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Fibonacci and Super Fibonacci Graceful Labeling of Some Graphs*

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Abstract: In the present work we discuss the existence and non-existence of Fibonacci and super Fibonacci graceful labeling for certain graphs. We also show that the graph obtained by switching a vertex in cycle C_n, (where $n \geq 6$) is not super Fibonacci graceful but it can be embedded as an induced subgraph of a super Fibonacci graceful graph.

Key words: Graceful Labeling; Fibonacci Graceful Labeling; Super Fibonacci Graceful Labeling

1. INTRODUCTION

Graph labeling where the vertices are assigned values subject to certain conditions. The problems arising from the effort to study various labeling schemes of the elements of a graph is a potential area of challenge. Most of the labeling techniques found their origin with 'graceful labeling' introduced by Rosa (1967). The famous graceful tree conjecture and many illustrious works on graceful graphs brought a tide of different graph labeling techniques. Some of them are Harmonious labeling, Elegant labeling, Edge graceful labeling, Odd graceful labeling etc. A comprehensive survey on graph labeling is given in Gallian (2010). The present work is aimed to provide Fibonacci graceful labeling of some graphs.

Throughout this work graph $G = (V(G), E(G))$ we mean a simple, finite, connected and undirected graph with p vertices and q edges. For standard terminology and notations in graph theory we follow Gross and Yellen (1998) while for number theory we follow Niven and Zuckerman (1972). We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1 A *vertex switching* G_v of a graph G is obtained by taking a vertex v of G, removing all edges incidence to v and adding edges joining v to every vertex which are not adjacent to v in G.

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Definition 1.2 Consider two copies of fan ($F_n = P_n + K_1$) and define a new graph known as *joint sum* of F_n is the graph obtained by connecting a vertex of first copy with a vertex of second copy.

Definition 1.3 A function f is called *graceful* labeling of graph if $f : V(G) \rightarrow \{0,1,2,.........q\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1,2,.........q\}$ defined as $f^*(e = uv) =| f(u) - f(v)|$ is bijective. A graph G is called graceful if it admits *graceful labeling*.

Definition 1.4 The *Fibonacci numbers* $F_0, F_1, F_2.....$ are defined by $F_0, F_1, F_2.....$ and $F_{n+1} = F_n + F_{n-1}$.

Definition 1.5 The function $f : V(G) \rightarrow \{0,1,2,.........F_q\}$ (where F_q is the q^{th} Fibonacci number) is said to be *Fibonacci graceful* if $f^* : E(G) \rightarrow \{F_1, F_2,......F_q\}$ defined by $f^*(uv) =| f(u) - f(v)|$ is bijective.

Definition 1.6 The function $f : V(G) \rightarrow \{0, F_1, F_2,.........F_q\}$ (where F_q is the q^{th} Fibonacci number) is said to be *Super Fibonacci graceful* if the induced edge labeling $f^* : E(G) \rightarrow \{F_1, F_2,......F_q\}$ defined by $f^*(uv) =| f(u) - f(v)|$ is bijective.

Above two concepts were introduced by Kathiresen and Amutha [5]. Deviating from the definition 1.1 they assume that $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5.....$ and proved that

- K_n is Fibonacci graceful if and only if $n \le 3$.
- If G is Eulerian and Fibonacci graceful then $q \equiv 0(mod 3)$.
- Every path P_n of length n is Fibonacci graceful.
- P_n^2 is a Fibonacci graceful graph.
- Caterpillars are Fibonacci graceful.
- The bistar $B_{m,n}$ is Fibonacci graceful but not Super Fibonacci graceful for $n \ge 5$.
- C_n is Super Fibonacci graceful if and only if $n \equiv 0(mod 3)$.
- Every fan F_n is Super Fibonacci graceful.
- If G is Fibonacci or Super Fibonacci graceful then its pendant edge extension G' is Fibonacci graceful.
- If G_1 and G_2 are Super Fibonacci graceful in which no two adjacent vertices have the labeling 1 and 2 , then their union $G_1 \cup G_2$ is Fibonacci graceful.
- If G_1, G_2, ...,G_n are super Fibonacci graceful graphs in which no two adjacent vertices are labeled with 1 and 2 then amalgamation of G_1, G_2, ...,G_n obtained by identifying the vertices having labels 0 is also a super Fibonacci graceful.

In the present work we prove that

• Trees are Fibonacci graceful.

• Wheels are not Fibonacci graceful.

• Helms are not Fibonacci graceful.

The graph obtained by

• Switching of a vertex in a cycle C_n is Fibonacci graceful.

• Joint Sum of two copies of fan is Fibonacci graceful.

• Switching of a vertex in a cycle C_n is super Fibonacci graceful except $n \geq 6$.

• Switching a vertex of cycle C_n for $n \geq 6$ can be embeded as an induced subgraph of a super Fibonacci graceful graph.

Observation 1.7 If in a triangle edges receives Fibonacci numbers from vertex labels than they are always consecutive.

2. MAIN RESULTS

Theorem 2.1 Trees are Fibonacci graceful.

Proof: Consider a vertex with minimum eccentricity as the root of tree T. Let this vertex be v . Without loss of generality at each level of tree T we initiate the labeling from left to right. Let $P^1, P^2, P^3, ...P^n$ be the children of v .

Define $f : V(T) \rightarrow \{0,1,2......F_q\}$ in the following manner.

$f(v) = 0$, $f(P^1) = F_1$

Now if $P^1_{1i}(1 \leq i \leq t)$ are children of P^1 then

$f(P^1_{1i}) = f(P^1) + F_{i+1}$, $1 \leq i \leq t$

If there are r vertices at level two of P^1 and out of these r vertices, r_1 be the children of P^1_{11} then label them as follows,

$f(P^1_{11i}) = f(P^1_{11}) + F_{t+1+i}$, $1 \leq i \leq r_1$

Let there are r_2 vertices, which are children of P^1_{12} then label them as follows,

$f(P^1_{12i}) = f(P^1_{12}) + F_{t+1+r_1+i}$, $1 \leq i \leq r_2$

Following the same procedure to label all the vertices of a subtree with root as P^1.

we can assign label to each vertex of the subtree with roots as $P^2, P^3, ...P^{n-1}$ and define $f(P^{i+1}) = F_{f_i+1}$, where F_{f_i} is the f_i^{th} Fibonacci number assign to the last edge of the tree rooted at P^i .

Now for the vertex P^n . Define $f(P^n) = F_q$

Let us denote $P_{ij}^{\,n}$, where i is the level of vertex and j is number of vertices at i^{th} level.

At this stage one has to be cautious to avoid the repeatation of vertex labels in right most branch. For that we first assign vertex label to that vertex which is adjacent to F_q and is a internal vertex of the path whose length is largest among all the paths whose origin is F_q (That is, F_q is a root). Without loss of generality we consider this path to be a left most path to F_q and continue label assignment from left to right as stated erlier.

If $P_{1i}^{n}(1 \le i \le s)$ be the children of P^n then define

$$f(P_{1i}^{n}) = f(P^n) - F_{q-i},\ 1 \le i \le s$$

If there are $P_{2i}^{n}(1 \le i \le b)$ vertices at level two of P^n and out of these b vertices, b_1 be the children of P_{11}^{n}. Then label them as follows.

$$f(P_{2i}^{n}) = f(P_{11}^{n}) - F_{q-s-i},\ 1 \le i \le b_1$$

If there are b_2 vertices, which are children of P_{12}^{n} then label them as follows,

$$f(P_{2(b_1+i)}^{1}) = f(P_{12}^{n}) - F_{q-s-b_1-i},\ 1 \le i \le b_2$$

We will also consider the situation when all the vertices of subtree rooted at F_q is having all the vertices of degree two after i^{th} level then we define labeling as follows.

$$f(P_{i1}^{n}) = f(P_{(i-1)1}^{n}) + (-1)^{i-1} F_{q-(labeled\ vertices\ in\ the\ branch)}$$

Continuing in this fashion unless all the vertices of a subtree with root as P^n are labeled.

Thus we have labeled all the vertices of each level. That is, T admits Fibonacci Graceful Labeling.

That is, trees are Fibonacci Graceful.

The following Figure 1 will provide better under standing of the above defined labeling pattern.

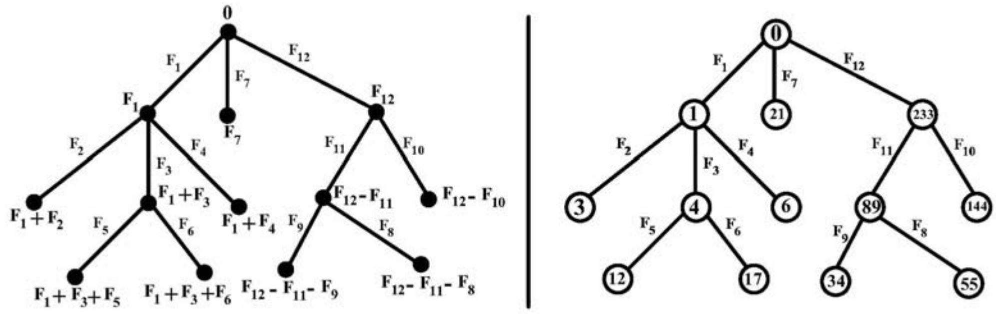

Figure 1: A Tree And its Fibonacci Graceful Labeling

Theorem 2.2 Wheels are not Fibonacci graceful.

Proof: Let v be the apex vertex of the wheel W_n and $v_1, v_2 ......v_n$ be the rim vertices.

Define $f: V(W_n) \rightarrow \{0, 1, 2......F_q\}$

We consider following cases.

Case 1: Let $f(v) = 0$

so, the vertices $v_1, v_2.......v_n$ must be label with Fibonacci numbers.

Let $f(v_1) = F_q$ then $f(v_2) = F_{q-1}$ or $f(v_2) = F_{q-2}$.

If $f(v_2) = F_{q-2}$ then $f(v_n) = F_{q-1}$ is not possible as $f(v_1 v_n) = f(vv_2) = F_{q-2}$.

If $f(v_2) = F_{q-1}$ then $f(v_n) \neq F_{q-2}$ otherwise $f(v_1 v_n) = f(vv_2) = F_{q-1}$.

If $f(v_n) = F_p$ be the Fibonacci number other then F_{q-1} and F_{q-2} then $|f(v_n) - f(v_1)| = |F_p - F_q|$ can not be Fibonacci number for $|p - q| > 2$

Case 2: If v_1 is a rim vertex then define $f(v_1) = 0$

If $f(v_2) = F_q$ then the apex vertex must be labeled with F_{q-1} or F_{q-2}.

<u>**Sub Case 1:**</u> Let $f(v) = F_{q-1}$

Now $f(v_n)$ must be labeled with either by F_{q-2} or by F_{q-3}.

If $f(v_n) = F_{q-2}$ then $f(v_1 v_n) = f(vv_2) = F_{q-2}$

and if $f(v_n) = F_{q-3}$ then $f(vv_n) = f(vv_2) = F_{q-2}$

<u>**Sub Case 2:**</u> Let $f(v) = F_{q-2}$

Now $f(v_n)$ must be label with either by F_{q-1} or by F_{q-3} or by F_{q-4}.

if $f(v_n) = F_{q-1}$ then $f(v_1 v_n) = f(vv_2) = F_{q-1}$

if $f(v_n) = F_{q-3}$ then

$f(v_1 v_2) = F_q$

$f(vv_1) = F_{q-2}$

$f(vv_2) = F_{q-1}$

$f(v_n v_1) = F_{q-3}$

$f(vv_n) = F_{q-4}$

For W_3, $f(v_2v_3)$ can not be Fibonacci number. Now for $n > 3$ let us assume that $f(v_3) = k$ which is not Fibonacci number because for $f(v_3) = F_{q-1}$, we have $f(vv_1) = f(v_2v_3) = F_{q-2}$.

now we have following cases. (1) $F_{q-2} < k < F_q$, (2) $k < F_{q-2} < F_q$

In (1) we have.....

$F_q - k = F_s$

$k - F_{q-2} = F_{s'}$

$F_q - F_{q-2} = F_s + F_{s'} \Rightarrow F_{q-1} = F_s + F_{s'}$ is possible only when $s = q-2$ and $s' = q-3$,

then $f(v_2v_3) = f(vv_1)$ and $f(vv_3) = f(v_1v_n)$

In (2) we have.....

$F_q - k = F_s$

$F_{q-2} - k = F_{s'}$

$F_q - F_{q-2} = F_s + F_{s'} \Rightarrow F_{q-1} = F_s + F_{s'}$ is possible only when $s = q-2$ and $s' = q-3$,

then $f(v_2v_3) = f(vv_1)$ and $f(vv_3) = f(v_1v_n)$

Thus, we can not find a number $f(v_3) = k$ for which $f(v_2v_3)$ and $f(vv_3)$ are the distinct Fibonacci numbers.

For $f(v_n) = F_{q-4}$ we can argue as above.

<u>Sub Case 3:</u> If $f(v) = F_q$

Then we do not have two Fibonacci numbers corresponding to $f(v_1)$ and $f(v_n)$ such that the edges will receive distinct Fibonacci numbers.

Thus we conclude that wheels are not Fibonacci graceful.

Theorem 2.3 Helms are not Fibonacci graceful.

Proof: Let H_n be the helm and v'_1, v'_2, v'_3 .... v'_n be the pendant vertices corresponding to it. If 0 is the label of any of the rim vertices of wheel corresponding to H_n then all the possibilities to admit Fibonacci graceful labeling is ruled out as we argued in above Theorem 2.2. Thus possibilities of 0 being the label of any of the pendant vertices is remained at our disposal.

Define $f: V(H_n) \rightarrow \{0, 1, 2......F_q\}$

Without loss of generality we assume $f(v'_1) = 0$ then $f(v_1) = F_q$

Let $f(v_2) = p$ and $f(v) = r$

In the following Figures $2(1)$ to $2(3)$ the possible labeling is demonstrated. In first two arrangements the possibility of H_3 being Fibonacci graceful is washed out by the similar arrangements for wheels are not Fibonacci graceful held in Theorem 2.2. For the remaining arrangement as shown in Figure $2(3)$ we have to consider following two possibilities.

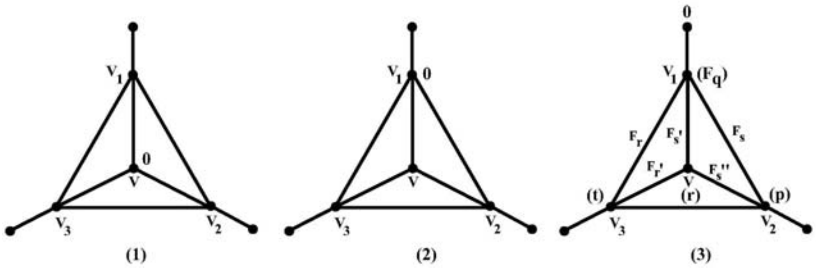

Figure 2: Ordinary Labeling in H3

Case 1: $p < r < F_q$

$F_q - p = F_s$

$F_q - r = F_{s'}$

$r - p = F_{s''}$ then

$F_{s'} + F_{s''} - F_s = 0 \Rightarrow F_s = F_{s'} + F_{s''}$

Case 2: $r < p < F_q$

$F_q - p = F_s$

$F_q - r = F_{s'}$

$p - r = F_{s''}$ then

$F_s + F_{s''} - F_{s'} = 0 \Rightarrow F_{s'} = F_s + F_{s''}$

Now let $f(v_3) = t$ then consider the case $p < r < t < F_q$,

$F_s = F_{s'} + F_{s''}$

$F_{s'} = F_r + F_{r'}$

From these two equations we have...

$F_{s'} = F_r + F_{r'} = F_s - F_{s''}$

so we have $F_r < F_{r'} < F_{s'} < F_{s''} < F_s$ and they are consecutive Fibonacci numbers according to Observation 1.7.

For $r \geq p, t$ we have $F_s = F_{s'} + F_{s''}$ and $F_r = F_{s'} + F_{r'}$ so we have

$F_{s'} = F_s - F_{s''}$ and $F_{s'} = F_r - F_{r'}$ which is not possible.

similar argument can be made for $r \leq p, t$.

i.e. we have either $p < r < t$ or $t < r < p$.

As $F_{s'} < F_{s''} < F_s$, so we can say that with $f(vv_2) = F_{s''}$ the edges of the triangle with vertices $f(v)$, $f(v_2)$ and $f(v_3)$ will not have Fibonacci numbers such that $F_{s''}$ = sum of two Fibonacci numbers.

Similar arguments can also be made for $t < r < p < F_q$.

Hence Helms are not Fibonacci graceful graphs.

Theorem 2.4 The graph obtained by switching of a vertex in cycle C_n admits Fibonacci graceful labeling.

Proof: Let $v_1, v_2, v_3, ...v_n$ be the vertices of cycle C_n and C'_n be the graph resulted from switching of the vertex v_1.

Define $f : V(C'_n) \rightarrow \{0,1,2......F_q\}$ as follows.

$f(v_1) = 0$

$f(v_2) = F_q - 1$

$f(v_3) = F_q$

$f(v_{i+3}) = F_{q-2i}$, $1 \leq i \leq n-3$

Above defined function *f* admits Fibonacci graceful labeling.

Hence we have the result.

Illustration 2.5 Consider the graph C'_8 . The Fibonacci graceful labeling is as shown in Figure 3.

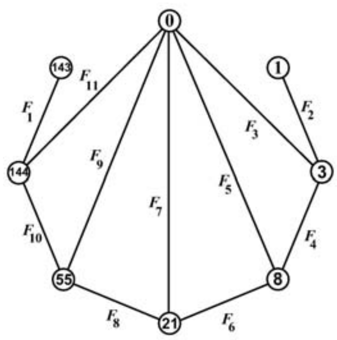

Figure 3: Fibonacci Graceful Labeling of C'_8

Theorem 2.6 The graph obtained by joint sum of two copies of fans $(F_n = P_n + K_1)$ is Fibonacci graceful.

Proof: Let $v_1, v_2, ...v_n$ and v'_1 , v'_2 , v'_3 .... v'_m be the vertices of F_n^1 and F_m^2 respectively. Let v be the apex vertex of F_n^1 and v' be the apex vertex of F_m^2 and let G be the joint sum of two fans.

Define $f : V(G) \rightarrow \{0,1,2......F_q\}$ as follows.

$f(v) = 0$

$f(v') = F_q$

$f(v_i) = F_{2i-1}$, $1 \leq i \leq n$

$f(v'_1) = F_q - F_{2n+1}$

$f(v'_2) = F_q - F_{2n+2}$

$f(v'_{2+i}) = F_q - F_{2n+2+2i}, 1 \leq i \leq m-2$

In view of the above defined pattern the graph G admits Fibonacci graceful labeling.

Illustration 2.7 Consider the Joint Sum of two copies of F_4. The Fibonacci graceful labeling is as shown in Figure 4.

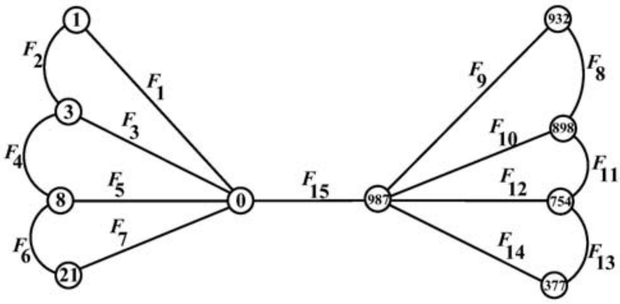

Figure 4: Fibonacci Graceful Labeling of Joint Sum of F_4

Theorem 2.8 The graph obtained by Switching of a vertex in a cycle C_n is super Fibonacci graceful except $n \geq 6$.

Proof: We consider here two cases.

case 1: $n = 3, 4, 5$

For $n = 3$ the graph obtained by switching of a vertex is a disconnected graph which is not desirable for the Fibonacci graceful labeling.

Super Fibonacci graceful labeling of switching of a vertex in C_n for $n = 4, 5$ is as shown in Figure 5.

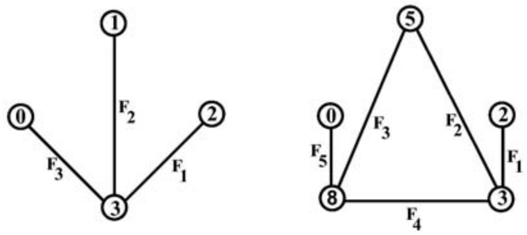

Figure 5: Switching of a Vertex in C_4 and C_5 and Super Fibobacci Graceful Labeling

case 2: $n \geq 6$ The graph shown in Figure 6 will be the subgraph of all the graphs obtained by switching of a vertex in $C_n(n \geq 6)$.

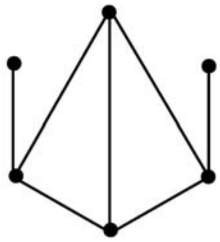

Figure 6: Switching of a Vertex in C_6

In Figure 7 all the possible assignment of vertex labels is shown which demonstrates the repetition of edge labels.

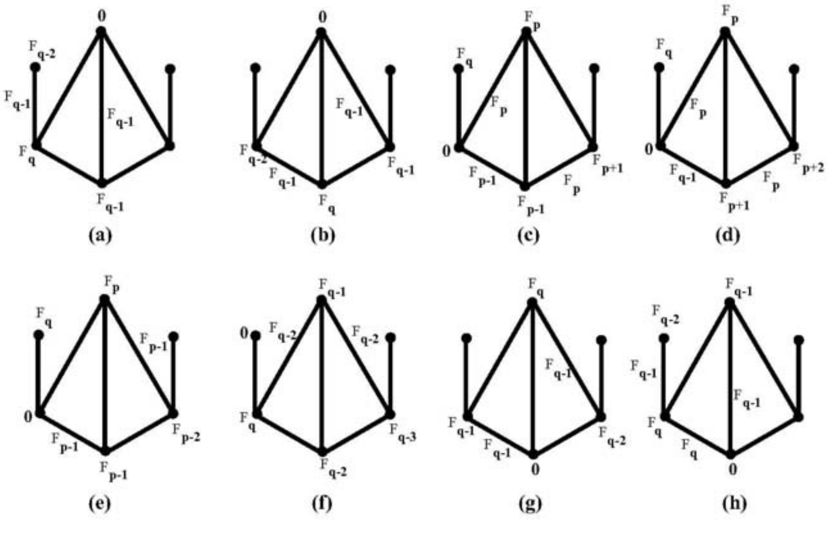

Figure 7: Possible Label Assignment for the Graph Obtained by Vertex Switching in C_6

(1) In *Fig8(a)* edge label F_{q-1} is repeated as

$|F_q - F_{q-2}| = F_{q-1}$ & $|F_{q-1} - 0| = F_{q-1}$

(2) In *Fig8(b)* edge label F_{q-1} is repeated as

$|F_q - F_{q-2}| = F_{q-1}$ & $|F_{q-1} - 0| = F_{q-1}$

(3) In *Fig8(c)* edge label F_p is repeated as $|F_{p+1} - F_{p-1}| = F_p$ & $|F_p - 0| = F_p$,

where F_p is any Fibonacci number.

(4) In *Fig8(d)* edge label F_p is repeated as $|F_{p+2} - F_{p+1}| = F_p$ & $|F_p - 0| = F_p$,

where F_p is any Fibonacci number.

(5) In *Fig8(e)* edge label F_{p-1} is repeated as $|F_p - F_{p-2}| = F_{p-1}$ & $|F_{p-1} - 0| =$

F_{p-1}, where F_p is any Fibonacci number.

(6) In *Fig8(f)* edge label F_{q-2} is repeated as

$|F_{q-1} - F_{q-3}| = F_{q-2}$ & $|F_q - F_{q-1}| = F_{q-2}$

(7) In *Fig8(g)* edge label F_{q-1} is repeated as

$|F_q - F_{q-2}| = F_{q-1}$ & $|F_{q-1} - 0| = F_{q-1}$

(8) In *Fig8(h)* edge label F_{q-1} is repeated as

$|F_q - F_{q-2}| = F_{q-1}$ & $|F_{q-1} - 0| = F_{q-1}$

Theorem 2.9 The graph obtained by Switching of a vertex in cycle C_n for $n \geq 6$ can be embedded as an induced subgraph of a super Fibonacci graceful graph.

Proof: Let $v_1, v_2, v_3 ....v_n$ be the vertices of C_n and v_1 be the switched vertex.

Define $f : V(G) \to \{0, F_1, F_2 ......F_{q+3}\}$

$f(v_1) = 0$

$f(v_{i+1}) = F_{2i-1}$, $1 \leq i \leq n-1$

Now it remains to assign Fibonacci numbers F_1, F_{q+2} and F_{q+3}. Put 3 vertices in the graph. Join first vertex v' labeled with F_2 to the vertex v_3. Now join second vertex v'' labeled with F_{q+3} to the vertex v_1 and vertex v''' labeled with F_{q+2} to the vertex v''.

Thus the resultant graph is a super Fibonacci graceful graph.

Illustration 2.10 In the following Figure 8 the graph obtained by switching of a vertex in cycle C_6 and its super Fibonacci graceful labeling of its embedding is shown.

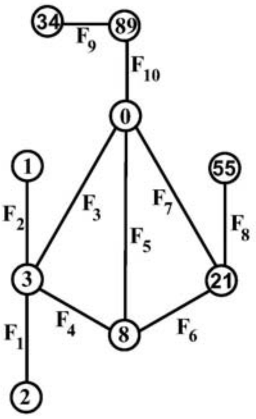

Figure 8: A Super Fibonacci Graceful Embedding

3. CONCLUDING REMARKS

Here we have contributed seven new results to the theory of Fibonacci graceful graphs. It has been proved that trees, vertex switching of cycle C_n, joint sum of two fans are Fibonacci graceful while wheels and helms are not Fibonacci graceful. We have also discussed super Fibonacci graceful labeling and show that the graph obtained by switching of a vertex in cycle $C_n (n \geq 6)$ does not admit super Fibonacci graceful labeling but it can be embedded as an induced subgraph of a super Fibonacci graceful graph.

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