A Fuzzy LP Approach to Option Portfolio

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Abstract: Owing to the fluctuation of financial market from time to time,the volatility and stock price may occur imprecisely in the real world. Therefore, it is natural to consider the fuzzy volatility and fuzzy stock price in the financial market. Under these assumptions,the theoretical price deduced by Black–Scholes formula are will turn into the fuzzy numbers, and the derivatives, called the Greek parameters delta, gamma, of the BS model are also fuzzy numbers. An option portfolio considering these fuzzy numbers will be more accord with actual situations. In this paper, we propose a fuzzy programming model of option portfolio based a ranking criterion of fuzzy numbers, which the fuzzy option portfolio model is converted into a classical linear programming problem. Finally, a numerical example is given to illustrate the validity of the method.

Key words: Ranking Criterion; Option portfolio; Fuzzy Linear Programming; Delta-Gamma Neutral

1. INTRODUCTION

As is well known, the Black–Scholes formula is the function of five parameters, such as stock price, volatility, interest rate, Strike Price, expiry date, in which strike price and the expiry date is precise,and the interest rate is also considered constant .These three parameters will not pose a risk. So, stock price and volatility are the source of risk. Because their future values are uncertain. That is, option trading risk control is how to minimize from the two aspects risk.

To calculate the delta, partial derivative with respect to S_t should be taken,

$$\Delta = \frac{\partial C}{\partial S_t} = N(d_1)$$

where

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$$d_1 = \frac{\ln\left(\dfrac{S_t}{K}\right) + \left(r + \dfrac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

N denotes the cumulative distribution function of a standard normal variable,and the stock price (S_t), strike price (K), volatility (σ), risk-free interest rate (r) and time to maturity (T-t),the call option price (C).

Delta hedging, due to Black and Scholes assumptions, means to hold one option contract and sell delta quantity of the underlying asset. The main idea is to create the option synthetically. Since the value of the delta quantity of the underlying asset and the option is always equal, the overall value of the transaction is always zero. Basically, to hedge the option – or a portfolio of options – the value of the delta should be zero. This argument is valid for all other Greek letters. A delta-neutral portfolio would capture the small changes in the price of the underlying. But the bigger changes cause deviations from delta-neutrality. Gamma-neutrality is used to avoid this problem. The gamma of the option is the second order partial derivative of the value of the option with respect to the underlying asset.

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{N'(d_1)}{S_t\sigma\sqrt{T-t}}$$

It is clear that gamma measures the rate of change of an option's delta just because it is calculated by taking the first order partial derivative of delta. The gamma always takes positive values.

Rendleman [1] developed a simple LP model to determine the optimal mix of securities to hold long and short, in a portfolio consisting of options and stocks as well. The LP model presented in this paper is very similar to that and is applied to Ericssons call and put options real data, as traded on February 13th 2001, at the Stockholm Stock Exchange. Papahristodoulou [2] developed a linear programming model to determine option strategies to achieve delta, gamma, theta, rho, vega neutrality. Mehmet Horasanlı[3] extended the model of Papahristodoulou to a multi-asset setting to deal with a portfolio of options and underlying assets. Their models are derived by BS formula, and the input parameters of the Black–Scholes formula are usually regarded as the precise real-valued data; that is, the input data are considered as the real numbers. However, in the real world, some parameters in the Black–Scholes formula cannot always be expected in a precise sense. For instance, the volatility, stock price, and also the Greek parameters delta, gamma. Therefore, the fuzzy sets theory proposed by this paper may be a useful tool for modeling this kind of imprecise problem.

This paper is organized as follows. In Section 2, the notions of fuzzy number and the arithmetics of fuzzy numbers are introduced. In Section 3, the notion of ranking criterion of fuzzy numbers is introduced. In Section 4, the fuzzy patterns of option portfolio are proposed. In Section 5, a numerical example is given to illustrate the validity of the method. Finally, the conclusions of this paper is depicted.

2. FUZZY NUMBER AND THE ARITHMETICS OF FUZZY NUMBERS

2.1 Fuzzy Set Theory[4]

Now we remind some facts about fuzzy sets and numbers:

Let X be a universal set and $\tilde{A}$ be a fuzzy subset of X, we denote by $\mu_{\tilde{A}}$ its membership function $\mu_{\tilde{A}} : X \to [0,1]$,and by $\tilde{A}_\alpha = \left\{X : \mu_{\tilde{A}} \geq \alpha\right\}$ the α − level set of ,where is the closure of the

set. Let $\tilde{a}$ be a fuzzy number. Then, under our assumptions, the $\alpha-$level set $\tilde{a}_\alpha$ is a closed interval,

$$\tilde{a}_\alpha = \left[\tilde{a}_\alpha^{\ L}, \tilde{a}_\alpha^{\ U} \right]$$

which can be denoted as .

The four arithmetic operations on closed intervals are defined as follows:

$[a,b]+[d,e]=[a+d,b+e]$

$[a,b]-[d,e]=[a-e,b-d]$

$[a,b]\cdot[d,e]=[\min\{ad,ae,bd,be\},\min\{ad,ae,bd,be\}]$ and provided that $0 \notin [d,e]$

if $a,b,d,e>0$, then $[a,b]\cdot[d,e]=[ad,be]$

$[a,b]/[d,e]=[a,b]\cdot[\dfrac{1}{d}-\dfrac{1}{e}]$ and provided that $0 \notin [d,e]$

2.2 Triangular Fuzzy Number

The membership function of a triangular fuzzy number $\tilde{a}$,which is denoted as $\tilde{a} = \left(a_L; a_C; a_R \right)$ is defined by:

$$\mu_{\tilde{a}}(x) = \begin{cases} (x-a_L)/(a_c-a_L), & \text{if } a_L \leq x \leq a_C \\ (a_R-x)/(a_R-a_C), & \text{if } a_C < x \leq a_R \\ 0, & \text{otherwise} \end{cases}$$

The α -level set of $\tilde{a}$ is ,

$$\tilde{a}_\alpha = \left[(1-\alpha)a_L + \alpha a_C, (1-\alpha)a_R + \alpha a_C \right]$$

$$\tilde{a}_\alpha^{\ L} = (1-\alpha)a_L + \alpha a_C, \tilde{a}_\alpha^{\ U} = (1-\alpha)a_R + \alpha a_C$$

3. THE NOTION OF RANKING CRITERION OF FUZZY NUMBERS

Dubois and Prade[5] proposed four different methods to compare the size of fuzzy number from the angle of probability and the "fuzzymax" "fuzzymin" arithmetic operators,which expanded the real arithmetic operators to fuzzy set field.[6] That is,

$$u_{\max\{\tilde{A},\tilde{B}\}}(r) = \sup_{r=\max\{s,t\}} \min\left(u_{\tilde{A}}(s), u_{\tilde{B}}(s) \right) \tag{1}$$

where $u_{\max\{\tilde{A},\tilde{B}\}}(r)$ denote the membership of $\max\{\tilde{A},\tilde{B}\}$.

We can get the simpler form applying (1) to the triangular fuzzy numbers, that is, if $\tilde{A}=(a_L,a_C,a_R)$, $\tilde{B}=(b_L,b_C,b_R)$ and $a_L\le b_L, a_C\le b_C, a_R\le b_R$,then $\tilde{A}<\tilde{B}$; $\tilde{A}=\tilde{B}$ if only if $a_L=b_L, a_C=b_C, a_R=b_R$ .Therefore, we can use the definition to define optimal function optimal solution.

4. THE FUZZY PATTERNS OF OPTION PORTFOLIO

It can be seen from Table 1 that the theoretical and market prices of options contracts differ. But due to the no arbitrage restrictions, price of every financial instrument converges to its theoretical value in the long run. For European type options the equivalence of the prices occurs at the expiry date.

For investors, the difference between the theoretical and the market prices can cause possible gains or losses. Therefore, the objective function consists of the multiplication of the difference between the theoretical and market price of options within various strike prices and the amount of each option in the portfolio. When one buys an option (no matter if it is call or put) to the market price, he will earn the difference between the theoretical and market price, when this gap disappears. Similarly, to issue call or put options would lead to lower premia received than the theoretical ones. As a consequence, the objective function is the sum of differences between the theoretical and market prices for all options. The objective function is given as follows:

$$Z_{\max}=\sum_{i=1}^{2}\sum_{j=1}^{N}\left(\tilde{p}_{theor,i,j}-p_{market,i,j}\right)x_{i,j} \tag{2}$$

The first indice i stand for the type of the transaction where 1 denotes buying and 2 denotes selling the corresponding option. The second indice j denotes the options written on different underlying assets taken in the portfolio. $p_{market,i,j}$ is the accurate market value, and $\tilde{p}_{theor}$ denotes the fuzzy theory option price, which is a triangular fuzzy numbers. $x_{i,j}$ is the proportion invested in various options when the optimal strategy is found. It is clear that it takes the positive value when the option is bought and negative ones when the same option is sold.

To hedge the portfolio against the possible risks caused from the changes in the underlying asset price, delta-neutrality – a position with a delta of zero – should be achieved. Delta-neutrality constraint is formulated as follows:

$$\sum_{i=1}^{2}\sum_{j=1}^{N}\tilde{d}_{j}x_{j}+Q-S=0 \tag{3}$$

The letter Q means the number of shares purchased and S means the number of shares sold. Both of them are to be sure. $\tilde{d}_{i,j}$ is the deltas of individual options in the portfolio, which is a triangular fuzzy numbers. The same arguments hold for gamma-neutrality：

$$\sum_{i=1}^{2}\sum_{j=1}^{N}\tilde{g}_{j}x_{j}=0 \tag{4}$$

where $\tilde{g}_{i,j}$ is the gammas of individual options in the portfolio, which is also a triangular fuzzy numbers.

For $x_{i,j}$,we have the constraint:

$$\sum_{i=1}^{2}\sum_{j=1}^{N}\left|x_{i,j}\right|=1 \tag{5}$$

It is easy to see that these constraints do not ensure a unique solution, a alternative method is to include a scale constraint to decide the relative size in the optimal solution. For instance:

$$\sum_{i=1}^{2}\sum_{j=1}^{N}\tilde{d}_{i,j}\,x_{i,j}+Q\leq1000 \tag{6}$$

5. NUMERICAL EXAMPLE

To illustrate the result of our fuzzy method for valuing option portfolio, we developed a program based on the data as table 1. $\tilde{p}_{theor,i,j}-p_{market,i,j}$, $\tilde{d}_j$ and $\tilde{g}_j$ are fuzzy numbers expressed as triangular fuzzy number noted in Table 2.

Table 1: The Greeks And The Theoretical Price of Ericssion's Options

Strike Price	Option Type	P(Market)	P(Theor)	Delta	Gamma
95	Call	10.25	11.8	0.5815	0.01407
110	Call	7.5	9.75	0.4448	0.01095
95	Put	8	10.1	-0.4185	0.01407
110	Put	20	22.3	-0.7264	0.01199

Table 2: The Triangular Fuzzy Number

$\tilde{p}_{theor,i,j}-p_{market,i,j}$			Fuzzy Delta Values			Fuzzy Gamma Values		
0.55	1.55	2.55	0.4815	0.5818	0.6815	0.00907	0.01407	0.01907
1.25	2.25	3.25	0.3448	0.4448	0.5448	0.00595	0.01095	0.01595
1.1	2.1	3.1	-0.5185	-0.4185	-0.3185	0.00907	0.01407	0.01907
1.3	2.3	3.3	-0.6552	-0.5552	-0.4552	0.00595	0.01095	0.01595

Based on Eqs. (2)–(6),we get several equivalent constraints in matrix form, by Matlab and Lingo, we get the results:

$x_{11}=0.9389174, x_{12}=-0.9681432, x_{13}=0.9389174, x_{14}=-0.8156736$

$x_{21}=-0.5702239, x_{22}=0.9389174, x_{23}=-0.4016273, x_{24}=0.9389174$

$S=0.4995688, Q=0$

The negative value of $x_{i,j}$ means that the option is sold. In our example, the optimum strategy is to buy 0.9389174 share call option with strike price 95, sell 0.9681432 share call option with strike price 110, buy 0.9389174 share put option with strike price 110, sell 0.8156736 share put option with strike price 110,and to sell 0.5702239 share call option with strike price 95, buy 0.9389174 share call option with strike price 110, sell 0.4016273 share put option with strike price 110,buy 0.9389174 share put option with strike price 110, and to buy 0.4995688 shares. This position yields to a profit with interval [2.3606, 3.986].

6. CONCLUSIONS

In this paper, the fuzzy pattern of option portfolio is proposed, which is more accord with actual situations. Considering the theory option price, delta and gamma as triangular fuzzy numbers owing to the fluctuation of financial market such as the volatility, we propose a fuzzy programming model of option portfolio based a ranking criterion of fuzzy numbers, which the fuzzy option portfolio model is converted into a classical linear programming problem. In the numerical example, we illustrate the validity of the method and get the optimum strategy for the option portfolio.

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