

Optimal Investment Strategy for a Defined Contributory Pension Plan in Nigeria Using Dynamic Optimization Technique

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Abstract: We consider an optimal investment strategy for a defined contributory pension plan in Nigeria using dynamic optimization technique. The Pension Plan Members (PPMs) make contributions continuously into the pension funds. The Pension Fund Administrator (PFA) propose to invest the contributions made by the PPMs into Federal Government of Nigeria (FGN) bond such as construction of roads in Nigeria. They propose that every road constructed must has a tollgate in order to collect toll and make more wealth for the PPMs at time $t \leq T$. We assume that there are Alternative Roads (AR) the drivers may take to their destination without paying toll. The AR may not be good enough for the vehicles to pass smoothly. We assume that Pension Plan Company (PPC) will make more wealth for the PPMs if the Company Roads (CR) are highly motorable. The PPC estimates some percentage of the Gross Returns (GR) at time t to be set aside as the Costs of Roads Construction (CRC). They also estimates some percentage of the Gross-Net Returns (GNR) (i.e. the returns after CRC has been deducted) as Maintenance Costs (MC). They further estimates some percentage of the Gross-Net-Net Returns (GNNR) (i.e. the returns after CRC and MC have been deducted) as Administrative Costs (AC) at time t . Our aim is to find the optimal value of wealth that will accrue to the PPMs over a period of time. We found that the optimal Net Returns (NR) accrued to the PPMs is $N6.6434 \times 10^{14}$ (N denotes Naira).

Key words: Optimal Investment; Defined Contributory; Pension Plan; Dynamic Optimization; Net Returns; Pension Plan Members; Pension Reform Act; Gross-Net-Net Returns; Pension Plan Company

1. INTRODUCTION

The Defined Contributory (DC) pension scheme was established by the Nigeria Pension Reform Act, 2004 which came into effect in June 25, 2004. The Nigerian Pension Reform Act, 2004 (known as the “Act”)

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establishes a DC pension scheme for payment of retirement benefits of employees of the public service of the Federation, the Federal Capital Territory and the private sector (see Section 1(1) of the Act). Before the Act, pension funds have been poorly managed. Consequently, pension funds which were managed by the employers generated a lot of problems. Retired workers faced the problem of non-payment of pension benefits. Many retired workers have suffered tremendously and died trying to collect their retirement benefits. The Nigeria Social Insurance Trust Fund (NSITF) to which employees make provident contributions for their retirement also failed to pay the retirees benefits as and when due (see Ahmad^{[1][2]}).

The DC pension scheme is contributory, fully funded, depends on individual accounts and level of risk that are privately managed by Pension Fund Administrators (PFAs) with the pension funds assets held by Pension Fund Custodians (PFCs) under direct regulation process. The Act provides that both the employee and employer should make equal contribution into the DC pension scheme. In Section 9(1) of the Act, provides that the employees should contribute a minimum of 7.5% of their Basic salary, Housing and Transport allowances and the employers should contribute 7.5% as well of the employees salary, Housing and Transport allowances in case of both the public and the private sectors. In the case of the military, 12.5%. An employer may elect to contribute on behalf of the employees such that the total contribution should not be less than 15% of the Basic salary, Housing and Transport allowances of the employees (see Section 9(2) of the Act).

The decision for allocation of asset is paramount in the PFA's investment management strategy. Within the investment guidelines provided by the Act, are three major asset classes. These are fixed income securities, equities and real estate securities. These classes of asset perform independently and have different risks and react differently to market conditions. The asset allocation decision is the most sensitive that PFAs must be very careful in making the decision. Nigerian Pension Commission (PenCom) provides asset allocation guidelines to the PFAs. It provides that 100% of the Retirement Savings Account (RSA) of the PPM may be invested in Federal Government securities, 20% into State Government (SG) securities, 30% into Corporate bonds/debt, 35% into money market instruments, 25% into ordinary shares and 5% into Open End and Closed End fund. Index bonds (or FGN/SG bonds) are securities issued by the Federal or State government to raise long-term funds from the Capital market for developmental projects. It is a long-term debt instrument, usually with a maturity of three years and above. The 1st Federal Government of Nigeria bond is a FGN certificate of indebtedness: it is backed by the "full faith and credit" of the FGN, and is regarded as default "risk-free" investment. It is issued by the Central Bank of Nigeria (CBN) on the authority of the Debt Management Office (DMO) and on behalf of the FGN. The main reason FGN issues bond is to finance its capital expenditures as well as develop the Nigerian Capital Market.

We consider an optimal investment strategy for a defined contributory pension plan in Nigeria. PPMs make contributions continuously into pension funds. The PFA propose to invest contributions into FGN bond such as roads construction. This, to a great extent will make more wealth for the PPMs and improve on the standard of living of the people by providing employment opportunity and good roads. The aim of the PPC is to build tollgates on the roads constructed in order to collect toll from the drivers that ply the roads. We assume that there are other roads for the drivers to ply without paying toll. But, these roads may not be good enough for the vehicles. We assume that the PPC will make more wealth for the PPMs when the roads are highly motorable. We also assume without loss of generality that the number of vehicles that pass through the various tollgates is random. The PPC estimated that some percentage of the gross returns should be set aside as costs of constructing the roads, some percentage of the GNR as maintenance costs and some percentage of the GNNR as administrative costs.

Cairns et al^[8] developed a pension plan accumulation programme designed to deliver a pension in retirement that is closely related to salary that the plan member received prior to retirement. Cairns et al^[7] considered the finding of the optimal dynamic asset allocation strategy for a DC pension plan, taking into account the stochastic features of the plan member's lifetime salary progression as well as the stochastic properties of the assets held in his accumulating pension fund. They emphasised that salary risk (the fluctuation in the plan member's earning in response to economic shocks) is not fully hedgeable using existing financial assets. They further emphasised that wage-indexed bonds could be used to hedge productivity and inflation shocks, but such bonds are not widely traded. They called the optimal dynamic asset allocation strategy stochastic lifestyling. They compare it against various static and deterministic

lifestyle strategies in order to calculate the costs of adopting suboptimal strategies. Their solution technique makes use of the present value of future contribution premiums into the plan. This technique can be found in Deelstra et al^[9], Korn and Krekel^[11] and Blake et al^[5]. Deterministic lifestyling which is the gradual switch from equities to bonds according to present rules is a popular asset allocation strategy during the accumulation phase of DC pension plans and is designed to protect the pension fund from a catastrophic fall in the stock market just prior to retirement (see Cairns et al^[7], Blake et al^[5]). Cairns et al^[7], Haberman and Vigna^[10] and Cairns et al^{[8][7]} analysed extensively the occupational DC pension funds, where the contribution rate is a fixed percentage of salary.

Cairns et al^[8] introduced non-hedgeable salary risk into the optimal allocation problem in accumulation phase of a DC pension plan. Cairns^[6] developed a continuous-time stochastic pension fund model with Markov control strategies over the contribution rate and asset-allocation. He found that the optimal proportions invested in each risky assets are constant relative to one another. Battocchio and Menoncin^[3] used a stochastic dynamic programming approach to model a DC pension fund in a complete financial market with stochastic investment opportunities and two background risks: salary risk and inflation risk. They gave a closed form solution to the asset allocation problem and analyze the behaviour of the optimal portfolio with respect to salary and inflation.

In this paper, we consider the problem as a dynamic program. Our approach builds on a previous research on dynamic programming technique. Mulvey and Vladimirov^[12] used the stochastic programming technique of dynamic programming in financial asset allocation problems for designing low-risk portfolios. Van Roy et al^[18] proposed the idea of using a parsimonious sufficient static in an application of approximate dynamic programming to inventory management. Powell^[16] adopted dynamic programming technique for large-scale asset management problems for both single and multiple assets. Topaloglu and Kunnumkal^[17] extended an approximate dynamic programming technique to optimize the distribution operations of a company manufacturing certain products at multiple production plants and shipping to different customer locations for sales. Nwozo and Nkeki^[15] adopted dynamic programming principle to considered the allocation of buses from single station to different routes in Nigeria for profit maximization. Nkeki^[13] considered the used of dynamic optimization technique for the allocation of buses from different stations to different routes by a transportation company in Nigeria. Nkeki and Nwozo^[14] considered the use of value iteration to minimize the costs of shipping different goods from the factories to the markets. In this paper, we consider the used of dynamic optimization technique to optimal investment strategy in a defined contributory pension plan in Nigeria.

The structure of the remainder of the paper is as follows. Section 2 presents the definition of the notations, problem formulation, the objective function of our problem and some useful results. In section 3, we presents the dynamic programming formulation of our problem. We develop the transformation equation which is a random variable and the optimality equation also in section 3. In section 4, we presents the computational work. In section 5, we give the discussion of the results obtained from our computational work. Finally, section 6 concludes the paper.

2. PROBLEM FORMULATION

In this section, we define the notations and construct the dynamics of our problem over a finite planning horizon. We assume that the wealth to be generated by the PPC depends on the number of vehicles the pass through the tollgates at time t . Again, the number of vehicles that pass through the tollgates depends on the nature of the roads. Our aim is to maximize the total expected returns over an infinite time horizon. We define the following notations:

γ - the discount factor, $0 < \gamma < 1$.

$S = \{s | s \text{ is the set of all vehicles in Nigeria} \}$.

$\lambda = \{s | s \text{ is the set of all additional vehicles that will decide take through the toll gates in Nigeria}\}$

$\tilde{\lambda}_t = \{s | s \text{ is the set of all additional vehicles that will decide take through the toll gates in Nigeria at time } t\}$

T - the set of time periods in the planning horizon.

$\pi : S \rightarrow X$ - is a rule which chooses an action $x \in X$ based on current state of the vehicles.

d - nature of the roads.

$x_{ij}^\pi(t) \in X$ - the expected number of vehicles to ply from State i through tollgate i to State j at time t under policy π .

$S_t = \{s | s \text{ is the number of vehicles that take through the toll gates at time } t\}$.

$\tilde{S}_t = \{s | s \text{ is the set of vehicles that take through the toll gates at time } t\}$

θ_i - the sum paid by a driver at tollgate i .

ω - the number of days.

$\Lambda_{it}^{\theta, \omega}(x_{ij}^\pi)$ - expected returns from the tollgates in State i at time t under policy π .

m - number of States in which the tollgate(s) is/are build.

n - number of tollgates.

α - the percentage of the GR that is set aside as costs of constructing the roads.

η - the percentage of the GNR that is set aside as MC.

β - the percentage of the GNNR that is set aside as AC.

S_{i0} - expected number of vehicles that will pass through the tollgate i at the beginning of the planning horizon.

$x_i^\pi(S_{i-1}(d))$ - the actual number of vehicles that ply road i through tollgate i at time $t-1$.

$G_t(S_t)$ - objective function of our problem.

2.1 One-Period Expected Return Function

If the returns from tollgate i in State i at time t is Λ_{it} , number of vehicles that pass through tollgate i to State j at time t under policy π is x_{ij}^π and the state of the vehicles through tollgate i at time t is S_{it} , θ_i is the sum paid by a driver at tollgate i and ω is the number of days, then the returns obtained by the PPC over T - horizon is

$$\sum_{t=1}^T \sum_{i=1}^n \sum_{j=1}^m \omega \theta_i \Lambda_{it}(x_{ij}^\pi(S_t(d))) = \sum_{t=1}^T \sum_{i=1}^n \sum_{j=1}^m \Lambda_{it}^{\omega, \theta}(x_{ij}^\pi(S_t(d))).$$

The expected maximum returns obtained under policy π , is

$$G_t^\pi(S_{t-1}) = E_t \left[\max_{x_t \in X(S_t)} \sum_{t=1}^T \sum_{i=1}^n \sum_{j=1}^m \gamma^t \Lambda_t^{\omega, \theta} (x_{t,ij}^\pi(S_{t-1}(d))) \right] \quad (1)$$

Subject to:

$$\sum_{i=1}^n \sum_{j=1}^m x_{ij}^\pi(S_{t-1}(d)) \leq S_t, \quad t = 1, \dots, T;$$

$$\sum_{j=1}^m x_{ij}^\pi(S_{i-1}(d)) \leq S_{it}, \quad t = 1, \dots, T, i = 1, \dots, n;$$

$$x_{ij}^\pi \geq 0, \quad t = 1, \dots, T; i = 1, \dots, n; j = 1, \dots, m,$$

where $X(S_t)$ is the set of all possible solution of Eq.(1). Conditioning Eq.(1) on $S_t \in S$, we obtain the following optimization problem

$$G_t^\pi(S_{t-1}) = E \left[\max_{x_t \in X(S_t)} \sum_{t'=t}^T \sum_{i=1}^n \sum_{j=1}^m \gamma^{t'} \Lambda_{t'}^{\omega, \theta} (x_{t',ij}^\pi(S_{t-1}(d))) \middle| S_t \in S \right] \quad (2)$$

Subject to:

$$\sum_{i=1}^n \sum_{j=1}^m x_{ij}^\pi(S_{t-1}(d)) \leq S_t, \quad t = 1, \dots, T$$

$$\sum_{j=1}^m x_{ij}^\pi(S_{i-1}(d)) \leq S_{it}, \quad t = 1, \dots, T, i = 1, \dots, n;$$

$$x_{ij}^\pi \geq 0, \quad t = 1, \dots, T; i = 1, \dots, n; j = 1, \dots, m,$$

For the returns function $\Lambda : S \rightarrow \mathfrak{R}^n$, if we accumulate the returns of the first T – stage and add to it the terminal returns

$$h_T^i(S_T) = \sum_{i=1}^n \sum_{j=1}^m \Lambda_{iT}(S_T),$$

then Eq.(2) becomes

$$G_t^\pi(S_{t-1}) = E \left[\max_{x_t \in X(S_t)} \sum_{t'=t}^T \sum_{i=1}^n \sum_{j=1}^m \gamma^{t'} \Lambda_{t'}^{\omega, \theta} (x_{t',ij}^\pi(S_{t-1}(d))) + \gamma^T h_T^i(S_T) \middle| S_t \in S \right]$$

Subject to:

$$\sum_{i=1}^n \sum_{j=1}^m x_{ij}^\pi(S_{t-1}(d)) \leq S_t, \quad t = 1, \dots, T$$

$$\sum_{j=1}^m x_{ij}^\pi(S_{i-1}(d)) \leq S_{it}, \quad t = 1, \dots, T, i = 1, \dots, n;$$

$$x_{ij}^\pi \geq 0, \quad t = 1, \dots, T; i = 1, \dots, n; j = 1, \dots, m.$$

Proposition 1: Let $V^{C^M A^-}(S_t)$ be the net returns accrued to PPMs at time t , then

$$V^{C^M A^-}(S_t) = (1 - \alpha)(1 - \eta)(1 - \beta)G_t^\pi(S_{t-1}).$$

Proof: Let α be the percentage of GR that is set aside as costs of constructing the roads, $CRC(t)$ at time t . Let η be the percentage of the GNR set aside as $MC(t)$ and β the percentage of the GNNR set aside as $AC(t)$. Then ,

$$\begin{aligned} CRC(t) &= \alpha E \left[\max_{x_t \in X(S_t)} \sum_{t'=t}^T \sum_{i=1}^n \sum_{j=1}^m \gamma^{t'} \Lambda_{t'}^{\omega, \theta} (x_{t'ij}^\pi(S_{t'-1}(d))) + \gamma^T h_T^i(S_T) \mid S_t \in S \right] \\ &= \alpha G_t^\pi(S_{t-1}). \end{aligned} \quad (3)$$

Let $V^{C^-}(S_t)$ be the total wealth after the costs of constructing the roads has been deducted at time t , then

$$\begin{aligned} V^{C^-}(S_t) &= G_t^\pi(S_{t-1}) - \alpha G_t^\pi(S_{t-1}) \\ &= (1 - \alpha)G_t^\pi(S_{t-1}). \end{aligned} \quad (4)$$

Now, the costs of maintenance is obtain as follows

$$MC(t) = \eta V^{C^-}(S_t) = \eta(1 - \alpha)G_t^\pi(S_{t-1}).$$

Let $V^{C^M^-}(S_t)$ be the GNR of the PPC, then

$$\begin{aligned} V^{C^M^-}(S_t) &= G_t^\pi(S_{t-1}) - CRC(t) - MC(t) \\ &= (1 - \alpha)(1 - \eta)G_t^\pi(S_{t-1}). \end{aligned}$$

Hence, the administrative costs is obtain as follows

$$AC(t) = \beta V^{C^M^-}(S_t) = \beta(1 - \alpha)(1 - \eta)G_t^\pi(S_{t-1})$$

Therefore, the net returns accrued to PPMs at time t is

$$\begin{aligned} V^{C^M A^-}(S_t) &= G_t^\pi(S_{t-1}) - CRC(t) - MC(t) - AC(t) \\ &= (1 - \beta)(1 - \alpha)(1 - \eta)G_t^\pi(S_{t-1}). \end{aligned} \quad (5)$$

This is the total profit gained from the decision made by the PFA of the company at time t . Next, we determine the optimal value of $G_t^\pi(S_{t-1})$ using dynamic programming technique.

3. DYNAMIC PROGRAMMING FORMULATION

Given S_t the state variable at time t , S the state space and x_t the decision variable at time t , we formulate the problem as a dynamic program. Let $0 \leq P_{ij} \leq 1$, for $i = 1, \dots, n; j = 1, \dots, m$ be the

probability that some drivers will not take the CR from State i to State j . Hence, the number of vehicles that are expected not to pass through the tollgates at time t is given by $P_{ij}x_{ij}, i = 1, \dots, n; j = 1, \dots, m; t \in T$. Therefore, the total number of vehicles that will not pass through

the tollgates from all the States at time t is given by $\sum_{i=1}^n \sum_{j=1}^m P_{ij}x_{ij}, t \in T$.

We assume that the PFA will try as much as possible to encourage those that will take AR to pass through the tollgates and pay toll by ensuring that the CR are attractive and motorable. The company further estimates some proportion of the drivers that belong to the class that ought not to take CR but decides to take the roads. Let δ be the fraction that decides to take CR, then

$$S_t = S_{t-1} + \delta \sum_{i=1}^n \sum_{j=1}^m P_{ij}x_{ij}, t \in T. \quad (6)$$

Eq.(6) is our transformation equation. The optimal value of our returns function can be found by computing the value functions through the optimization problem

$$F_T^\pi(S_{t-1}) = \max_{x_t \in X(S_t)} \left\{ (1-\alpha)(1-\eta)(1-\beta) \left[\sum_{t'=t}^{T-1} \sum_{i=1}^n \sum_{j=1}^m \Lambda_{t'-1}^{\omega, \theta} (x_{ij}^\pi(S_{t-1}(d))) + \gamma E(F_{T-1}(S_t(d))) \right] S_t \in S \right\} \quad (7)$$

Subject to:

$$\sum_{i=1}^n \sum_{j=1}^m x_{ij}^\pi(S_{t-1}(d)) \leq S_t, t = 1, \dots, T. \quad (8)$$

$$\sum_{j=1}^m x_{ij}^\pi(S_{t-1}(d)) \leq S_{it}, t = 1, \dots, T, i = 1, \dots, n. \quad (9)$$

Equivalently,

$$\lambda_t + \sum_{i=1}^n \sum_{j=1}^m x_{ij}^\pi(S_{t-1}(d)) = S_t, t = 1, \dots, T, \lambda_t \geq 0, \lambda_t \in \tilde{\lambda}_t \subseteq \lambda, \quad (10)$$

$$\lambda_{it} + \sum_{j=1}^m x_{ij}^\pi(S_{t-1}(d)) = S_{it}, t = 1, \dots, T, \lambda_{it} \geq 0, \lambda_{it} \in \tilde{\lambda}_{it}, \lambda_{it} \in S, S_{it} \in \tilde{S}_t. \quad (11)$$

$$x_{ij}^\pi \geq 0, t = 1, \dots, T; i = 1, \dots, n; j = 1, \dots, m.$$

Hence, Eq.(10) can be expressed as

$$\sum_{i=1}^n \sum_{j=1}^m x_{ij}^\pi(S_{t-1}(d)) = S_t - \lambda_t, t = 1, \dots, T, \lambda_t \geq 0, \lambda_t \in \lambda, S_t \in \tilde{S}_t, \lambda_t \in \tilde{\lambda}_t,$$

where

$$\lambda_t = \{s | s \text{ is the number of vehicles that decide to take through the toll gates at time } t\}$$

This implies that

$$\lambda_t = \delta \sum_{i=1}^n \sum_{j=1}^m P_{ij} x_{ij}^\pi, t \in T.$$

But, $S'_t \in S, S'_t \notin \tilde{S}_t, \lambda_t \in S$, where

$$S'_t = \{s | s \text{ is the set of vehicles that will not take through the toll gates at time } t\}$$

Then, for $\tilde{S}_{t-1} \cup \tilde{\lambda}_t = \{s | s \in \tilde{S}_{t-1} \vee s | s \in \tilde{\lambda}_t\} \subseteq \tilde{S}_t$, $\tilde{S}_{t-1} \cup \tilde{\lambda}_t \cup S'_t \subseteq S$ and $\tilde{S}_{t-1} \cup \tilde{\lambda}_t \cap S'_t = \phi$, we have that

$\tilde{S}_{t-1} \cup \tilde{\lambda}_t \cap S = \tilde{S}_{t-1} \cup \tilde{\lambda}_t$. If $\lambda_t = 0$, it implies that there is no additional vehicles that pass through the tollgates at time t . If $\lambda_t < 0$, it implies that the number of vehicles that are expected to pass through the tollgates decreases and this will be too dangerous for the company. Therefore, the aim of the company is to ensure that $\lambda_t \geq 0$ at all time t . Hence, $\delta \geq 0$.

Theorem 1: The optimization problem $V^{C^M-A^-}(S_t)$ is equivalent to the optimality equation $F_T^\pi(S_{t-1})$.

Proof: (see Nkeki [3], Powell [6]).

From now on, we will be using Eq.(5) and Eq.(7) interchangeably.

Lemma 1: Let $B(S)$ be a Banach space and $\Gamma : B(S) \rightarrow B(S)$ be a contraction mapping, then

(i) there exists a unique solution $F^* \in B(S)$ such that $\Gamma F^* = F^*$.

(ii) Given $F^0 \in B(S)$, the sequence of iteration $\{\Gamma F^0\}$ defined by $F^{n+1} = \Gamma^{n+1} F^0$ is convergent to the fixed point of Γ , F^* say.

Proof: Given $F^* \in B(S)$ consider the sequence $F^1 = \Gamma F^0, F^2 = \Gamma^2 F^0, \dots, F^n = \Gamma^n F^0, \dots$

For $n > m$,

$$\begin{aligned} \|F^{n+1} - F^{m+1}\| &\leq \gamma \|F^n - F^m\| \\ &\leq \gamma^{m+1} \|F^{n-m} - F^0\| \end{aligned}$$

But,

$$\|F^{n-m} - F^0\| \leq \frac{\gamma^{n-m-1}(1-\gamma^m)}{1-\gamma} \|F^1 - F^0\|$$

Therefore,

$$\|F^{n+1} - F^{m+1}\| \leq \frac{\gamma^n(1-\gamma^m)}{1-\gamma} \|F^1 - F^0\|$$

Now, as $m \rightarrow \infty, n \rightarrow \infty$ and $\gamma < 1$, we have that $\frac{\gamma^n(1-\gamma^m)}{1-\gamma} \rightarrow 0$.

Therefore, $\|F^{n+1} - F^{m+1}\| \rightarrow 0$, as $m \rightarrow \infty$.

Hence, the sequence of iteration $\{F^n\}$ is a Cauchy sequence, since $B(S)$ is complete, it must be that F^n has a limit point, F^* in $B(S)$ say. We now conclude that

$$\lim_{n \rightarrow \infty} F^n \rightarrow F^*.$$

We now show that F^* is a fixed point of the mapping Γ . To show this, we observe that

$$\begin{aligned} 0 \leq \|\Gamma F^* - F^*\| &\leq \|\Gamma F^* - \Gamma F^n\| + \|\Gamma F^n - F^*\| \\ &\leq \gamma \|F^* - F^{n-1}\| + \gamma \|F^n - F^*\| \end{aligned}$$

Since $\{F^n\}$ converges to F^* , we deduce that

$$\|\Gamma F^* - F^*\| = 0,$$

which implies that $\Gamma F^* = F^*$.

Next, we show that F^* is a unique fixed point of Γ . Suppose that there exists another solution \tilde{F}^* such that

$$\Gamma \tilde{F}^* = \tilde{F}^*.$$

Then,

$$\|F^* - \tilde{F}^*\| \leq \gamma \|F^* - \tilde{F}^*\|$$

Since $0 < \gamma < 1$, then

$$\|F^* - \tilde{F}^*\|(1-\gamma) \leq 0.$$

Therefore, $F^* = \tilde{F}^*$.

Our aim is to find

$$\begin{aligned} V^{C^M A^-}(S_t) &= F^*(S_t) = \max_{\pi} F^{\pi}(S_t) \\ &= (1-\alpha)(1-\eta)(1-\beta) \max_{\pi} G_t^{\pi}(S_t). \end{aligned}$$

This is obtained by solving the optimality equation

$$F_T(S_{t-1}) = \max_{x^{\pi} \in X(S_t)} \left[(1-\alpha)(1-\eta)(1-\beta) \left\{ \sum_{i=1}^n \sum_{j=1}^m \Lambda_{ij}^{\omega, \theta} (x_{ij}^{\pi}(S_{t-1}(d))) + \gamma E(F_{T-1}(S_t)) \right\} \middle| S_t \in S \right]$$

The next result shows that as $T \rightarrow \infty, F^*(S) \rightarrow \Gamma^T F(S)$, for all $S_0 \in S$. Thus, the returns of the PPMs per stage must be bounded i.e.

$$\left| \sum_{i=1}^n \sum_{j=1}^m \Lambda_{ij} \left(x_{ij}^{\pi} (S_{t-1}(d)) \right) \right| \leq K, \text{ where } K \text{ is a positive constant.}$$

Theorem 2: Let $F : S \rightarrow \mathfrak{R}^n$ be a bounded returns function for the PPMs, then for all initial number of vehicles that pass through the tollgates, $S_0 \in S$, we have that

$$\text{as } T \rightarrow \infty, (\Gamma^T G)(S_0) \rightarrow G^*(S_0), \text{ then } F_T(S_0) \rightarrow F^*(S_0), \text{ as } T \rightarrow \infty.$$

Proof: Given that $F^{\pi}(S_0) = (1 - \alpha)(1 - \eta)(1 - \beta)G^{\pi}(S_0)$. Let $Q > 0, S_0 \in S$ and a policy π , we can decompose the returns function of the PPMs

$$G^{\pi}(S_0) = \lim_{T \rightarrow \infty} E \left[\sum_{t=0}^{T-1} \sum_{i=1}^n \sum_{j=1}^m \gamma^t \Lambda_{ij}^{\omega, \theta} \left(x_{ij}^{\pi} (S_{t-1}(d)) \right) \right]$$

into the portion received over the first Q stages and over the remaining stages as follows:

$$G^{\pi}(S_0) = E \left[\sum_{t=0}^{Q-1} \sum_{i=1}^n \sum_{j=1}^m \gamma^t \Lambda_{ij}^{\omega, \theta} \left(x_{ij}^{\pi} (S_{t-1}(d)) \right) \right] + \lim_{T \rightarrow \infty} E \left[\sum_{t=Q}^{T-1} \sum_{i=1}^n \sum_{j=1}^m \gamma^t \Lambda_{ij}^{\omega, \theta} \left(x_{ij}^{\pi} (S_{t-1}(d)) \right) \right]$$

$$\text{But, } \lim_{T \rightarrow \infty} E \left[\sum_{t=Q}^{T-1} \sum_{i=1}^n \sum_{j=1}^m \gamma^t \Lambda_{ij}^{\omega, \theta} \left(x_{ij}^{\pi} (S_{t-1}(d)) \right) \right] \leq \frac{\gamma^Q K}{1 - \gamma}.$$

$$\text{Now, } G^{\pi}(S_0) \leq E \left[\sum_{t=0}^{Q-1} \sum_{i=1}^n \sum_{j=1}^m \gamma^t \Lambda_{ij}^{\omega, \theta} \left(x_{ij}^{\pi} (S_{t-1}(d)) \right) + \frac{\gamma^Q K}{1 - \gamma} \right]$$

It then follows that

$$\begin{aligned} G^{\pi}(S_0) - \frac{\gamma^Q K}{1 - \gamma} - \gamma^Q \sup_{S_0 \in S} |\Lambda(S_0)| &\leq E \left[\sum_{t=0}^{Q-1} \sum_{i=1}^n \sum_{j=1}^m \gamma^t \Lambda_{ij}^{\omega, \theta} \left(x_{ij}^{\pi} (S_{t-1}(d)) \right) + \gamma^Q h(S_Q) \right] \\ &\leq G^{\pi}(S_0) + \frac{\gamma^Q K}{1 - \gamma} + \gamma^Q \sup_{S_0 \in S} |\Lambda(S_0)|. \end{aligned}$$

Taking the supremum over π , we obtain for all S_0 and Q

$$\begin{aligned} G^*(S_0) - \frac{\gamma^Q K}{1 - \gamma} - \gamma^Q \sup_{S_0 \in S} |\Lambda(S_0)| &\leq E \left[\sum_{t=0}^{Q-1} \sum_{i=1}^n \sum_{j=1}^m \gamma^t \Lambda_{ij}^{\omega, \theta} \left(x_{ij}^{\pi} (S_{t-1}(d)) \right) + \gamma^Q h(S_Q) \right] \\ &\leq G^*(S_0) + \frac{\gamma^Q K}{1 - \gamma} + \gamma^Q \sup_{S_0 \in S} |\Lambda(S_0)|. \end{aligned}$$

Taking the limit as $Q \rightarrow \infty$, we obtain

$$G^*(S_0) \leq \lim_{Q \rightarrow \infty} (\Gamma^Q G)(S_0) \leq G^*(S_0).$$

Therefore, $G^*(S_0) = \lim_{Q \rightarrow \infty} (\Gamma^Q G)(S_0)$, for all $S_0 \in S$.

This result shows that our optimization problem converges to a fixed point G^* in an infinite horizon.

Hence, it follows that

$$(\Gamma G^*)(S_0) - \frac{\gamma^{T+1}K}{1-\gamma} \leq (\Gamma^{T+1}G)(S_0) \leq (\Gamma G^*)(S_0) + \frac{\gamma^{T+1}K}{1-\gamma}$$

as $T \rightarrow \infty, (\Gamma^{T+1}G)(S_0) \rightarrow G^*(S_0)$.

Therefore, $G^* = \Gamma G^*$.

Hence, $F^*(S_0) = \max_{x \in X(S_0)} E[(1-\alpha)(1-\eta)(1-\beta)G^*(S_0)]$, for all $S_0 \in S$.

$$= \max_{x \in X(S_0)} E \left[(1-\alpha)(1-\eta)(1-\beta) \left\{ \sum_{i=1}^n \sum_{j=1}^m \Lambda^{\omega, \theta} (x_{ij}(S_0(d))) + \gamma E(F^*(S_0(d))) \right\} \right], \text{ for all } S_0 \in S.$$

4. COMPUTATIONAL WORK

A pension plan company in Nigeria proposed to construct some roads and build tollgates on the roads constructed in order collect toll and make more wealth for the PPMs. We therefore define the following:

- FB – the fraction (or probability) of vehicles that choose to take AR from Edo State to other States;
- FL – the fraction (or probability) of vehicles that choose to take AR from Lagos State to other States;
- FA – the fraction (or probability) of vehicles that choose to take AR from Abuja to other States;
- FP – the fraction (or probability) of vehicles that choose to take AR from Rivers State to other States;
- FE – the fraction (or probability) of vehicles that choose to take AR from Enugu State to other States;
- FC – the fraction (or probability) of vehicles that choose to take AR from Cross Rivers State to other States;
- FI – the fraction (or probability) of vehicles that choose to take AR from Ibadan City (Oyo State) to other States;
- FW – the fraction (or probability) of vehicles that choose to take AR from Delta State to other States;
- FO – the fraction (or probability) of vehicles that choose to take AR from Ondo State to other States;
- FKW – the fraction (or probability) of vehicles that choose to take AR from Kwara State to other States;
- FM – the fraction (or probability) of vehicles that choose to take AR from Borno State to other States;
- FS – the fraction (or probability) of vehicles that choose to take AR from Sokoto State to other States;
- FN – the fraction (or probability) of vehicles that choose to take AR from Niger State to other States;
- FK – the fraction (or probability) of vehicles that choose to take AR from Kano State to other States.
- AB – the expected amount at the first stage of the planning horizon from Edo State tollgates;
- AL – the expected amount at the first stage of the planning horizon from Lagos State tollgates;
- AA – the expected amount at the first stage of the planning horizon from Abuja tollgates;
- AP – the expected amount at the first stage of the planning horizon from Rivers State tollgates;
- AE – the expected amount at the first stage of the planning horizon from Enugu State tollgates;

- AC – the expected amount at the first stage of the planning horizon from Cross Rivers State tollgates;
- AI – the expected amount at the first stage of the planning horizon from Ibadan tollgates;
- AW – the expected amount at the first stage of the planning horizon from Delta State tollgates;
- AO – the expected amount at the first stage of the planning horizon from Ondo State tollgates;
- AKW – the expected amount at the first stage of the planning horizon from Kwara State tollgates;
- AM – the expected amount at the first stage of the planning horizon from Borno State tollgate;
- AS – the expected amount at the first stage of the planning horizon from Sokoto State tollgates;
- AN – the expected amount at the first stage of the planning horizon from Niger State tollgates;
- AK – the expected amount at the first stage of the planning horizon from Kano State tollgates.
- XB – the decision variable representing the number of vehicles that will pass through tollgates in Edo State;
- XB – the decision variable representing the number of vehicles that will pass through tollgates in Lagos State;
- XA – the decision variable representing the number of vehicles that will pass through tollgates in Abuja;
- XP – the decision variable representing the number of vehicles that will pass through tollgates in Rivers State;
- XE – the decision variable representing the number of vehicles that will pass through tollgates in Enugu State;
- XC – the decision variable representing the number of vehicles that will pass through tollgates in Cross Rivers State;
- XI – the decision variable representing the number of vehicles that will pass through tollgates in Ibadan;
- XW – the decision variable representing the number of vehicles that will pass through tollgates in Delta State;
- XO – the decision variable representing the number of vehicles that will pass through tollgates in Ondo State;
- XKW – the decision variable representing the number of vehicles that will pass through tollgates in Kwara State;
- XM – the decision variable representing the number of vehicles that will pass through tollgates in Borno State;
- XS – the decision variable representing the number of vehicles that will pass through tollgates in Sokoto State;
- XN – the decision variable representing the number of vehicles that will pass through tollgates in Niger State;
- XK – the decision variable representing the number of vehicles that will pass through tollgates in Kano State.

Table 1 indicate the positions the tollgates (TG) were situated in the various positions in the States of Nigeria. The first row, second column tells us that tollgate should be situated on the road in Edo State towards Lagos State. In the first row, third column tells us that on the road in Edo State towards Abuja, tollgate should be build and so on. This implies that Edo State will have five tollgates, Lagos State will have three tollgates, Abuja seven tollgates, Rivers State four tollgates, Enugu State five tollgates, Cross Rivers State two tollgates, Ibadan four tollgates, Delta State four tollgates, Ondo State four tollgates, Kwara State five tollgates, Borno State one tollgate, Sokoto State three tollgates, Niger State four tollgates and Kano State four tollgates.

Table 1: The Flow of Vehicles through Tollgates from One State Towards Other States

| | | | | | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| | B | L | A | P | E | C | I | W | O | KW | M | S | N | K |
| B | | TG | TG | TG | TG | | | | TG | | | | | |
| L | TG | | | | | | TG | | | TG | | | | |
| A | TG | | | TG | TG | | | | | TG | | TG | TG | TG |
| P | TG | | TG | | TG | TG | | | | | | | | |
| E | TG | | TG | TG | | TG | | TG | | | | | | |
| C | | | | TG | TG | | | | | | | | | |
| I | TG | TG | | | | | | | TG | TG | | | | |
| W | TG | | TG | TG | TG | | | | | | | | | |
| O | TG | TG | | | | | TG | | | TG | | | | |
| KW | | TG | TG | | | | TG | | TG | | | | TG | |
| M | | | | | | | | | | | | | | TG |
| S | | | TG | | | | | | | | | | TG | TG |
| N | | | TG | | | | | | | TG | | TG | | TG |
| K | | | TG | | | | | | | | TG | TG | TG | |

The matrices below represents the probability of vehicles that may not take CR. The first row, second column represents the probability of vehicles that will not take one of the tollgates in Edo State towards Lagos State, the third column in the first row represents the probability of vehicles that will not take through one of the tollgates in Edo State towards Abuja and so on. The order of arrangement follows from Table 1.

$$\begin{pmatrix}
 0 & \frac{1}{40} & \frac{2}{45} & \frac{1}{30} & \frac{1}{36} & 0 & 0 & 0 & \frac{1}{15} & 0 & 0 & 0 & 0 & 0 \\
 \frac{1}{30} & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & 0 & 0 & \frac{1}{19} & 0 & 0 & 0 & 0 \\
 \frac{1}{33} & 0 & 0 & \frac{2}{35} & \frac{1}{25} & 0 & 0 & 0 & 0 & \frac{2}{45} & 0 & \frac{1}{27} & \frac{1}{29} & \frac{1}{49} \\
 \frac{1}{25} & 0 & \frac{2}{45} & 0 & \frac{1}{20} & \frac{3}{125} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \frac{2}{39} & 0 & \frac{1}{41} & \frac{1}{35} & 0 & \frac{1}{60} & 0 & \frac{3}{65} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & \frac{2}{55} & \frac{3}{73} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \frac{1}{21} & \frac{5}{118} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{50} & \frac{1}{30} & 0 & 0 & 0 & 0 \\
 \frac{3}{95} & 0 & \frac{1}{28} & \frac{2}{27} & \frac{1}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \frac{1}{18} & \frac{2}{55} & 0 & 0 & 0 & 0 & \frac{1}{26} & 0 & 0 & \frac{1}{40} & 0 & 0 & 0 & 0 \\
 0 & \frac{1}{21} & \frac{1}{23} & 0 & 0 & 0 & \frac{1}{31} & 0 & \frac{5}{88} & 0 & 0 & 0 & \frac{1}{30} & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{33} \\
 0 & 0 & \frac{1}{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{27} & \frac{3}{50} \\
 0 & 0 & \frac{1}{35} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & 0 & \frac{2}{51} & 0 & \frac{1}{18} \\
 0 & 0 & \frac{1}{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{29} & \frac{1}{32} & \frac{2}{51} & 0
 \end{pmatrix}$$

The matrices below represents the number of vehicles that are expected to take CR at the beginning of the planning horizon. The first row, second column represents the number of vehicles that will take one of

the tollgates in Edo State towards Lagos State, the third column in the first row represents the number of vehicles that will take through one of the tollgates in Edo State towards Abuja and so on. The order of arrangement also follows from Table 1.

| | | | | | | | | | | | | | |
|-------|-------|-------|-------|-------|------|-------|------|-------|-------|------|------|------|------|
| 0 | 14000 | 15630 | 12804 | 9906 | 0 | 0 | 0 | 4010 | 0 | 0 | 0 | 0 | 0 |
| 17630 | 0 | 0 | 0 | 0 | 0 | 15804 | 0 | 0 | 13822 | 0 | 0 | 0 | 0 |
| 12224 | 0 | 0 | 8300 | 10102 | 0 | 0 | 0 | 0 | 3014 | 0 | 2416 | 2700 | 3634 |
| 9030 | 0 | 14030 | 0 | 3402 | 2020 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10204 | 0 | 14418 | 7012 | 0 | 4220 | 0 | 2416 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 7200 | 4004 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3018 | 8216 | 0 | 0 | 0 | 0 | 0 | 0 | 10400 | 6808 | 0 | 0 | 0 | 0 |
| 10002 | 0 | 15802 | 12100 | 9004 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6032 | 7196 | 0 | 0 | 0 | 0 | 10224 | 0 | 0 | 4738 | 0 | 0 | 0 | 0 |
| 0 | 14678 | 10402 | 0 | 0 | 0 | 6900 | 0 | 4596 | 0 | 0 | 0 | 3002 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3618 |
| 0 | 0 | 5438 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1810 | 2002 |
| 0 | 0 | 6902 | 0 | 0 | 0 | 0 | 0 | 0 | 4304 | 0 | 1498 | 0 | 4444 |
| 0 | 0 | 9634 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1978 | 1464 | 2890 | 0 |

We define the following vectors which is our decision variables.

$$X_t B = (x_t^{BB}, x_t^{BL}, x_t^{BA}, x_t^{BP}, x_t^{BE}, x_t^{BC}, x_t^{BI}, x_t^{BW}, x_t^{BO}, x_t^{BKW}, x_t^{BM}, x_t^{BS}, x_t^{BN}, x_t^{BK})$$

$$X_t L = (x_t^{LB}, x_t^{LL}, x_t^{LA}, x_t^{LP}, x_t^{LE}, x_t^{LC}, x_t^{LI}, x_t^{LW}, x_t^{LO}, x_t^{LKW}, x_t^{LM}, x_t^{LS}, x_t^{LN}, x_t^{LK})$$

$$X_t A = (x_t^{AB}, x_t^{AL}, x_t^{AA}, x_t^{AP}, x_t^{AE}, x_t^{AC}, x_t^{AI}, x_t^{AW}, x_t^{AO}, x_t^{AKW}, x_t^{AM}, x_t^{AS}, x_t^{AN}, x_t^{AK})$$

$$X_t P = (x_t^{PB}, x_t^{PL}, x_t^{PA}, x_t^{PP}, x_t^{PE}, x_t^{PC}, x_t^{PI}, x_t^{PW}, x_t^{PO}, x_t^{PKW}, x_t^{PM}, x_t^{PS}, x_t^{PN}, x_t^{PK})$$

$$X_t E = (x_t^{EB}, x_t^{EL}, x_t^{EA}, x_t^{EP}, x_t^{EE}, x_t^{EC}, x_t^{EI}, x_t^{EW}, x_t^{EO}, x_t^{EKW}, x_t^{EM}, x_t^{ES}, x_t^{EN}, x_t^{EK})$$

$$X_t C = (x_t^{CB}, x_t^{CL}, x_t^{CA}, x_t^{CP}, x_t^{CE}, x_t^{CC}, x_t^{CI}, x_t^{CW}, x_t^{CO}, x_t^{CKW}, x_t^{CM}, x_t^{CS}, x_t^{CN}, x_t^{CK})$$

$$X_t I = (x_t^{IB}, x_t^{IL}, x_t^{IA}, x_t^{IP}, x_t^{IE}, x_t^{IC}, x_t^{II}, x_t^{IW}, x_t^{IO}, x_t^{IKW}, x_t^{IM}, x_t^{IS}, x_t^{IN}, x_t^{IK})$$

$$X_t W = (x_t^{WB}, x_t^{WL}, x_t^{WA}, x_t^{WP}, x_t^{WE}, x_t^{WC}, x_t^{WI}, x_t^{WW}, x_t^{WO}, x_t^{WKW}, x_t^{WM}, x_t^{WS}, x_t^{WN}, x_t^{WK})$$

$$X_t O = (x_t^{OB}, x_t^{OL}, x_t^{OA}, x_t^{OP}, x_t^{OE}, x_t^{OC}, x_t^{OI}, x_t^{OW}, x_t^{OO}, x_t^{OKW}, x_t^{OM}, x_t^{OS}, x_t^{ON}, x_t^{OK})$$

$$X_t KW = (x_t^{KWB}, x_t^{KWL}, x_t^{KWA}, x_t^{KWP}, x_t^{KWE}, x_t^{KWC}, x_t^{KWI}, x_t^{KWW}, x_t^{KWO}, x_t^{KWKW}, x_t^{KWM}, x_t^{KWS}, x_t^{KWN}, x_t^{KWK})$$

$$X_t M = (x_t^{MB}, x_t^{ML}, x_t^{MA}, x_t^{MP}, x_t^{ME}, x_t^{MC}, x_t^{MI}, x_t^{MW}, x_t^{MO}, x_t^{MKW}, x_t^{MM}, x_t^{MS}, x_t^{MN}, x_t^{MK})$$

$$X_t S = (x_t^{SB}, x_t^{SL}, x_t^{SA}, x_t^{SP}, x_t^{SE}, x_t^{SC}, x_t^{SI}, x_t^{SW}, x_t^{SO}, x_t^{SKW}, x_t^{SM}, x_t^{SS}, x_t^{SN}, x_t^{SK})$$

$$X_t N = (x_t^{NB}, x_t^{NL}, x_t^{NA}, x_t^{NP}, x_t^{NE}, x_t^{NC}, x_t^{NI}, x_t^{NW}, x_t^{NO}, x_t^{NKW}, x_t^{NM}, x_t^{NS}, x_t^{NN}, x_t^{NK})$$

$$X_t K = (x_t^{KB}, x_t^{KL}, x_t^{KA}, x_t^{KP}, x_t^{KE}, x_t^{KC}, x_t^{KI}, x_t^{KW}, x_t^{KO}, x_t^{KKW}, x_t^{KM}, x_t^{KS}, x_t^{KN}, x_t^{KK})$$

The company further estimates that 25% percentage of the vehicles that ought not to take the CR will decide otherwise as a result of good maintenance and provision of security on the roads. The aim of the company is to determine the optimal returns that will accrue to the PPMs at time t . Therefore, our optimality equation becomes

$$F_T^j(S_{jt-1}) = \max_{x_t \in X(S_t)} \sum_{i=1}^n \Lambda_{it}^{\theta, \omega} (x_{it-1}^\pi(S_{it-1}(d))) + \gamma [F_{T-1}^j(S_{jt}) S_t], t = T, T-1, \dots, 1; j = 1, 2, \dots, m.$$

$$x_{it-1}^\pi \geq 0, t = T, T-1, \dots, 1; i = 1, 2, \dots, n.$$

Hence,

$$\left. \begin{aligned} F_T^B(S_{t-1}^B) &= \Psi \max_x \left[AB(X_t, B)^{Tr} + \gamma F_{T-1}^B \left(S_{t-1}^B + \frac{1}{4} \{FB(X_t, B)^{Tr}\} \right) \right] \\ F_T^L(S_{t-1}^L) &= \Psi \max_x \left[AB(X_t, L)^{Tr} + \gamma F_{T-1}^L \left(S_{t-1}^L + \frac{1}{4} \{FL(X_t, L)^{Tr}\} \right) \right] \\ F_T^A(S_{t-1}^A) &= \Psi \max_x \left[AA(X_t, A)^{Tr} + \gamma F_{T-1}^A \left(S_{t-1}^A + \frac{1}{4} \{FA(X_t, A)^{Tr}\} \right) \right] \\ F_T^P(S_{t-1}^P) &= \Psi \max_x \left[AP(X_t, P)^{Tr} + \gamma F_{T-1}^P \left(S_{t-1}^P + \frac{1}{4} \{FP(X_t, P)^{Tr}\} \right) \right] \\ F_T^E(S_{t-1}^E) &= \Psi \max_x \left[AE(X_t, E)^{Tr} + \gamma F_{T-1}^E \left(S_{t-1}^E + \frac{1}{4} \{FE(X_t, E)^{Tr}\} \right) \right] \\ F_T^C(S_{t-1}^C) &= \Psi \max_x \left[AB(X_t, C)^{Tr} + \gamma F_{T-1}^C \left(S_{t-1}^C + \frac{1}{4} \{FC(X_t, C)^{Tr}\} \right) \right] \\ F_T^I(S_{t-1}^I) &= \Psi \max_x \left[AI(X_t, I)^{Tr} + \gamma F_{T-1}^I \left(S_{t-1}^I + \frac{1}{4} \{FI(X_t, I)^{Tr}\} \right) \right] \\ F_T^W(S_{t-1}^W) &= \Psi \max_x \left[AB(X_t, W)^{Tr} + \gamma F_{T-1}^W \left(S_{t-1}^W + \frac{1}{4} \{FW(X_t, W)^{Tr}\} \right) \right] \\ F_T^O(S_{t-1}^O) &= \Psi \max_x \left[AO(X_t, O)^{Tr} + \gamma F_{T-1}^O \left(S_{t-1}^O + \frac{1}{4} \{FO(X_t, O)^{Tr}\} \right) \right] \\ F_T^{KW}(S_{t-1}^{KW}) &= \Psi \max_x \left[AKW(X_t, KW)^{Tr} + \gamma F_{T-1}^{KW} \left(S_{t-1}^{KW} + \frac{1}{4} \{FKW(X_t, KW)^{Tr}\} \right) \right] \\ F_T^M(S_{t-1}^M) &= \Psi \max_x \left[AM(X_t, M)^{Tr} + \gamma F_{T-1}^M \left(S_{t-1}^M + \frac{1}{4} \{FM(X_t, M)^{Tr}\} \right) \right] \\ F_T^S(S_{t-1}^S) &= \Psi \max_x \left[AS(X_t, S)^{Tr} + \gamma F_{T-1}^S \left(S_{t-1}^S + \frac{1}{4} \{FS(X_t, S)^{Tr}\} \right) \right] \\ F_T^N(S_{t-1}^N) &= \Psi \max_x \left[AN(X_t, N)^{Tr} + \gamma F_{T-1}^N \left(S_{t-1}^N + \frac{1}{4} \{FN(X_t, N)^{Tr}\} \right) \right] \\ F_T^K(S_{t-1}^K) &= \Psi \max_x \left[AK(X_t, K)^{Tr} + \gamma F_{T-1}^K \left(S_{t-1}^K + \frac{1}{4} \{FK(X_t, K)^{Tr}\} \right) \right] \end{aligned} \right\} (11)$$

$$t = T, T-1, \dots, 1.$$

Subject to:

$$\sum_{i=1}^n \sum_{j=1}^m x_{ij}^{\pi}(S_{t-1}) = S_t - \lambda_t = S_{t-1} = X_t, t = T, T-1, \dots, 1;$$

$$x_{ij}^{\pi} \geq 0, t = T, T-1, \dots, 1; i = 1, 2, \dots, n; j = 1, 2, \dots, m,$$

$$X_t = X_t^B + X_t^L + X_t^A + X_t^P + X_t^E + X_t^C + X_t^I + X_t^W + X_t^O + X_t^{KW} + X_t^M + X_t^S + X_t^N + X_t^K$$

,

where,

$$X_t^B \geq 0, X_t^L \geq 0, X_t^A \geq 0, X_t^P \geq 0, X_t^E \geq 0, X_t^C \geq 0, X_t^I \geq 0, X_t^W \geq 0,$$

$$X_t^O \geq 0, X_t^{KW} \geq 0, X_t^M \geq 0, X_t^S \geq 0, X_t^N \geq 0, X_t^K \geq 0.$$

This is the parametric linear programming problem of 14 variables. Note that Tr denotes transpose.

We now set
 $\alpha = 0.65, \eta = 0.10, \beta = 0.03, \gamma = 0.80, \delta = 0.25, \theta = N100, \Psi = (1 - \alpha)(1 - \eta)(1 - \beta)$

and use MATLAB to solve the problem. The results are represented in Table 2.

Table 2: The Table Containing the Optimal Net Returns, Optimal Costs of Constructing the Roads, Optimal Maintenance Costs and Optimal Administrative Costs

| The Station in Each States Of Nigeria | NR $\times 10^{14}$ (in Naira) | CRC $\times 10^{14}$ (in Naira) | MC $\times 10^{13}$ (in Naira) | AC $\times 10^{12}$ (in Naira) | Average Net Returns from each Tollgates $\times 10^{13}$ (in Naira) |
|---------------------------------------|-----------------------------------|------------------------------------|-----------------------------------|-----------------------------------|--|
| Edo State Station | 1.0644 | 2.2643 | 1.2192 | 3.2919 | 2.1288 |
| Lagos State Station | 1.0046 | 2.1370 | 1.1507 | 3.1069 | 3.3486 |
| Abuja Station | 0.6244 | 1.3284 | 0.7153 | 1.9312 | 0.8920 |
| Rivers State Station | 0.4829 | 1.0273 | 0.5532 | 1.4936 | 1.2073 |
| Enugu State Station | 0.6641 | 1.4128 | 0.7607 | 2.0540 | 1.3283 |
| Cross Rivers State Station | 0.0973 | 0.2070 | 0.1115 | 0.3010 | 0.4866 |
| Ibadan Station | 0.3557 | 0.7567 | 0.4075 | 1.1002 | 0.8893 |
| Delta State Station | 0.8942 | 1.9023 | 1.0243 | 2.7656 | 2.2355 |
| Ondo State Station | 0.3479 | 0.7401 | 0.3985 | 1.0759 | 0.8697 |
| Kwara State Station | 0.7025 | 1.4944 | 0.8047 | 2.1727 | 1.4050 |
| Borno State Station | 0.0158 | 0.0336 | 0.0181 | 0.0488 | 0.1577 |
| Sokoto State Station | 0.0611 | 0.1299 | 0.0699 | 0.1889 | 0.2035 |
| Niger State Station | 0.1426 | 0.3033 | 0.1633 | 0.4410 | 0.3564 |
| Kano State Station | 0.1859 | 0.3955 | 0.2130 | 0.5750 | 0.4648 |

5. DISCUSSION

Table 2 gives the optimal net returns, construction costs, maintenance costs and administrative costs of the PPC. The result shows that the maximum amount of money to be spent on the five roads in Edo State leading to other States is $N2.2643 \times 10^{14}$. We can see that Borno State has only one tollgate. It has a net returns of $N1.5800 \times 10^{12}$, CRC of $N3.3600 \times 10^{12}$, maintenance costs of $N1.8100 \times 10^{11}$ and

administrative costs of $N4.8800 \times 10^{10}$. The result also shows that the maximum sum of $N1.2192 \times 10^{13}$ and $N3.2919 \times 10^{12}$ should be spent on roads maintenance and administration in Edo State, respectively. The same interpretation is also applicable to the remaining States. We have that the total sum of $N1.4133 \times 10^{15}$ should be the maximum amount to be spent on roads construction by the PPC. The total sum of $N7.6099 \times 10^{13}$ should be spent on roads maintenance and the total sum of $N2.0547 \times 10^{13}$ should be spent on administration. Hence, the gross total (maximum) amount to be generated by the investment is $N2.1743 \times 10^{15}$. This shows that, on the average, each of the five tollgates in Edo State will generate an optimal net returns of $N2.1288 \times 10^{13}$, each of the three tollgates in Lagos State will generate an optimal net returns of $N3.3486 \times 10^{13}$, in Abuja, each of the seven tollgates will generate an optimal net returns of $N8.9205 \times 10^{12}$, in Rivers State, each of the four tollgates will generate an optimal net returns of $N1.2073 \times 10^{13}$, in Enugu State, each of the five tollgates will generate an optimal net returns of $N1.3282 \times 10^{13}$, in Cross River State, each of the two tollgates will generate an optimal net returns of $N4.8665 \times 10^{12}$, in Ibadan, each of the four tollgates will generate an optimal net returns of $N8.8929 \times 10^{12}$, in Delta State, each of the four tollgates will generate an optimal net returns of $N2.2355 \times 10^{13}$, in Ondo State, each of the four tollgates will generate an optimal net returns of $N8.6972 \times 10^{12}$, in Kwara State, each of the five tollgates will generate an optimal net returns of $N1.4050 \times 10^{13}$, in Borno State, the only tollgate will generate an optimal net returns of $N1.5774 \times 10^{12}$, in Sokoto State, each of the three tollgates will generate an optimal net returns of $N2.0354 \times 10^{12}$, in Niger State, each of the four tollgates will generate an optimal net returns of $N3.5644 \times 10^{12}$ and in Kano State, each of the four tollgates will generate an optimal net returns of $N4.6481 \times 10^{12}$.

6. CONCLUSION AND RECOMMENDATION

This paper dealt with the optimal investment strategy for a defined contributory pension plan in Nigeria. We found that the PPC needs a total sum (maximum) of $N1.4133 \times 10^{15}$ to start the operation at the beginning of the planning horizon. This sum total amount is for the construction of the roads. We also found that the optimal net returns that will be accrued to the PPMs is $N6.6434 \times 10^{14}$. We observed that some States yield higher returns than the others. We therefore recommend that the States with higher returns such as Lagos State, Delta State, Edo State, Enugu State, Kwara State, Rivers State, Abuja, Ibadan and Ondo State should be consider first to invest the PPMs contributions into, before other States since they yield the highest returns.

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