pullback of the category of L-fuzzy quantales are investigated, Also, some properties of their are discussed.

Key words
L-fuzzy quantales; Category; Equalizers; Coproduct; Pullback

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1. INTRODUCTION

Quantale was introduced by C.J.Mulvey in 1986 in order to provide a lattice theoretic setting for studying non-commutative C*-algebras[1], as well as a constructive foundations of quantum logic. A quantale-besed (non-commutative logic theoretic) approach to quantum mechanics was developed by Piazza. It is known that quantales are one of the semantics of linear logic. The systematic introduction of quantale theory came from the book [2], which written by K.I.Rosenthal in 1990. Quantale theory provides a powerful tool in studying noncommutative structures, it has a wide applications, especially in studying noncommutative C*-algebra theory [3], the ideal theory of commutative ring[4], linear logic [5] and so on. Following C.J.Mulvey, the quantale theory have been studied by many researches [6-21].

Since coproducts is very important concept in many categories, and their coproducts product have been studied systemically. In this paper, the concrete forms of the coproducts of unital quantales is obatined. For notions and concepts concerned, but explained, please refer to [2,22].

2. PRELIMINARIES

Definition 2.1.[2] A *quantale* is a complete lattice Q with an associative binary operation "&" satisfying: $a\&(\bigvee\limits_{i\in I} b_i) = \bigvee\limits_{i\in I}(a\&b_i)$ and $(\bigvee\limits_{i\in I} b_i)\&a = \bigvee\limits_{i\in I}(b_i\&a)$, for all $a, b_i \in Q$, where I is a set, 0 and 1 denote the smallest element and the greatest element of Q, respectively.

A quantale Q is said to be *unital* if there is an element $u \in Q$ such that $u\&a = a\&u = a$ for all $a \in Q$.

Definition 2.2.[2] Let Q and P be quantales. A function $f : Q \longrightarrow P$ is a homomorphism of quantale if f preserves arbitrary sups and the operation "&". If Q and P are unital, then f is unital homomorphism if in addition to being a homomorphism, it satisfies $f(u_Q) = u_P$, where u_Q and u_P are units of Q and P, respectively.

Definition 2.3.[2] Let Q be a quantale. A subset $S \subseteq Q$ is a *subquantale* of Q iff the inclusion $S \hookrightarrow Q$ is a quantale homomorphism, i.e., S is closed under sups and "&".

Definition 2.4. Let Q be a quantale,and L be a complete lattice. A mapping $\overline{Q} : Q \longrightarrow L$ is said to be L-fuzzy quantale if $\overline{Q}_p = \{x \in Q \mid \overline{Q}(x) \geq p\}$ be a subquantale of Q for all $p \in L$.

Definition 2.5. Let Q and K be quantales, L be a complete lattice. $\overline{Q} : Q \longrightarrow L$ and $\overline{K} : K \longrightarrow L$ be L-fuzzy quantales. A function $\psi : Q \longrightarrow K$ is said to be L-fuzzy quantales homomorphism provided that ψ is a quantale homomorphism and $\overline{Q}(x) \leq \overline{K}(\psi(x))$ for all $x \in Q$.

3. EQUALIZER AND COEQUALIZER OF THE CATEGORY LFQUANT

In this section, we study the equalizre, coequalizre, product in the category of L-fuzzy quantales and some properties are discussed.

Let L be a complete lattice, **LFquant** denote the category of L-fuzzy quantales and L-fuzzy quantale homomorphism.

Theorem 3.1. The constant morphism in **LFquant** are exactly zero mapping.

Proof. Sufficiency: Let and be L-fuzzy quantales. The mapping be a zero function,i.e., $f(x) = 0_K$ for all $x \in Q$. It is not hard to see that f is a L-fuzzy quantale homomorphism. For all L-fuzzy quantale homomorphism$\overline{P} : P \longrightarrow L$, and L-fuzzy quantale homomorphism $r, s : P \longrightarrow Q$ such that $f \circ s = f \circ r$,i.e,.zero mapping is constant morphism in **LFquant**.

Necessity: Let $\overline{Q} : Q \longrightarrow L$ and $\overline{Q} : Q \longrightarrow L$ be L-fuzzy quantales, $f : Q \longrightarrow K$ is a constant morphism in **LFquant**. It is easy to check that the mapping $id_Q : Q \longrightarrow Q$ and zero mapping $0_Q : Q \longrightarrow Q$ be L-fuzzy quantales homomorphisms, then$f \circ id_Q = f \circ 0_Q$, i.e,. $(f \circ id_Q)(x) = (f \circ 0_Q)(x)$ for all $x \in Q$, hence $f(x) = 0_K$.

Theorem 3.2. The coconstant morphism in **LFquant** are exactly zero mapping.

corollary 3.3. The category **LFquant** is pointed.

Theorem 3.4. The category **LFquant** has initial object.

Proof. Let L be a complete lattice, it is to easy to check that mapping $\overline{0} : \{0\} \longrightarrow L$, $\overline{0}(0) = 0_L$ be a L-fuzzy quantale. For all L-fuzzy quantale $\overline{Q} : Q \longrightarrow L$ and L-fuzzy quantale homomorphism $f : \{0\} \longrightarrow Q$, then $f(0) = 0_Q$. Hence f is the unique L-fuzzy quantale homomorphism from $\{0\}$ to Q. Therefore the mapping $\overline{0} : \{0\} \longrightarrow L$ be a inital object in **LFquant**.

Theorem 3.5. The category **LFquant** has terminal object.

corollary 3.6. The category **LFquant** is connected.

Theorem 3.7. The category **LFquant** has equalizer.

$$\begin{tikzcd}
P \arrow[rr, "\overline{P}"] \arrow[d, "\overline{e}"'] \arrow[dr, "e"] & & L \\
Q' \arrow[r, "i"'] & Q \arrow[ur, "\overline{Q}"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & K \arrow[u, "\overline{K}"']
\end{tikzcd}$$

Proof. Let L be a complete lattice, $\overline{Q} : Q \longrightarrow L$ and $\overline{Q} : Q \longrightarrow L$ be L-fuzzy quantales, $f, g : Q \longrightarrow K$ be L-fuzzy quantale homomorphisms.

Define$Q' = \{x \in Q \mid f(x) = g(x)\}$. Next, we will prove that Q' is a subquantale of Q.

At first, since $f(0) = g(0)$, we have$0 \in Q' \neq 0$.

Secondly, for all $x_1, x_2 \in Q'$, $f(x_1\&x_2) = f(x_1)\&f(x_2) = g(x_1)\&g(x_2) = g(x_1\&x_2)$, then $f(x_1\&x_2) = f(x_1)\&f(x_2) = g(x_1)\&g(x_2) = g(x_1\&x_2)$, hence $x_1\&x_2 \in Q'$.

At last, for all $\{x_i\}_{i\in I}$, then $f(\bigvee\limits_{i\in I} x_i) = \bigvee\limits_{i\in I} f(x_i) = \bigvee\limits_{i\in I} g(x_i) = g(\bigvee\limits_{i\in I} x_i)$, hence $\bigvee\limits_{i\in I} x_i \in Q'$.

Let $i : Q' \longrightarrow Q$ is the inclusion mapping, we can prove that i is a quantale homomorphism, and $f \circ i = g \circ i$. Assume that $\overline{Q'} = \overline{Q} \mid^{Q'} Q' \longrightarrow L$, then $\overline{Q'}$ be a L-fuzzy quantale.

For any L-fuzzy quantale $\overline{P} : P \longrightarrow L$ and $e : P \longrightarrow Q$ such that $f \circ e = g \circ e$, then $e(P) = \{e(p) \mid p \in P\} \subseteq Q$.

Define $e' : P \longrightarrow Q'$ such that $e'(p) = e(p)$ for all $p \in P$, then $f(e(p)) = g(e(p))$. Hence e' is well defined, and e' is a L-fuzzy quantale homomorphism such that $e = i \circ e$.

Let $e'' : P \longrightarrow Q'$ be a L-fuzzy quantale homomorphism such that $e = i \circ e''$ for all $p \in P$, then $e''(p) = (i \circ e'')(p) = e(p) = e'(p)$ for all $p \in P$. Hence $e'' = e'$. So, e' is the unique L-fuzzy quantale homomorphism satisfy. Therefore, (Q', i) is the equalizer of f and g.

Remark 3.8. By theorem 3.7, we have that the mapping i is a regular monmorphism in the category **LFquant**.

Definition 3.9. Let L be a complete lattice, be a L-fuzzy quantale, The set R is said to be a congruence of L-fuzzy quantale on L if R satisfies:

(1) R is an equivalence relation on Q;

(2) If $(x_i, y_i) \in R$ for all $i \in I$, then $(\bigvee\limits_{i\in I} x_i, \bigvee\limits_{i\in I} y_i) \in R$;

(3) If $(x, y) \in R$, $(s, t) \in R$, then $(x\&s, y\&t) \in R$;

(4) For all $(x, y) \in R$, we have $\overline{Q}(x) = \overline{Q}(y)$.

Let Con($\overline{Q}$) the set of all congruence on , then be a complete lattice with respect to the inclusion order.

Let L be a complete lattice, be a L-fuzzy quantale, and R is a congruence of L-fuzzy quantale on L. Define the order relation " $\leq$ " on and operation $\bigvee^{Q/R}$, & on Q/R, $\forall\ [x], [y] \subseteq Q/R$, $\forall\ \{[x_i]\}_{i\in I} \subseteq Q/R$,

$$[x] \leq [y] \Longleftrightarrow [x \vee y] = [y]; \quad \bigvee_{i\in I}^{Q/R} [x_i] = [\bigvee_{i\in I} x_i]; \quad [x]\&_{Q/R}[y] = [x\&y].$$

We can prove that $(Q/R, \bigvee^{Q/R}, \&_{Q/R})$ be a quantale.

Theorem 3.10. Let L be a complete lattice, $\overline{Q} : Q \longrightarrow L$ be a L-fuzzy quantale, R be a L-fuzzy quantale congruence on $\overline{Q}$. Define $\overline{Q/R} : Q/R \longrightarrow L$ such that $\overline{Q/R}([x]) = \overline{Q}(x)$ for all $[x] \in Q/R$. Then $\overline{Q/R} : Q/R \longrightarrow L$ be a L-fuzzy quantale, and $\pi : Q \longrightarrow Q/R\ x \longmapsto [x]$ be a L-fuzzy quantale homomorphism.

Theorem 3.11. Let L be a complete lattice, $\overline{Q} : Q \longrightarrow L$ be a L-fuzzy quantale, then $\triangle = \{(x, x) \mid x \in M\}$ be a L-fuzzy quantale congruence on $\overline{Q}$.

Theorem 3.12. Let L be a complete lattice, $\overline{Q} : Q \longrightarrow L$ and $\overline{K} : K \longrightarrow L$ be L-fuzzy quantale, $f : Q \longrightarrow K$ be a L-fuzzy quantale homomorphism, R be a L-fuzzy quantale congruence on $\overline{K}$. Then $f^{-1}(R) = \{(x, y) \in M \times M \mid (f(x), f(y)) \in R\}$ be a L-fuzzy quantale congruence on $\overline{Q}$.

Theorem 3.13. Let L be a complete lattice, $\overline{Q} : Q \longrightarrow L$ be a L-fuzzy quantale, R be a L-fuzzy quantale congruence on $\overline{Q}$. Then there exists a smallest L-fuzzy quantale congruence containing R, which is the intersection all the L-fuzzy quantale congruence contain R on $\overline{Q}$. We said that this congruence is generated by R.

Theorem 3.14. The category **LFquant** has coequalizer.

$$\begin{tikzcd}
Q \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\overline{Q}"'] & K \arrow[r, "h"] \arrow[d, "\pi"] \arrow[dl, "\overline{K}"] & P \\
L & E \arrow[l, "\overline{E}"] \arrow[ur, "h'"'] &
\end{tikzcd}$$

Proof. Let L be a complete lattice, $\overline{Q} : Q \longrightarrow L$ and $\overline{K} : K \longrightarrow L$ be L-fuzzy quantales, $f, g : Q \longrightarrow K$ be L-fuzzy quantale homomorphisms. Suppose R is the smaiiest congruence of L-fuzzy quantale on $\overline{K}$, which contain $\{(f(x), g(x)) \mid x \in Q\}$. Let $E = K/R$, $\pi : K \longrightarrow E$ is the canonical mapping, then mapping $\overline{K/R} : E \longrightarrow L$ be L-fuzzy quantale, and $\pi : K \longrightarrow E$ is the L-fuzzy quantale homomorphism by theorem 3.10.

We will prove that (π, E) is the coequalier of f and g. In fact,

(1) $\pi \circ f = \pi \circ g$ is clear;

(2) Let $\overline{P} : P \longrightarrow L$ be a L-fuzzy quantale, $h : K \longrightarrow P$ be a L-fuzzy quantale homomorphism such that $h \circ f = h \circ g$. Let $R_1 = h^{-1}(\Delta)$, where $\Delta = \{(x, x) \mid x \in P\}$. By theorem 3.11, we can see that R_1 is a congruence of L-fuzzy quantale on $\overline{K}$. Since $h(f(x)) = h(g(x))$ for all $x \in Q$, then $(f(x), g(x)) \in R_1$. Define $h' : E \longrightarrow P$ such that $h'([k]) = h(k)$ for all $[k] \in K/R = E$. Let $(k_1, k_2) \in R$, then $(k_1, k_2) \in R_1$, i.e,. $h(k_1) = h(k_2)$ Therefore h' is well defined.

Next, we will prove that h' is a L-fuzzy quantale homomorphism.

(1) For all $[k_1], [k_2] \in E$, then $h'([k_1]\&[k_2]) = h(k_1\&k_2) = h(k_1)\&h(k_2) = h'([k_1])\&h'([k_2])$;

(2) For all $\{[k_i]\}_{i\in I} \subseteq E$, then $h'(\bigvee_{i\in I}[k_i]) = h'([\bigvee_{i\in I} k_i]) = h(\bigvee_{i\in I} k_i) = \bigvee_{i\in I} h(k_i) = \bigvee_{i\in I} h'([k_i])$;

(3) For all $[k] \in E$, since $h : K \longrightarrow P$ is a L-fuzzy quantale homomorphism, we can see that $\overline{E}([k]) = \overline{K}(k) \leq \overline{P}(h(k)) = \overline{P}(h'([k]))$ and $h = h' \circ \pi$.

It is easy to prove that h' is the unique L-fuzzy quantale homomorphism satisfy $h = h' \circ \pi$. Therefore $(\pi,\ E)$ is the coequalizer of f and g.

Theorem 3.15. The category **LFquant** has multiple coequalizer.

4. PRODUCT, INTERSECTION AND PULLBACK OF THE CATEGORY LFQUANT

In this section, the Product, intersection and pullback of the category of L-fuzzy quantales are investigated, the Concrete structure of Product, intersection and pullback of the category of L-fuzzy quantales is obtained.

Theorem 4.1. The category **LFquant** has product.

$$\begin{tikzcd}
\prod_{k\in K} Q_k \arrow[r, "\pi_k"] \arrow[d, "\overline{\pi}"'] & Q_k \arrow[dl, "\overline{Q_k}"] \\
L & Q \arrow[l, "\overline{Q}"] \arrow[u, "f_k"'] \arrow[ul, "f'"]
\end{tikzcd}$$

Proof. Let $\{(\overline{Q_k}, L) \mid k \in K\}$ be a family of L-fuzzy quantales. Define $\overline{\pi} : \prod_{k\in K} \overline{Q_k} \longrightarrow L$ such that $\overline{\pi}(f) = \bigwedge_{k\in K} \overline{Q_k}(f_k)$ for all $f \in \prod_{k\in K} \overline{Q_k}$.

At first, we will prove that $\overline{\pi} : \prod_{k\in K} \overline{Q_k} \longrightarrow L$ be a L-fuzzy quantale. Obviously, $\overline{\pi}$ is well defined.

(1) $\overline{\pi}(0) = \bigwedge_{k\in K} \overline{Q_k}(0_k) = \bigwedge_{k\in K} 1 = 1$;

(2) For all $f, g \in \prod_{k\in K} \overline{Q_k}$, then $\overline{\pi}(f\&g) = \bigwedge_{k\in K} \overline{Q_k}(f_k\&g_k) \geq \bigwedge_{k\in K} (\overline{Q_k} f_k) \wedge \overline{Q_k}(g_k)) = (\bigwedge_{k\in K} \overline{Q_k} f_k) \wedge (\bigwedge_{k\in K} \overline{Q_k}(g_k)) = \overline{\pi}(f) \wedge \overline{\pi}(g)$;

(3) For all $\{f^j\} \subseteq \prod_{k\in K} \overline{Q_k}$, then $\overline{\pi}(\bigvee_{j\in J} f^j) = \bigwedge_{k\in K} (\overline{Q_k}((\bigvee_{j\in J} f^j)_k)) = \bigwedge_{k\in K} (\overline{Q_k}(\bigvee_{j\in J} f_k^j)) \geq \bigwedge_{k\in K} (\bigwedge_{j\in J} \overline{Q_k}(f_k^j)) = \bigwedge_{j\in J} (\bigwedge_{k\in K} \overline{Q_k}(f_k^j)) = \bigwedge_{j\in J} \overline{Q_k}(f^j)$.

Secondly, we will prove that $\pi_k : \prod_{k\in K} \overline{Q_k} \longrightarrow \overline{Q_k}, f \longmapsto f_k$ is a L-fuzzy quantale homomorphism.

(1) For all $f, g \in \prod\limits_{k\in K} \overline{Q_k}$, $\forall \{f^j\}_{j\in J} \subseteq \prod\limits_{k\in K} \overline{Q_k}$, then

$\pi_k(f\&g) = (f\&g)_k = f_k\&g_k = \pi_k(f)\&\pi_k(g)$; $\pi_k(\bigvee\limits_{j\in J} f^j) = (\bigvee\limits_{j\in J} f^j)_k = \bigvee\limits_{j\in J} f_k^j = \bigvee\limits_{j\in J} \pi_k(f^j)$;

(2) For all $f \in \prod\limits_{k\in K} \overline{Q_k}$, then $\overline{\pi}(f) = \bigwedge\limits_{k\in K} \overline{Q_k}(f_k) \leq \overline{Q_k}(f_k) = \overline{Q_k}(\pi_k(f))$, hence $\pi_k : \prod\limits_{k\in K} \overline{Q_k} \longrightarrow \overline{Q_k}$ is a L-fuzzy quantale homomorphism.

Define $f' : Q \longrightarrow \prod\limits_{k\in K} \overline{Q_k}$ such that $f'(x) = (f_k(x))_{k\in K}$,i.e., $(f'(x))_k = f_k(x)$ for all $x \in Q$. Obviously, $\pi_k \circ f' = f_k$.

(1) For all $x, y \in Q$,$\{x_i\}_{i\in I}$, we have $(f'(x\&y))_k = f_k(x\&y) = f_k(x)\&f_k(y) = (f'(x))_k\&(f'(y))_k$;

$(f'(\bigvee\limits_{i\in I} x_i))_k = f_k(\bigvee\limits_{i\in I} x_i) = \bigvee\limits_{i\in I} f_k(x_i) = (f'(x_i))_k$;

(2) For all $x \in Q$, $\forall\ k \in K$, since f_k be a L-fuzzy quantale homomorphism, we can see that $\overline{Q_k}(x) \leq \overline{Q_k}(f_k(x))$, hence $\overline{Q_k}(x) \leq \bigwedge\limits_{k\in K} \overline{Q_k}(f_k(x))$, i.e., $\overline{Q_k}(x) \leq \overline{\pi}(f'(x))$.

Thus $f' : Q \longrightarrow \prod\limits_{k\in K} \overline{Q_k}$ is a unique L-fuzzy quantale homomorphism. Therefore $(\overline{\pi},\ \{\pi_k\}_{k\in K})$ is the product of $\{(\overline{Q_k}, L) \mid k \in K\}$ in **LFquant**.

Theorem 4.2. The category **LFquant** has intersection.

$$\begin{tikzcd}
& A_i \arrow[r, "m_i"] \arrow[ddl, "\overline{A_i}"'] & B \\
& D \arrow[u, "d_i"] \arrow[ur, "d"] \arrow[dl, "\overline{D}"'] & C \arrow[l, "f"] \arrow[u, "g"'] \arrow[ul, "g_i"] \arrow[dll, "\overline{C}"] \\
L & &
\end{tikzcd}$$

Proof. Let $\overline{B} : B \longrightarrow L$ be a L-fuzzy quantale, $\{\overline{A_i} : A_i \longrightarrow L\}_{i\in I}$ is family subobjects of $\overline{B}$. We can see that $m_i(A_i)$ is subquantale of B, and $A_i \cong m_i(A_i)$.

Let $m_i^{-1} : m_i(A_i) \longrightarrow A_i m_i$, and $D = \bigvee\limits_{i\in I} m_i(A_i)$, then D is the subquantale of B. Define $\overline{D} : D \longrightarrow L$, then $\overline{D}$ is a L-fuzzy quantale. Assume $d : D \longrightarrow B$ is a inclusion mapping. Hence $d : \overline{D} \longrightarrow \overline{B}$ is a L-fuzzy quantale homomorphism.

Next, we will prove that $(\overline{D},\ d)$ is the intersection of $\{\overline{A_i} : A_i \longrightarrow L\}_{i\in I}$ in the category of L-fuzzy quantales.

(1) For all $i \in I$, let $d_i : (m_i^{-1}) \mid_D: D \longrightarrow A_i$, then d_i is a L-fuzzy quantale homomorphism, and $d = m_i \circ d_i$.

(2) Let $g : C \longrightarrow B$ and $g_i : C \longrightarrow A_i$ be L-fuzzy quantale homomorphism, and $g = m_i \circ g_i$ for all $i \in I$. At the same time, we can see that $g_i(C)$ is a subquantale of A_i, then $g(C) = m_i(g_i(C))$ is a subquantale of $m_i(A_i)$. Hence $g(C)$ is subquantale of D. Let $f = g \mid^D$, then f is the unique L-fuzzy quantale homomorphism such that $d \circ f = g$. Therefor, $(\overline{D},\ d)$ is the intersection of $\{\overline{A_i} : A_i \longrightarrow L\}_{i\in I}$ in the category of L-fuzzy quantales.

Theorem 4.3. The category **LFquant** has pullback square.

$$\begin{tikzcd}
M \arrow[dr, "h"] \arrow[drr, "e_i"] \arrow[ddr, "f"'] & & \\
& E \arrow[r, "p_i"] \arrow[d, "p_0"] & D_i \arrow[d, "f_i"] \\
& B \arrow[r, "g_i"] & B_i
\end{tikzcd}$$

Proof. Let L be a complete lattice, $\overline{X} : X \longrightarrow L$ and $\overline{Y} : Y \longrightarrow L$ be L-fuzzy quantales, $f : X \longrightarrow P$ and $g : Y \longrightarrow P$ be L-fuzzy quantale homomorphisms.

Let $E = \{(x, y) \mid f(x) = g(y)\}$, then E is the subquantale of $X \times Y$. p_X and p_Y are projection from $X \times Y$ to X and Y, respectively. Define $\overline{E} = \overline{\pi} \mid^E : E \longrightarrow L$ such that $\overline{E}(x, y) = \overline{X}(x) \wedge \overline{Y}(y)$ for all $(x, y) \in E$, then $\overline{E}$ is L-fuzzy quantale homomorphisms.

Next, we will prove that p_X and p_Y are L-fuzzy quantale homomorphisms.

(1) p_X and p_Y are quantale homomorphisms is clear;

(2) For all $a = (x, y) \in E$, then $\overline{E}(a) = \overline{X}(x) \wedge \overline{Y}(y) \leq \overline{X}(x) = \overline{X}(p_X(a))$,i.e., $\overline{E}(a) \leq \overline{X}(p_X(a))$. Similarly, $\overline{E}(a) \leq \overline{X}(p_Y(a))$.

Let be a L-fuzzy quantale, and are L-fuzzy quantale homomorphisms, such that, then for all, hence.

Define such that for all, then is a quantale homomorphism, and,. Since and are L-fuzzy quantale homomorphisms, we are see that and for all. Hence , and is unique L-fuzzy quantale homomorphism satisfies above conditions.

Therefore, the category **LFquant** has pullback square.

Theorem 4.4. The category **LFquant** has multiple pullback square.

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REFERENCES

[1] Mulvey C. J. (1986). *Suppl. Rend. Circ. Mat. Palermo Ser., 12*, 99-104.

[2] Rosenthal K.I. (1990). *Quantales and Their Applications*. London: Longman Scientific and Tech-nical

[3] Nawaz M. (1985). Quantales: Quantale Sets. Ph.D.Thesis, University of Sussex.

[4] Brown C. & Gurr D. (1993) A Representation Theorem for Quantales. *Journal of Pure and Applied Algebra, 85*, 27-42.

[5] Resende P. (2001). Quantales, Finite Observations And Strong Bisimulation *Theoretical Computer Science, 254*, 95-149.

[6] Coniglio M.E., Miraglia F. (2001). Modules in the Category of Sheaves over Quantales. *Annals of Pure and Applied Logic, 108*, 103-136.

[7] Resende P. (2002). Tropological Systems are Points of Quantales. *Journal of Pure and Applied Algebra,173*, 87-120.

[8] LI Yongming, ZHOU Meng & LI Zhihui (2002). Projectives and Injectives in the Category of Quantales. *Journal of Pure and Applied Algebra, 176*(2), 249-258.

[9] Picado J. (2004). The Quantale of Galois Connections. *Algebra Universalis, 52*, 527-540.

[10] Resende P. (2007). Etale Groupoids and Their Quantales. *Advances in Mathematics, 208*(1), 147-209.

[11] Russo C. (2007). *Quantale Modules, with Applications to Logic and Image Processing.* Ph.D.Thesis, Salerno: University of Salerno.

[12] Solovyov S. (2008). On the Category Q-Mod. *Algebra Universalis, 58*(1), 35-58.

[13] LIU Z.B., ZHAO B. (2006). Algebraic Properties of Category of Quantale. *Acta Mathematica Sinica* (Chinese Series), *49*(6), 1253-1258.

[14] L. Roman (2005). Residuated Semigroups and the Algebraic Foundations of Quantum Mechains. *International Journal of Theoretical Physics, 44*(12), 2259-2266.

[15] N. Kamide (2004). Quantized Linear Logic, Involutive Quantales and Strong Negation. *Studia Logica, 77*(3), 355-384.

[16] D. Kruml, P. Resende (2004). On Quantales that Classify C∗-algebra. *Cahiers de Topologie et Geometrie Differentielle Categoriques, 45*(4), 287-296.

[17] P. Resende (2007). Etale Groupoids and Their Quantales. *Advances in Mathematics, 208*(1), 147-209.

[18] B. Banaschewski, C. J . Mulvey (2006). A Globalization of the Gelfand Duality Theorem. *Annals of Pure and Applied Logic, 137*(1-3), 62-103.

[19] Han S. W, Zhao B. (2009). The Quantic Conuclei on Quantales. *Algebra Universalis, 61*(1), 97-114.

[20] Zhao B., Liang S. H. (2009). The Categorgy of Double Quantale Modules. *Acta Mathematica Sinica* (Chinese Series), *52*(4), 821-832.

[21] Liang S. H., Zhao B. (2008). Continuous Quantale and Its Category Properties. *Journal of Shanxi Normal University* (Natural Science Edition)(In Chinese), *36*(5), 1-5.

[22] Ad'amek, J. (1990). Abstrat and Concrete Categories. Wiley, New York.