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On Behaviour of a Host-vector Epidemic Model with Non-linear Incidence

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Abstract
In this paper we find the possible phase portraits and bifurcations for a general class of host-vector epidemic models with non-linear incidence function generalizing the Ross model.

Key words
Epidemics; Non-linear incidence; Global analysis; Bifurcations
MSC2010: 34C05, 34C23, 34D23, 92D30

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INTRODUCTION

In this paper we consider a modified Ross model of vector-borne diseases with non-linear incidence function. The model is given as a a two dimensional system of ODE:

$$\begin{aligned} x' &= g_1(y)h_1(x) - c_1\mu_1(x) \\ y' &= g_2(x)h_2(y) - c_2\mu_2(y) \end{aligned} \tag{1}$$

Here x and y represent the infective host and vector populations. The terms $g_1(y)h_1(x)$ and $g_2(x)h_2(y)$ correspond to the incidence functions. In the original Ross model obtained from ideas given in [8], the functions g_i and h_i are linear. The effect of different non-linear incidence functions for usual (without vector) epidemic models have been studied by many authors. Models with incidence function of type kI^pS^q have been studied in [5, 6], and models with incidence function, where $g_i(I)$ are of type $\frac{kI^p}{1+\alpha I^q}$, have been studied in [3, 7, 9]. Results for some general type of non-linear incidence functions are obtained in [1, 2, 4].

1. MAIN THEOREM

Equations in (1) are considered for $0 \le x, y \le 1$ because x and y are supposed to correspond to the relative infectious population of host and vector. We assume that the functions g_i, h_i and μ_i satisfy the following conditions.

Condition 1. g_i, h_i, μ_i are continuously differentiable in $(0,1]$ and continuous in $[0,1]$. Moreover $h_i'(z) < 0$ and $g_i'(z), \mu_i'(z) > 0$ for $i = 1,2$. The functions satisfy the boundary conditions $g_i(0) = 0$, $\mu_i(0) = 0$ and $h_i(1) = 0$ and parameters c_i are positive for $i = 1,2$.

Let $\phi_i(z) = \frac{-z f_i'(z)}{f_i(z)}$ and $\theta_i(z) = \frac{z g_i'(z)}{g_i(z)}$ where $f_i(z) = \frac{h_i(z)}{\mu_i(z)}$ and $0 < z < 1$.

Condition 2. ϕ_i is increasing and θ_i is non-increasing and $\lim_{z\to 0_+} \phi_i(z) = b_i$ and $\lim_{z\to 0_+} \theta_i(z) = a_i$, where a_i and b_i are positive.

Conditions 1 and 2 are satisfied for many known epidemical models and incidence functions, for example, if $g_i(z) = \frac{kz^p}{1+\alpha z^q}$ and $h_i(z) = (1-z)^b$ and μ_i are linear.

We now formulate our main theorem.

Theorem. Suppose system (1) satisfies Conditions 1 and 2, then we have the following structurally stable phase portraits.

Case 1. If $a_1a_2 > b_1b_2$ then there is a number $c_m(c_1)$ depending on c_1 such that for $c_2 > c_m(c_1)$ the origin is a global attractor and for $c_2 < c_m(c_1)$ there are two equilibria, a saddle P_s and a stable one P_e except the origin. The stable set of P_s divides the phase space into two parts, one in the basin of attraction of the origin and the other in the basin of attraction of P_e.

Case 2. If $a_1a_2 = b_1b_2$ then there are two possibilities. The first one is that there is a c depending on c_1 such that when $c_2 < c$ there is an equilibrium P_e (except the origin) attracting all trajectories except the origin and when $c_2 \geq c$ the origin is a global attractor. The second possibility is that for any c_2 there is exactly one equilibrium P_e (except the origin) attracting all trajectories except the origin.

Case 3. If $a_1a_2 < b_1b_2$ then there is always exactly one equilibrium P_e (except the origin) attracting all trajectories except the origin.

In order to prove our main theorem we need the following lemmas. To formulate the lemmas we introduce notations convenient for using in our proof.

Notation. We consider positive-valued continuously differentiable functions g defined in an interval $(0, B]$, where $B > 0$.

We denote by $Q(a,\infty)$, the set of functions g such that when $z \to 0_+$ then $\frac{g(z)}{z^a} \to A$, where either $A \in R_+$ or equal ∞ and $\frac{g(z)}{z^\alpha} \to 0$ for $\alpha < a$, where a is a real number.

Similarly we denote by $Q(a,0)$, the set of functions g such that when $z \to 0_+$ then $\frac{g(z)}{z^a} \to A$, where either $A \in R_+$ or equals 0 and $\frac{g(z)}{z^\alpha} \to \infty$ for $\alpha > a$, where a is a real number.

For these Q-classes the following is known to hold:

Lemma 1. If g is differentiable invertible with inverse g^{-1} and $a > 0$ then

$$g \in Q(a,\infty) \Leftrightarrow g^{-1} \in Q(a^{-1}, 0).$$

Lemma 2. If $g \in Q(a,\infty)$, $h \in Q(b,0)$ and $a < 0 < b$ then $g \circ h \in Q(ab,\infty)$.

If $g \in Q(a,0)$, $h \in Q(b,0)$ and $a, b > 0$ then $g \circ h \in Q(ab, 0)$.

Lemma 3. If $g \in Q(a,0)$ and $h \in Q(b,\infty)$ then $\frac{g}{h} \in Q(a-b, 0)$.

Lemma 4. If $g \in Q(a,\infty)$ and $h \in Q(b,\infty)$ then $gh \in Q(a+b,\infty)$.

The proofs of these lemmas are obtained by straightforward calculations. More details and also details of other parts of this preprint are available from authors.

We now state another lemma connecting the Q-classes and the θ-function defined by $\theta(z) = \frac{zg'(z)}{g(z)}$. We introduce a known lemma and give a short proof of it.

Lemma 5. If θ is non-increasing and $\theta(z) \to a$ as $z \to 0_+$ then $g \in Q(a,\infty)$.

Proof of Lemma 5. We denote by u the function defined by $u(z) = \frac{g(z)}{z^\alpha}$ and by η the function defined by $\eta(z) = \frac{zu'(z)}{u(z)}$. Then $\eta(z) = \theta(z) - \alpha$.

If $\alpha < a$ then $\eta(z) > b$ for some positive b in a neighbourhood of zero. Integrating the inequality $\frac{u'}{u} > \frac{b}{z}$ from u to $u_0 = u(z_0)$ to left and from z to z_0 to right and using monotonicity of logarithm we obtain

$$\frac{u_0}{u} > \left(\frac{z_0}{z}\right)^b,$$

implying $u(z) \to 0$ for $z \to 0_+$.

We now assume $\alpha = a$. Since η is non-increasing and $\eta \to 0$ for $z \to 0_+$, the derivative u' is non-positive and the limit set of u when $z \to 0_+$ cannot contain more than one point and this cannot be zero.

We are now ready to prove our main theorem. The proof of the main theorem consists of three parts. First part examines the number of equilibria from intersections of zero-isoclines. This part needs the lemmas. The second part examines the type of the equilibria found. The final part makes the global analysis using sign analysis of the right hand sides of system 1.

Proof of main theorem. We start by finding the number of equilibria and their position in relation to each other.

First we see that c_2 can be considered as a function of the x-coordinate at equilibrium and analyze the behaviour at endpoints 0 and x_1 of the interval of definition. Secondly we differentiate that function to find out the behaviour inside the interval $(0, x_1)$.

Let c_1 be fixed and suppose (x, y) is a point on the isocline $x' = 0$. Then from Condition 1 it follows that $y = p_1(x) = g_1^{-1}\left(\frac{1}{f_1(x)}\right)$ is an increasing function of x and $p_1(0) = 0$ and $p_1(x) \to \infty$ for $x \to 1_-$. Thus, there is an x_1 between 0 and 1 such that $p_1(x_1) = 1$ and the isocline $x' = 0$ is given by the function p_1 defined in $[0, x_1]$.

In this part of the proof calculating the limit behaviour when $z \to 0_+$ we consider functions g_i as defined only for $z > 0$.

From Condition 2 and Lemma 5 it follows that $g_i \in Q(a_i, \infty)$ and $f_i \in Q(-b_i, \infty)$. From Lemma 3 it follows that $\frac{1}{f_1} \in Q(b_1, 0)$ and from Lemma 1 it follows that $g_1^{-1} \in Q(\frac{1}{a_1}, 0)$. However from Lemma 2 (second part) it follows that $p_1 \in Q(\frac{b_1}{a_1}, 0)$.

We now suppose (x, y) is also on the isocline $y' = 0$, that is (x, y) is an equilibrium point. We then calculate c_2 as a function of x i.e. $c_2 = g_2(x) f_2(p_1(x))$. From Lemma 2 it follows that the composition of f_2 and p_1 belongs to $Q(-\frac{b_2 b_1}{a_1}, \infty)$ and finally Lemma 4 implies, that c_2 as a function of x, belongs to $Q(a_2 - \frac{b_2 b_1}{a_1}, \infty)$.

We conclude that for $x \to 0_+$ we get $c_2(x) \to 0$ in case 1 where $a_1 a_2 > b_1 b_2$ and $c_2(x) \to \infty$ in case 3 where $a_1 a_2 < b_1 b_2$ and either $c_2(x) \to \infty$ or $c_2(x) \to c \in R_+$ in case 2 where $a_1 a_2 = b_1 b_2$.

Because $h_2(1) = 0$, we conclude that in all cases $c_2(x) \to 0$ for $x \to x_1$.

We have now finished examining the behaviour of c_2 at the endpoints.

To find out when $c_2(x)$ is growing or decreasing we calculate the derivative.

Differentiating $c_1 = g_1(y) f_1(x)$ with respect to x and solving for y' we get

$$y' = \frac{-g_1(y) f_1'(x)}{g_1'(y) f_1(x)}.$$

Differentiating $c_2 = g_2(x) f_2(y)$ with respect to x and substituting our expression for y' we get

$$\frac{dc_2}{dx} = g_2'(x) f_2(y) \left(1 - \frac{\phi_1(x)\phi_2(y)}{\theta_2(x)\theta_1(y)}\right). \tag{2}$$

From the boundary behaviour of c_2 and the derivative, we make conclusions about the behaviour of c_2 between 0 and x_1 and from there we find the number of equilibria depending on c_2. From condition 1 and 2 it follows that $\frac{dc_2}{dx}$ is always decreasing.

We consider case 1 when $a_1 a_2 > b_1 b_2$. From conditions 1 and 2 it follows that $\frac{dc_2}{dx}$ is positive near 0 and becomes negative near x_1. Then there is an $x_m(c_1)$ such that $\frac{dc_2}{dx} = 0$ for $x = x_m(c_1)$ and $\frac{dc_2}{dx} > 0$ for $x < x_m(c_1)$ and $\frac{dc_2}{dx} < 0$ for $x > x_m(c_1)$.

We denote by $c_m(c_1)$, the maximum value of c_2 for fixed c_1.

In the case when $c_2 > c_m(c_1)$, we cannot solve for x and thus have no equilibrium apart from the origin.

If $c_2 = c_m(c_1)$, we have only one solution for x i.e. $x = x_m(c_1)$ and thus one equilibrium (in addition to the origin).

As $c_2(x) \to 0$ for $x \to x_1$ and $x \to 0_+$ for the case $c_2 < c_m(c_1)$ we obtain two solutions for x, one with $x < x_m(c_1)$ and the other with $x > x_m(c_1)$ and thus have two equilibria (in addition to the origin).

Let us denote the equilibrium at the origin by P_0. If $c_2 < c_m(c_1)$, we denote the equilibrium when $x > x_m(c_1)$ by P_e and the equilibrium when $x < x_m(c_1)$ by P_s.

The equilibrium P_s tends to P_0 (disease-free) and P_e gets it's maximal size as c_2 tends to zero. At $c_2 = c_m(c_1)$ there is a saddle-node bifurcation with both equilibria coinciding and disappearing after that.

Next we consider case 3, where $a_1a_2 < b_1b_2$. There $\frac{dc_2}{dx} < 0$ for all x and c_2 is decreasing from infinity to zero when x is increasing from zero to x_1. Thus, there is always exactly one non-trivial equilibrium denoted by P_e. The equilibrium P_e tends to zero when c_2 grows to infinity.

Finally we consider case 2, where $a_1a_2 = b_1b_2$. In this case $\frac{dc_2}{dx} \le 0$ and if $c_2(x) \to \infty$ for $x \to 0_+$, we have a situation analogous to the one in case 3. If $c_2(x) \to c$ then c_2 is decreasing from c to zero when x is increasing from zero to x_1. Thus for $c_2 < c$ there is always exactly one non-trivial equilibrium denoted by P_e. The equilibrium P_e tends to zero when $c_2 \to c_-$. In this case there is no non-trivial (endemic) equilibrium for $c_2 > c$. At $c_2 = c$ there is a transcritical bifurcation.

The first part of the proof is now complete and we begin with the second part to find the type of the equilibria.

The Jacobian matrix for system (1) is given by

$$J = \begin{bmatrix} g_1(y)h_1'(x) - c_1\mu_1'(x) & g_1'(y)h_1(x) \\ g_2'(x)h_2(y) & g_2(x)h_2'(y) - c_2\mu_2'(y) \end{bmatrix}.$$

Using that $h_i(z) = f_i(z)\mu_i(z)$ and at equilibrium $c_1 = g_1(y)f_1(x)$ and $c_2 = g_2(x)f_2(y)$ after some calculations the Jacobian matrix becomes

$$J = \begin{bmatrix} g_1(y)f_1'(x)\mu_1(x) & g_1'(y)f_1(x)\mu_1(x) \\ g_2'(x)f_2(y)\mu_2(y) & g_2(x)f_2'(y)\mu_2(y) \end{bmatrix}$$

for $x, y \neq 0$. We now calculate the trace and determinant of the Jacobian matrix.

From Condition 1, it follows that $f_i'(z) < 0$ and

$$Trace(J) = g_1(y)f_1'(x)\mu_1(x) + g_2(x)f_2'(y)\mu_2(y) < 0.$$

Calculations show that the determinant D of the Jacobian matrix is equal to

$$D = -\mu_1(x)\mu_2(y)g_1'(y)f_1(x)\frac{dc_2}{dx}.$$

We consider case 1.

We have two equilibria in the case $c_2 < c_m(c_1)$. When $x > x_m(c_1)$ then $\frac{dc_2}{dx} < 0$. This implies $D > 0$. We note that $g_i'(z) > 0$ from Condition 1. Thus the determinant is positive at P_e. When $x < x_m(c_1)$ then $\frac{dc_2}{dx} > 0$. This implies $D < 0$. Thus the determinant is negative at P_s.

Since the trace is negative and the determinant is positive at P_e, it is a sink. Also, since the determinant is negative at P_s, it is a saddle.

In cases 2 and 3 we conclude in the same way that P_e is always stable when it exists.

The type of equilibrium P_0 cannot always be found from Jacobian matrix, as the derivatives of the functions might not exist at 0. Anyhow a lot is known about origin from global analysis below.

The second part of proof is now complete and we begin with the last part and examine the global behaviour. We do this by using sign analysis of x' and y'.

We start with case 1, which has the most complicated behaviour.

We notice that in the xy-space above the isocline $x' = 0$, the sign of x' is positive and below negative.To the left of the isocline $y' = 0$, the sign of y' is negative and to the right it is positive.

We consider the situation when $c_2 > c_m(c_1)$. Here the isoclines do not intersect in the xy-plane. The isocline $x' = 0$ is above and to the left from the isocline $y' = 0$. The isoclines divide the phase space into three parts as seen in Figure 1. These parts are defined as follows:

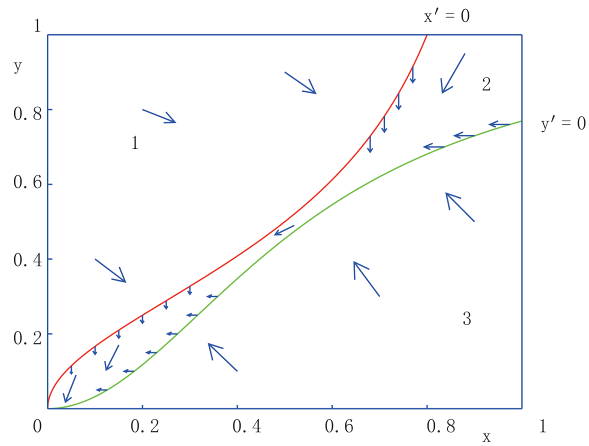

Figure 1
Sign Analysis for System $x' = y^2(1-x) - 0.25x, y' = x^2(1-y) - 0.3y$

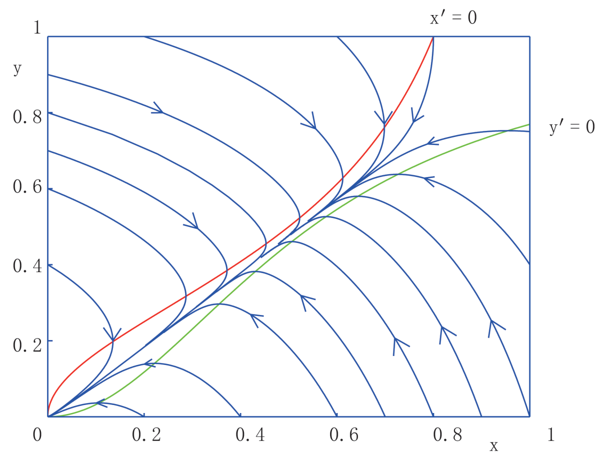

Figure 2
Phase Portrait and Zero-Isoclines for System $x' = y^2(1-x) - 0.25x, y' = x^2(1-y) - 0.3y$

1. The region where $y' < 0 < x'$.
2. The region where $x', y' < 0$.
3. The region where $x' < 0 < y'$.

In region 1 the x-coordinate of the trajectory is increasing and the y-coordinate is decreasing. Thus the trajectory cannot remain in the region, but has to hit the isocline $x' = 0$ entering region 2 after some time . In region 3 the x-coordinate of the trajectory is decreasing and the y-coordinate is increasing. In that way the trajectory cannot remain in the region, but has to hit the isocline $y' = 0$ entering region 2 after some time. In region 2, the x- and y-coordinates are decreasing and the trajectory cannot escape from region 2 against the direction field on the boundaries $x' = 0$ and $y' = 0$ and also on $x = 1$ and $y = 1$. In region 2, trajectories can be attracted only to the origin. We conclude that in the first and the third region the trajectories hit either the isocline $x' = 0$ or $y' = 0$ and afterwards they remain in region 2 where they are all attracted to P_0. Thus the disease-free origin is a global attractor. One example of such a phase portrait is given in Figure 2.

We now consider the situation $c_2 < c_m(c_1)$. Here the isoclines intersect and divide the phase space into five parts as shown in Figure 3. These regions are defined as follows:

1. The region where $y' < 0 < x'$.
2. The region where $x', y' < 0$ and x is less than the x-coordinate of P_s.
3. The region where $x', y' < 0$ and x is greater than the x-coordinate of P_e.

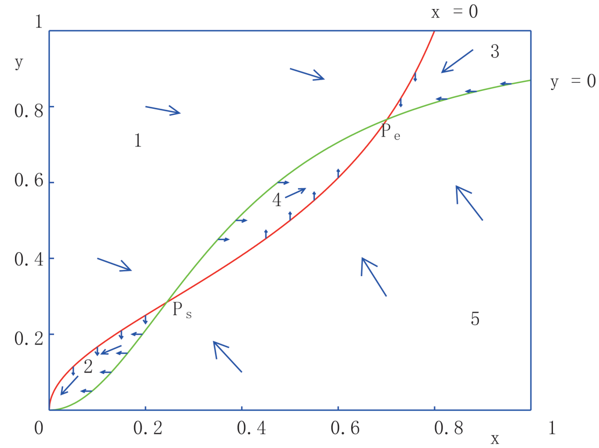

Figure 3
Sign Analysis for System $x' = y^2(1-x) - 0.25x, y' = x^2(1-y) - 0.15y$

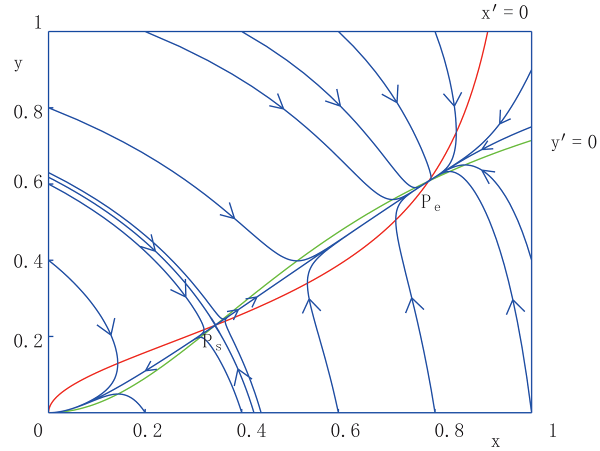

Figure 4
Phase Portrait and Zero-Isoclines for System $x' = y^2(1-x) - 0.1x, y' = x^2(1-y) - 0.4y$

4. The region where $x', y' > 0$.
5. The region where $x' < 0 < y'$.

In region 1, the x-coordinate of the trajectory is increasing and the y-coordinate is decreasing. This means the trajectory cannot remain in the region, but has to hit either the isocline $x' = 0$ or $y' = 0$ entering one of regions 2, 3 or 4 after some time, or the trajectory is attracted to P_s or P_e. In region 5, the x-coordinate of the trajectory is decreasing and the y-coordinate is increasing. This means the trajectory cannot remain in the region, but has to hit either the isocline $x' = 0$ or $y' = 0$ entering one of regions 2, 3 or 4 after some time, or the trajectory is attracted to P_s or P_e. In region 2, the x- and y-coordinates are decreasing and the trajectory cannot escape from region 2 against the direction field on the boundaries $x' = 0$ and $y' = 0$. In region 2, trajectories can be attracted only to the origin. In region 4, the x- and y-coordinates are increasing and the trajectory cannot escape from region 4 against the direction field on the boundaries $x' = 0$ and $y' = 0$. In region 4, trajectories can be attracted only to P_e. In region 3, the x- and y-coordinates are decreasing and the trajectory cannot escape from region 3 against the direction field on the boundaries $x' = 0$ and $y' = 0$ and also on $x = 1$ and $y = 1$. In region 3, trajectories can be attracted only to P_e.

We conclude that in the first and the fifth region trajectories after some time either hit the isocline $x' = 0$ or $y' = 0$ or tend directly to some equilibrium without visiting other parts. If they go through one of the isoclines they come into one of regions 2, 3 or 4 and remain in the region they enter. Trajectories in region

2 are attracted to P_0 and in regions 3 and 4 to P_e. Thus the stable set of the saddle P_s divides the phase space into two parts, one where trajectories are attracted to origin and another where they are attracted to P_e. One example of such a phase portrait is given in Figure 4.

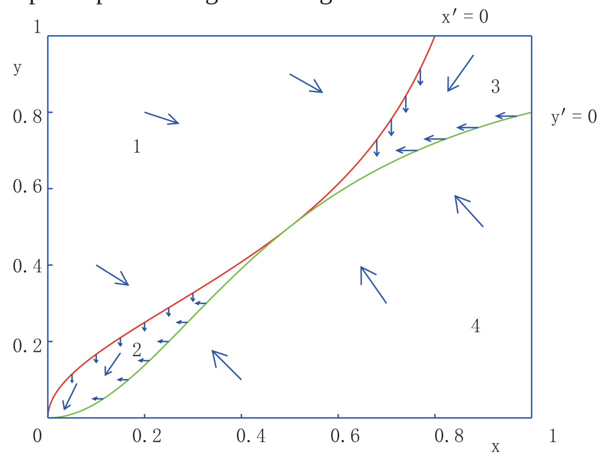

Figure 5
Sign Analysis for System $x' = y^2(1-x) - 0.25x, y' = x^2(1-y) - 0.25y$

In the case $c_2 = c_m(c_1)$ there is a saddle-node bifuraction dividing the parameter space into two parts with different qualitative behaviour described in the two sitautions above. In this case, the isoclines intersect, except at origin at a tangency point which is an equilibrium. The phase space is divided into four parts as shown in Figure 5. The regions are defined as follows:

1. The region where $y' < 0 < x'$.
2. The region where $x', y' < 0$ and x is less than the x-coordinate of the equilibrium.
3. The region where $x', y' < 0$ and x is greater than the x-coordinate of the equilibrium.
4. The region where $x' < 0 < y'$.

In region 1, the x-coordinate of the trajectory is increasing and the y-coordinate is decreasing. Thus the trajectory cannot remain in the region, but has to hit either the isocline $x' = 0$ entering one of region 2 or 3 after some time or the trajectory is attracted to the equilibrium point at tangency of isoclines. In region 4 the x-coordinate of the trajectory is decreasing and the y-coordinate is increasing. In that way the trajectory cannot remain in the region but has to hit after some time either the isocline $y' = 0$ entering one of region 2 or 3 or the trajectory is attracted to the equilibrium point. Trajectories in region 2 cannot escape against the direction field on the boundary and they are all attracted by the origin. For the same reason, the trajectories in region 3 cannot escape and they must be attracted by the equilibrium at tangency.

We conclude that trajectories in regions 1 and 4 either hit one of the isoclines $x' = 0$ or $y' = 0$ after some time or are attracted to the equilibrium at tangency. The equilibrium is a saddle-node and the boundary of it's stable set divides the phase space into two parts, to the left, the trajectories are in the basin of attraction of the origin and to the right, we have the stable set of the equilibrium including the boundary of itself. Figure 6 shows one example of such a phase portrait.

In case 2, where $a_1 a_2 = b_1 b_2$ and in the situation $c_2 \geq c$ the sign analysis can be carried out in the same way as in case 1 when $c_2 > c_m(c_1)$ and we get origin as global attractor.

In case 2 and in the situation when $c_2 < c$ we have the typical situation in the endemic case in the Ross model. The zero-isoclines divide the phase space into four regions as shown in Figure 7. The regions are defined as:

1. The region where $y' < 0 < x'$.
2. The region where $x', y' > 0$ (here x is less than the x-coordinate of the endemic equilibrium).
3. The region where $x', y' < 0$ (here x is greater than the x-coordinate of the endemic equilibrium).
4. The region where $x' < 0 < y'$.

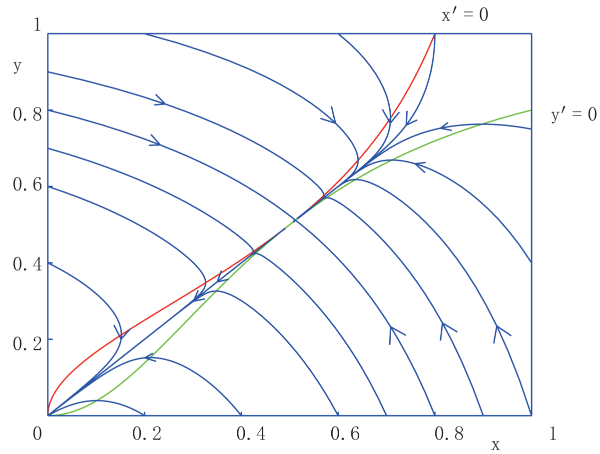

Figure 6
Phase Portrait and Zero-Isoclines for System $x' = y^2(1 - x) - 0.25x, y' = x^2(1 - y) - 0.25y$

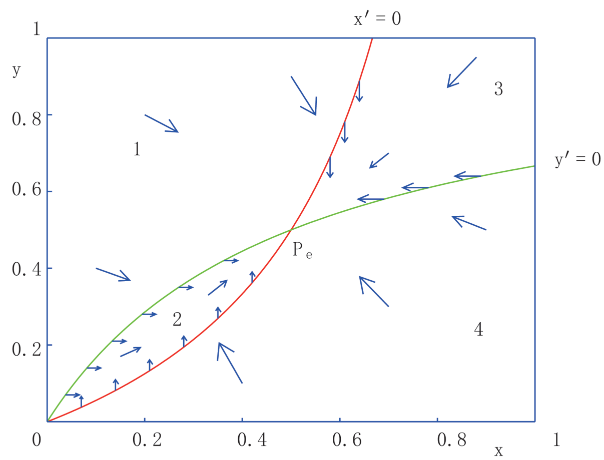

Figure 7
Sign Analysis for System $x' = y(1 - x) - 0.5x, y' = x(1 - y) - 0.5y$

As before we conclude that the trajectories in regions 1 and 4 hit one of the isoclines after some time or are attracted directly by the endemic equilibrium. After hitting one isocline they either enter region 2 or 3 where they are attracted to the endemic equilibrium. Thus the endemic equilibrium is a global attractor, attracting everything except the origin. One example of such a phase portrait is given in Figure 8.

In case 3, where $a_1a_2 < b_1b_2$ similar sign analysis as in the previous case shows that P_e is a global attractor attracting everything except the origin.

2. SOME EXAMPLES

We now study special cases of the functions g_i, h_i and μ_i in system (1). These are often used in models with non-linear incidence. We assume the functions have the form $g_i(z) = z^a$, $h_i(z) = (1 - z)^b$ and $\mu_i(z) = z$, $i = 1, 2$, which gives system

$$\begin{aligned} x' &= y^a(1 - x)^b - c_1x \\ y' &= x^a(1 - y)^b - c_2y. \end{aligned} \tag{3}$$

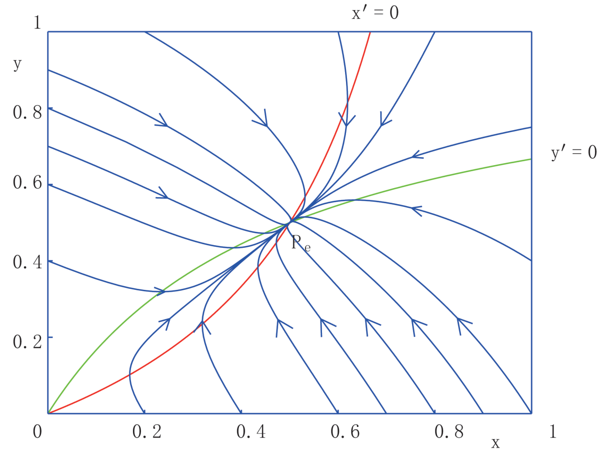

Figure 8
Phase Portrait and Zero-Isoclines for System $x' = y(1-x) - 0.5x, y' = x(1-y) - 0.5y$

We suppose $a > 1$ which here will imply case 1, a saddle-node bifurcation and two possible main types of phase portraits.

It is possible to prove that this system satisfies Conditions 1-2 by direct calculations.

Calculations give $\theta_i(z) = a$ and $\phi_i(z) = \frac{1+dz}{1-z}$, where $d = b - 1$ and $a_1 = a_2 = a$ and $b_1 = b_2 = 1$. Thus, we can apply case 1 in our theorem.

The saddle-node bifurcation occurs when $\frac{dc_2}{dx} = 0$ in (2) thereby giving us

$$\phi_1(x)\phi_2(y) = \theta_2(x)\theta_1(y) \tag{4}$$

Equation (4) must be satisfied for an equilibrium point (x, y) in order to get a saddle-node bifurcation.

Equation (4) for bifurcation in our example (3) becomes

$$\frac{1+dx}{1-x}\frac{1+dy}{1-y} = a^2, \tag{5}$$

and solving for y we get:

$$y = \frac{-(d+a^2)x - 1 + a^2}{(d^2 - a^2)x + d + a^2}. \tag{6}$$

For any equilibrium in system (3) we must have

$$c_1 = \frac{y^a(1-x)^b}{x}, c_2 = \frac{x^a(1-y)^b}{y}. \tag{7}$$

Substituting (6) into (7) we obtain a parameter representation for the saddle-node bifurcation curve in the c_1c_2-space if a and b are known. Some examples of such bifurcation curves are shown in Figures 9 and 10.

In some special cases, it is possible to get algebraic formulas for calculating c_1 and c_2 or the equilibrium (x, y) at bifurcation.

In the case where $b = 1$, it is possible to calculate c_1, c_2 and the equilibrium at bifurcation if the product $c_| c_2$ is given.

In this case bifurcation equation (5) becomes

$$(1-x)(1-y) = \frac{1}{a^2}. \tag{8}$$

Multiplying equalities (7) we get

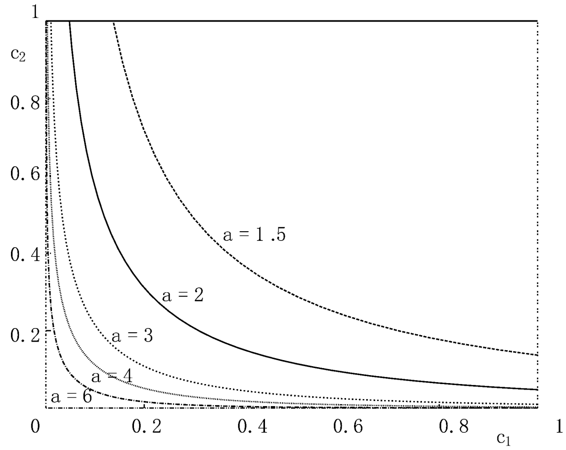

Figure 9
Bifurcation Curves for $a = 1.5, 3, 4, 6$ and $b = 1$ of System 3.

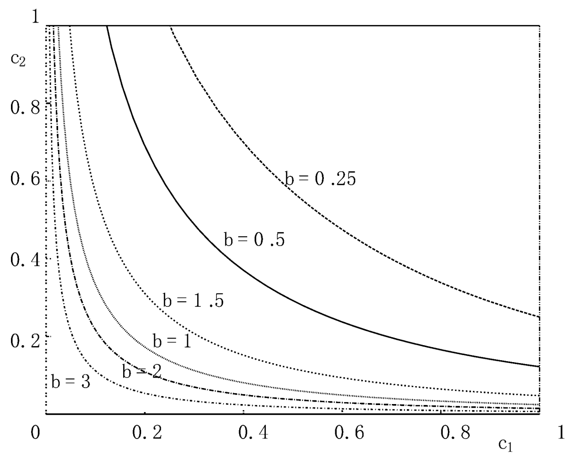

Figure 10
Bifurcation Curves for $b = 0.25, 0.5, 1, 1.5, 2, 3$ and $a = 2$ of System 3.

$$c_1c_2 = (xy)^{a-1}(1-x)(1-y) \tag{9}$$

and using (8), we obtain an expression for xy:

$$xy = \left(a^2c_1c_2\right)^{\frac{1}{a-1}}. \tag{10}$$

Using expression (10) in expansion of (8) gives

$$x + y = 1 - \frac{1}{a^2} + \left(a^2c_1c_2\right)^{\frac{1}{a-1}}. \tag{11}$$

Solving for y from (8) and substituting into (11), we obtain a second order equation for x if a and c_1c_2 are known. Knowing x we can solve y from (8) and finally we can calculate c_1 and c_2 from equalities (7).

Finally we consider a special case where $a = 2$ and $b = 1$. Here it is possible to get an expression for c_1 or c_2 at bifurcation, if we know one of them. Also the bifurcation point (x, y) can be easily calculated from explicit algebraic expressions afterwards.

In this case equality (10) takes on a simple form

$$xy = 4c_1c_2, \tag{12}$$

and (11) takes on the form

$$4x + 4y = 3 + 16c_1c_2. \tag{13}$$

Subsituting $a = 2$ and $b = 1$ into the expression for y in (6) we get

$$y = \frac{3 - 4x}{4(1 - x)}. \tag{14}$$

From (7) using (14) we see that

$$\frac{c_1 x}{y^2} = 1 - x = \frac{3 - 4x}{4y}, \tag{15}$$

which simplifies to

$$3y - 4xy = 4c_1 x. \tag{16}$$

Pluging (12) in (16) we obtain

$$-4c_1 x + 3y = 16c_1c_2. \tag{17}$$

Solving for x and y from (13) and (17) we obtain expressions

$$x = \frac{9 - 16c_1c_2}{16c_1 + 12}, \quad y = \frac{(16c_1^2 + 16c_1)c_2 + 3c_1}{4c_1 + 3} \tag{18}$$

Substituting now expressions (18) for x and y into (12) we see after simplifications that the bifurcation curve in the c_1c_2-space satisfies the condition

$$256(c_1^2c_2^2 + c_1c_2^2 + c_1^2c_2) + 288c_1c_2 = 27. \tag{19}$$

From this we can easily solve for c_1 or c_2 from a second order equation, if one of them is known. And knowing c_1 and c_2, we can calculate the coordinates for the equilibrium at bifurcation from formulas (18).

CONCLUSION

We have examined a generalized Ross model for a large class of non-linear incidence functions and found possible phase portraits and bifurcations. Many known incidence functions are inside this class. There are three types of structurally stable types of phase portraits. One type has the disease-free origin as a global attractor. A second one has the endemic equilibrium as a global attractor. In the third type both disease-free origin and endemic equilibrium are attractors and there is a saddle equilibrium with stable set forming the boundary between the basins of attractions of the both attractors. The possible bifurcations are the usual saddle-node and transcritical bifurcations.

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