Studies in Mathematical Sciences **ISSN 1923-8444 [Print]**
Vol. 2, No. 2, 2011, pp. 01-23 **ISSN 1923-8452 [Online]**
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Delaunay-like Hypersurfaces in $\mathbf{S}^{n+1}$

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Abstract: Consider $\mathbf{S}^{n+1} \subset \mathbf{R}^2 \times \mathbf{R}^n$ and allow the subgroup $O(n) \subset O(n+2)$ to act on $\mathbf{S}^{n+1}$ by its action on the last n coordinates. Then one asks for CMC surfaces of $\mathbf{S}^{n+1}$ that are invariant by the action of $O(n)$. The resulting hypersurfaces are the so-called rotational CMC hypersurfaces of $\mathbf{S}^{n+1}$ and the Delaunay-like hypersurfaces constructed in [1] are examples of such surfaces with small necksize. The main aim of this paper is to construct Delaunay-like hypersurfaces with slightly larger necksize

Key Words: Euclidean space; Hypersurfaces; Constant mean curvature; Approximate solution.

1. INTRODUCTION

Classical examples of non-trivial constant mean curvature (CMC) surfaces in three dimensional Euclidean space $\mathbf{R}^3$ are the sphere, the cylinder and the Delaunay surfaces, and for a long while they were only known CMC surfaces. In 1984 Wente discovered a family of compact CMC tori immersed in $\mathbf{R}^3$. The technique used by Wente have culminated in a Weierstrass representation for CMC surfaces in $\mathbf{R}^3$. Amongst later developments is the gluing technique for constructing CMC hypersurfaces in $\mathbf{R}^3$ from simple building blocks. This technique was pioneered by Kapouleas [4] and used perturbation arguments from the theory of geometric partial differential equations to construct many new CMC surfaces: e.g. compact surfaces of higher genus and non-compact surfaces with arbitrary number of ends by fusing together spheres and Delaunay surfaces. In [6], the building blocks for gluing techniques become two orientable, immersed, compact, nondegenerate CMC surfaces with nonempty boundary and a catenoidal neck inserted between them. The catenoids are truncated at the right scale so that their boundaries fit as well as possible with the small curves produced by exciting small balls around the points where the two surfaces are closest to each other. In [5] CMC surfaces of genus g with k ends are constructed by attaching Delaunay ends to complete minimal surfaces of finite total curvature in $\mathbf{R}^3$ of genus g with k ends. In [7], half-Delaunay surfaces are attached to arbitrary points of any nondegenerate CMC surfaces to construct new nondegenerate CMC surfaces.

The corresponding theory of CMC hypersurfaces of higher dimensions or in other ambient manifolds is not progressed as far as it is in $\mathbf{R}^3$. The theory on CMC hypersurfaces in hyperbolic space has developed e.g. in [8],[9],[10], which is however not such a vast departure from theory of CMC hypersurfaces in $\mathbf{R}^{n+1}$, due to the non-compactness of the hyperbolic space. Much less is known when the ambient space is the sphere. The classically known examples in $\mathbf{S}^{n+1}$ are the hyperspheres obtained from intersecting $\mathbf{S}^{n+1}$ with affine hyperplanes, and the so-called generalized Clifford tori which are products of lower-dimensional spheres of the form $\mathbf{T}^{p,q}_\alpha := \mathbf{S}^p(\cos\alpha) \times \mathbf{S}^q(\sin\alpha)$ for $p + q = n$ and $\alpha \in (0, \pi/2)$.

In [5], [6] and [7], gluing techniques are adapted in order to construct CMC hypersurfaces in $\mathbf{S}^{n+1}$.

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†Received 10 December 2010; accepted 9 April 2011.

The central idea is to position hyperspheres and/or generalized Clifford tori of the same mean curvature H throughout $\mathbf{S}^{n+1}$ in various ways such that each building block is separated from its neighbors by a small amount. After appropriate small modifications of this initial configuration, catenoidal necks are then inserted between the building blocks at the points where they come closest to each other. This approximate solution $\widetilde{S}$ is then perturbed until it has exactly CMC. One crucial step is to choose an appropriate Banach space so as to express a small perturbation of the approximately CMC surface $\widetilde{S}$ as a normal graph over $\widetilde{S}$ whose graphing function belongs to this Banach space. Once this is done, the desired perturbation is shown to exist by applying the Banach space inverse function theorem to the mean curvature operator. In applying the Banach space inverse function theorem, the symmetries of the initial configuration play the important role in ruling out the existence of Jacobi fields which are non-trivial elements of the kernel of the linearized mean curvature operator of the constituents of $\widetilde{S}$.

1.1 Delaunay-Like Hypersurfaces of $\mathbf{S}^{n+1}$

Let S_α be the hypersphere obtained by intersecting $\mathbf{S}^{n+1}$ by an affine hyperplane passing a distance $\cos\alpha \in (0,1)$ from the origin. The mean curvature of S_α is the constant $H_\alpha := n\cot\alpha$. The construction of the Delaunay-like hypersurfaces begins with defining the rotation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & I_n \end{pmatrix},$$

where I_n is the $n\times n$ identity matrix. The rotation R_θ generates the geodesic γ formed by intersecting $\mathbf{S}^n$ with the (x_0, x_1)-plane. Any pair of rotated hyperspheres $R^k_{2\alpha+\tau}(S_\alpha)$ and $R^{k+1}_{2\alpha+\tau}(S_\alpha)$ are separated by a distance τ. The following result is proved in [1].

Theorem 1 ([1], Main Theorem 1) *Suppose γ is the great circle in $\mathbf{S}^{n+1}$ generated by the one-parameter family of rotations $R_\theta \in SO(n+2)$. For every $\alpha \in (0,\pi/2)$ and sufficiently small $\tau > 0$, there exists a CMC hypersurface $\Lambda_{\alpha,\tau}$ of mean curvature $H_\alpha := n\cot\alpha$ which is approximately equal to a union of hyperspheres of the form $R^k_{2\alpha+\tau}(S_\alpha)$ that are separated by a distance τ from each other and connected by small catenoidal necks.*

As $\tau \to 0$, the hypersurfaces $\Lambda_{\alpha,\tau}$ converges in the C^∞ topology to the union of hyperspheres of mean curvature H_α positioned end-to-end along γ.

The Delaunay-like hypersurfaces constructed in Theorem 1 are either non-compact and immersed, compact and immersed, or compact and embedded depending on the values of α and τ.

1.2 Statement of Results

Based on Theorem 1, the main aim of this paper is to prove the following result.

Theorem 2 (Main Theorem) *For every $\alpha_1 \in (0,\pi/2)$ and sufficiently small τ_1, let $\Lambda_{\alpha_1,\tau_1}$ be as constructed in Theorem 1, and let $\Lambda_{\alpha_1,\tau_1} = \bigcup_k R^k_{2\alpha_1+\tau_1}(\Lambda^0_{\alpha_1,\tau_1})$, the central part of $\Lambda^0_{\alpha_1,\tau_1}$ being around the neck and the rotation R_θ being introduced in Theorem 1. Then for sufficiently small $\tau_* > 0$, there exists a CMC hypersurface $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ of mean curvature $H_{\alpha_1} := n\cot\alpha_1$ which is of the form $\bigcup_k R^k_{2\alpha_1+\tau_1+\tau_*}(\widehat{\Lambda}^0_{\alpha_1,\tau_1,\alpha_2,\tau_*})$; the constituent piece $\widehat{\Lambda}^0_{\alpha,\tau_1,\alpha_2,\tau_*}$ is a deformation of the approximate hypersurface obtained by first separating the upper and lower halves of $\Lambda^0_{\alpha_1,\tau_1}$ by a distance τ_* and then connecting them by small Delaunay-like necks included in $\Lambda_{\alpha_2,\tau_2}$, for some α_2, τ_2 determined by τ_1 and τ_*, $\alpha_2 > \alpha_1$, $\tau_2 < \tau_1$.*

As $\tau_* \to 0$, the hypersurfaces $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ converge in the C^∞ topology to the hypersurfaces $\Lambda_{\alpha_1,\tau_1}$.

From this, we obtain immediately the following.

Corollary 3 *For every $\alpha \in (0, \pi/2)$, there exists a one parameter family of rotational CMC hypersurfaces of mean curvature $H_\alpha := n \cot \alpha$; these hypersurfaces are periodic and cylindrically bounded, and they converge, on one side, to a sequence of spheres, and on the other side, to a cylinder-like hypersurface.*

2. REVIEW THE PROOF OF THEOREM 1

It is essential for the proof of Main Theorem 1 to have a thorough understanding of the construction in the proof of Theorem 1 [1]. Thus here we give a sketch of the construction.

2.1 Parametrization of S_α. Normal Graph over S_α

Let p be the point $p = (1, 0, \cdots, 0)$ and let $p^\pm := R_\alpha^{\pm 1}(p) \in S_\alpha$. Choose a parametrization of $S_\alpha \setminus \{p^+, p^-\}$ in which rotational symmetry around the geodesic γ is in evidence; namely, the parametrization

$$(\mu, \Theta) \in (0, \pi) \times \mathbf{S}^{n-1} \mapsto (\cos \alpha, \sin \alpha \cos \mu, \sin \alpha \sin \mu \Theta), \tag{1}$$

where $\Theta : \mathbf{S}^{n-1} \to \mathbf{R}^n$ is a parametrization of $\mathbf{S}^{n-1}$ of the unit sphere in $\mathbf{R}^n$.

Let $f : S_\alpha \to \mathbf{R}$ be a function on S_α. Then one can parametrize the normal graph over S_α corresponding to f by

$$(\mu, \Theta) \mapsto (\cos(\alpha + f(\mu, \Theta)), \sin(\alpha + f(\mu, \Theta)) \cos \mu, \sin(\alpha + f(\mu, \Theta)) \sin \mu \Theta).$$

2.2 Stereographic Coordinates Adapted to a Pair of Hyperspheres

The construction in [1] uses canonical coordinates that are well adapted to the pair of the hyperspheres $R^k_{2\alpha+\tau}(S_\alpha)$ and $R^{k+1}_{2\alpha+\tau}(S_\alpha)$. These are defined as follows. First, note that the points of closest approach between the rotated spheres $R^k_{2\alpha+\tau}(S_\alpha)$ and $R^{k+1}_{2\alpha+\tau}(S_\alpha)$ are $R^k_{2\alpha+\tau}(p^+) \in R^k_{2\alpha+\tau}(S_\alpha)$ and $R^{k+1}_{2\alpha+\tau}(p^-) \in R^{k+1}_{2\alpha+\tau}(S_\alpha)$, and therefore the point $R^{2k+1}_{\alpha+\tau/2}(p)$ lies on the geodesic γ at the midpoint between these two hyperspheres. Now let $K : \mathbf{S}^{n+1} \to \{-p\} \to \mathbf{R}^{n+1}$ denote the stereographic projection centered at p defined by

$$K(x^0, x^1, \cdots, x^{n+1}) := \left(\frac{x^1}{1 + x^0}, \cdots, \frac{x^{n+1}}{1 + x^0}\right).$$

Then the desired adapted coordinates are given by the inverse of the mapping $K \circ R^{-(2k+1)}_{\alpha+\tau/2} : \mathbf{S}^{n+1} \setminus \{-R^{(2k+1)}_{\alpha+\tau/2}(p)\} \to \mathbf{R}^{n+1}$.

The coordinate image of the geodesic γ is the y^1-axis. The coordinate images of the two hyperspheres $R^k_{2\alpha+\tau}(S_\alpha)$ and $R^{k+1}_{2\alpha+\tau}(S_\alpha)$ are two hyperspheres symmetrically located on either side of the origin centered at two points on the y^1-axis. Indeed, the coordinate image $(y^1, \widehat{y})$ of any point of $\mathbf{S}^{n+1}$ of the form (1) lies on the locus of points satisfying the equation

$$(y^1 + d)^2 + \|\widehat{y}\|^2 = r^2, \tag{2}$$

where $r = r(\alpha, \tau) := \frac{\sin \alpha}{\cos \alpha + \cos(\alpha + \tau/2)}$, and $d = d(\alpha, \tau) := \frac{\sin(\alpha+\tau/2)}{\cos \alpha + \cos(\alpha+\tau/2)}$. Observe that $d - r = \tan(\tau/4)$ is displacement from the origin of the hypersphere determined by (2) in these coordinates.

2.3 Normal Perturbation of the Hyperspheres

Each hypersphere in the initial configuration is perturbed slightly in the normal direction. In order to preserve symmetry with respect to $R_{2\alpha+\tau}$, the same perturbation will be used for each hypersphere. Thus it suffices to explain how S_α is perturbed.

The normal perturbation begins with the choice of a function $G : S_\alpha \to \mathbf{R}$ which determines the normal perturbation. For this, recall that the linearized mean curvature operator on the space of normal graphs over the hypersurface Λ in $\mathbf{S}^{n+1}$ is $\mathcal{L}_\Lambda := \Delta_\Lambda + \|B_\Lambda\|^2 + n$, where Δ_Λ is the Laplacian of Λ and B_Λ is the second fundamental form of Λ. In the case $\Lambda = S_\alpha$, then $\mathcal{L}_\alpha := \sin^{-2}\alpha(\Delta_{\mathbf{S^n}} + n)$. Now we choose G to be the Θ-independent solution of the equation $\mathcal{L}_\alpha(G) = 0$ which is singular at $\mu = 0$ and $\mu = \pi$ and symmetric with respect to $\mu \mapsto \pi - \mu$. Explicitly, this function is

$$G(\mu) := -\sin\mu - \cos\mu \int_{\pi/2}^{\mu} \frac{1 - \sin^{n-1}\sigma}{\cos^2\sigma \sin^{n-1}\sigma} d\sigma.$$

whose asymptotic expansion at $\mu = 0$ is

$$G(\mu) = \begin{cases} -1 + \log 2 - \log\mu + O(\mu^2|\log\mu|), & n = 2 \\ \frac{1}{(n-2)\mu^{n-2}} + O(\mu^{4-n}), & n \geq 3 \end{cases}$$

Choose a small parameter $\varepsilon > 0$ and define the normal graph

$$\widetilde{S}_\alpha := \exp(\varepsilon^{n-1} G N_\alpha)(S_\alpha \setminus \{p^+, p^-\})$$

where N_α is the outward unit normal vector field of S_α. The coordinates $y(\mu, \Theta) \in \mathbf{R}^{n+1}$ of a point in the stereographic projection of the perturbed hyperspheres $R^k_{2\alpha+\tau}(\widetilde{S}_\alpha)$ satisfy

$$y^1(\mu) = -D(\mu) + \sqrt{[R(\mu)]^2 - \|\widehat{y}\|^2}$$

where $R(\mu) := \frac{\sin(\alpha\varepsilon^{n-1} + G(\mu))}{\cos(\alpha\varepsilon^{n-1} G(\mu)) + \cos(\alpha+\tau/2)}$, and $D(\mu) := \frac{\sin(\alpha+\tau/2)}{\cos(\alpha\varepsilon^{n-1} G(\mu)) + \cos(\alpha+\tau/2)}$. From this, together with the invertibility of the relation between μ and $\|\widehat{y}\|$ whenever both $\varepsilon^{n-1}G(\mu)$ and μ are small, one finds that

$$y^1(\|\widehat{y}\|) = G_\varepsilon(\|\widehat{y}\|) := -D(\mu(\|\widehat{y}\|)) + \sqrt{[R(\mu(\|\widehat{y}\|))]^2 - \|\widehat{y}\|^2},$$

whenever both $\varepsilon^{n-1}G(\mu)$ and μ are small.

2.4 Inserting Truncated Catenoids. Assembling the Approximate Solution

The next task is to find a truncation and rescaling of the catenoid that fits exactly within the gap between the two perturbed hyperspheres. For this, observe that the $\widetilde{\varepsilon}$-scaled catenoid $\widetilde{\varepsilon}\Sigma$ in $\mathbf{R} \times \mathbf{R}^n$ can be written as the union of two graphs over the $\mathbf{R}^n$ factor. That is, $\widetilde{\varepsilon}\Sigma = \Sigma^+_{\widetilde{\varepsilon}} \cup \Sigma^-_{\widetilde{\varepsilon}}$, where $\Sigma^\pm_{\widetilde{\varepsilon}} := \{(\pm F_{\widetilde{\varepsilon}}(\|\widehat{y}\|), \widehat{y}) : \|\widehat{y}\| \geq \widetilde{\varepsilon}\}$; the function $F_{\widetilde{\varepsilon}} : \{x \in \mathbf{R} : \mathbf{x} \geq \widetilde{\varepsilon}\} \to \mathbf{R}$ is defined by $F_{\widetilde{\varepsilon}}(x) = \widetilde{\varepsilon}F(x/\widetilde{\varepsilon})$ where $F(x) := \int_1^x (\sigma^{2n-2} - 1)^{-1/2} d\sigma$. Comparing the asymptotic expansion of $G_\varepsilon(\|\widehat{y}\|)$ and $\widetilde{\varepsilon}F(\|\widehat{y}\|/\widetilde{\varepsilon})$, one finds that the matching is optimal if

$$\lim_{\|\widehat{y}\| \to \infty} \widetilde{\varepsilon}F(x/\widetilde{\varepsilon}) = \widetilde{\varepsilon}c_n = \tan(\tau/4)$$

in the dimensions $n \geq 3$ and

$$\widetilde{\varepsilon}\log(2/\widetilde{\varepsilon}) = \tan(\tau/4) + \varepsilon c_2$$

in dimension $n = 2$, for some constant c_n, $n \geq 2$, depending on n and τ. Moreover, observing that for constants C_n, $n \geq 2$, depending on n and τ, the choice $\widetilde{\varepsilon} = \varepsilon C_n^{1/(n-1)}$ in dimensions $n \geq 3$ makes the coefficients of the term $\|\widehat{y}\|^{2-n}$ in the expansion of $G_\varepsilon(\|\widehat{y}\|)$ and $\widetilde{\varepsilon}F(x/\widetilde{\varepsilon})$ coincide and the choice $\widetilde{\varepsilon} = \varepsilon$ in the dimension $n = 2$ makes the coefficients of the term $\log\|\widehat{y}\|$ in the expansion of $G_\varepsilon(\|\widehat{y}\|)$ and $\widetilde{\varepsilon}F(x/\widetilde{\varepsilon})$ coincide.

Once ε and $\widetilde{\varepsilon}$ have been found, the error $|\widetilde{\varepsilon}F(\|\widehat{y}\|/\widetilde{\varepsilon}) - G_\varepsilon(\|\widehat{y}\|)|$ is made smallest when one chooses $\|\widehat{y}\| = O(\rho_\varepsilon)$, where $\rho_\varepsilon := \varepsilon^{(3n-3)/(3n-2)}$.

To assemble the approximate solution, denote by $S_\pm$ the stereographic coordinate images of a pair of perturbed hyperspheres near the y^1-axis. Let $\eta : [0,\infty) \to \mathbf{R}$ be a smooth, monotone cut-off function satisfying $\eta(s) = 0$, for $s \in [0, 1/2]$ and $\eta(s) = 1$, for $s \in [2,\infty)$. Define the function $\widetilde{F}_{\alpha,\tau} : \widetilde{B}_{2\rho_\varepsilon}(0) \setminus B_\varepsilon(0) \subseteq \mathbf{R}^n \to \mathbf{R}$ by

$$\widetilde{F}_{\alpha,\tau}(\widehat{y}) = \varepsilon(1 - \eta(\|\widehat{y}\|/\rho_\varepsilon))F(\|\widehat{y}\|/\varepsilon) + \eta(\|\widehat{y}\|/\rho_\varepsilon)G_\varepsilon(\|\widehat{y}\|).$$

Then define the hypersurfaces $\widetilde{\Sigma}_\varepsilon^\pm = \{(\pm\widetilde{F}_{\alpha,\tau}(\widehat{y}), \widehat{y}) : \|\widehat{y}\| \in [\varepsilon, \rho_\varepsilon]\}$ so that $\widetilde{\Sigma}_\varepsilon := \widetilde{\Sigma}_\varepsilon^+ \cup \widetilde{\Sigma}_\varepsilon^-$ is a smooth hypersurface connecting $S_+ \setminus (\mathbf{R} \times B_{2\rho_\varepsilon}(0))$ to $S_- \setminus (\mathbf{R} \times B_{2\rho_\varepsilon}(0))$ through the catenoid. Note that there exists a radius $\widetilde{\rho}_\varepsilon$ so that the boundary of $R^k_{\alpha+2\tau}(S_\alpha \setminus B_{\widetilde{\rho}_\varepsilon}(p^+)) \cup R^{k+1}_{\alpha+2\tau}(S_\alpha \setminus B_{\widetilde{\rho}_\varepsilon}(p^-))$ under the stereographic projection $R^{2k+1}_{\alpha+\tau/2} \circ K$ coincides with $\partial(\widetilde{\Sigma}_\varepsilon^+ \cup \widetilde{\Sigma}_\varepsilon^-)$. Clearly $\widetilde{\rho}_\varepsilon = O(\rho_\varepsilon)$. The approximate solution with parameters α and τ is the hypersurface

$$\widetilde{\Lambda}_{\alpha,\tau} := \left[\bigcup_{k=0}^{\infty} R^k_{2\alpha+\tau}\Big(S_\alpha \setminus ((B_{\widetilde{\rho}_\varepsilon}(p^-) \cup B_{\widetilde{\rho}_\varepsilon}(p^+))\Big)\right] \cup \left[\bigcup_{k=0}^{\infty} R^{2k+1}_{\alpha+\tau/2} \circ K^{-1}(\widetilde{\Sigma}_\varepsilon)\right]$$

where $\varepsilon = \varepsilon(\tau)$ is the scale parameter associated to τ.

2.5 Deforming the Approximate Solution. Banach Space Inverse Function Theorem

Let $Cyl(\rho) := \{(y^1, \widehat{y}) : \mathbf{R} \times \mathbf{R}^n : \|\widehat{y}\| < \rho\}$ denote the cylinder of radius ρ parallel to the y^1-axis in $\mathbf{R} \times \mathbf{R}^n$. The approximate solution $\widetilde{\Lambda}_{\alpha,\tau}$ is divided into the following three regions.

- Let $\mathcal{N}_\varepsilon^k := R^{2k+1}_{\alpha+\tau/2} \circ K^{-1}(\widetilde{\Sigma} \cap Cyl(\rho_\varepsilon/2))$, and $\mathcal{N}_\varepsilon = \cup_{k=0}^\infty \mathcal{N}_\varepsilon^k$ be the neck region of $\widetilde{\Lambda}_{\alpha,\tau}$.
- Let $\mathcal{T}_\varepsilon^{k,\pm} := R^{2k+1}_{\alpha+\tau/2} \circ K^{-1}(\widetilde{\Sigma}_\varepsilon^\pm \cap [Cyl(2\rho_\varepsilon) \setminus Cyl(\rho_\varepsilon/2)])$ and $\mathcal{T}_\varepsilon = \cup_{k=0}^\infty \mathcal{T}_\varepsilon^{k,+} \cup \mathcal{T}_\varepsilon^{k,-}$ be the transition region of $\widetilde{\Lambda}_{\alpha,\tau}$.
- Let $\mathcal{E}_\varepsilon^k := R^k_{2\alpha+\tau}(S_\alpha \setminus (B_{\widetilde{\rho}_\varepsilon}(p^+) \cup B_{\widetilde{\rho}_\varepsilon}(p^-))$ and $\mathcal{E}_\varepsilon = \cup_{k=0}^\infty \mathcal{E}_\varepsilon^k = \widetilde{\Lambda}_{\alpha,\tau} \setminus [\mathcal{N}_\varepsilon \cup \mathcal{T}_\varepsilon]$ be the exterior region of $\widetilde{\Lambda}_{\alpha,\tau}$.

The approximate solution $\widetilde{\Lambda}_{\alpha,\tau}$ has mean curvature almost equal to H_α everywhere except in the neck and transition regions where the mean curvature transitions to zero. To deform $\widetilde{\Lambda}_{\alpha,\tau}$ into an exactly CMC hypersurface, choose a function $f : \widetilde{\Lambda}_{\alpha,\tau} \to \mathbf{R}$ and then consider the deformation $\Phi_f : \widetilde{\Lambda}_{\alpha,\tau} \to \mathbf{S}^{n+1}$ given by $\Phi_f(q) := \exp_q(f(q)N(q))$, where $\exp_q$ is the exponential map at the point q and $N(q)$ is the outward unit normal vector field of $\widetilde{\Lambda}_{\alpha,\tau}$ at the point q. Finding an exactly CMC normal graph near $\widetilde{\Lambda}_{\alpha,\tau}$ therefore consists of finding a function f satisfying the equation $H_{\Phi_f(\widetilde{\Lambda}_{\alpha,\tau})} = H_\alpha$, where H_Λ denotes the mean curvature of a hypersurface Λ. Let $\Phi_{\alpha,\tau}$ be the operator

$$\Phi_{\alpha,\tau} : f \mapsto H_{\Phi_f(\widetilde{\Lambda}_{\alpha,\tau})} - H_\alpha.$$

This is a quasi-linear, second order partial differential operator for the function f whose zero gives the desired deformation of $\widetilde{\Lambda}_{\alpha,\tau}$. Finding a solution of the equation $\Phi_{\alpha,\tau}(f) = 0$ when τ is small is accomplished by invoking the Banach space inverse function theorem.

Theorem 4 (Banach Space Inverse Function Theorem) *Let $\Phi : Y \to Z$ be a smooth map of Banach spaces with the noem $\|\cdot\|$. Set $\Phi(0) := E$ and define the linearized operator*

$$\mathcal{L}(u) := D\Phi(0)(u) = \frac{d}{ds}\Phi(f + su)\Big|_{f=0,s=0}.$$

Suppose $\mathcal{L}$ is bounded and either $\mathcal{L}$ is invertible and satisfies $\|\mathcal{L}^{-1}(z)\| \leq C\|z\|$, for all $z \in Z$; or else $\mathcal{L}$ is surjective and possesses a bounded right inverse $\mathcal{R} : Z \to Y$ satisfying

$$\|\mathcal{R}(z)\| \leq C\|z\|, \quad \forall z \in Z. \tag{3}$$

Choose ρ so that if $y \in B_\rho(0) \subseteq Y$, then

$$\|\mathcal{L}(x) - D\Phi(y)(x)\| \leq \frac{1}{2C}\|x\|, \quad \forall x \in Y, \tag{4}$$

where $C > 0$ is a constant. Then if $z \in Z$ is such that

$$\|z - E\| \leq \frac{\rho}{2C}, \tag{5}$$

there exists a unique $x \in B_\rho(0)$ so that $\Phi(x) = z$. Moreover, $\|x\| \leq 2C\|z - E\|$.

The inverse function theorem states that, if an appropriate Banach space is so chosen that the *linearization* of $\Phi_{\alpha,\tau}$ at $f = 0$ is surjective with uniformly bounded right inverse, then $\Phi_{\alpha,\tau}$ can be inverted on a small neighborhood of $\Phi_{\alpha,\tau}(0)$. Thus if $\Phi_{\alpha,\tau}(0)$ is sufficiently small, i.e. that the mean curvature of $\widetilde{S}$ deviates very little from H_α with respect to the norm of the Banach space, then there exists f so that $\Phi_{\alpha,\tau}(f) = 0$.

However the operator $D\Phi_{\alpha,\tau}(0)$ is not surjective with uniformly bounded right inverse on arbitrary Banach spaces and so the inverse function theorem does not apply in general. The obstructions to invertibility come from the Jacobi fields of the approximate solution. These are the eigenfunctions of the operator $D\Phi_{\alpha,\tau}(0)$ with zero or small eigenvalues tending to zero as the approximate solution becomes singular. Their origin is geometric: first, the isometries of the ambient $\mathbf{S}^{n+1}$ preserve mean curvature and thus all infinitesimal isometries are in the kernel of $D\Phi_{\alpha,\tau}(0)$; second, when the surface consists of several constituent pieces separated by small necks, as in the present case, then those motions of the surface corresponding to an infinitesimal isometry on one of the constituents and keeping the others fixed (with transition on the neck region), generate for $D\Phi_{\alpha,\tau}(0)$ small eigenvalues. These phenomena ensure that $D\Phi_{\alpha,\tau}(0)$ fails to be bounded below by a positive constant on any Banach space which is not transverse to the kernel and approximate kernel of $D\Phi_{\alpha,\tau}(0)$.

These obstructions to controllable invertibility is avoided by exploiting the natural symmetry of $\widetilde{\Lambda}_{\alpha,\tau}$ and deforming $\widetilde{\Lambda}_{\alpha,\tau}$ equivariantly (i.e. deformations of $\widetilde{\Lambda}_{\alpha,\tau}$ are forced to preserve all symmetries). The controllable invertibility of $D\Phi_{\alpha,\tau}(0)$ is contingent on whether the Jacobi fields –both the global ones and those on the individual constituents of $\widetilde{\Lambda}_{\alpha,\tau}$– possess these additional symmetries or not. If it turns out that, on each summand of $\widetilde{\Lambda}_{\alpha,\tau}$, there are no Jacobi fields possessing the symmetries, then the space of equivariant deformations of $\widetilde{\Lambda}_{\alpha,\tau}$ is transverse to the kernel and approximate kernel associated to small eigenvalues, and then $D\Phi_{\alpha,\tau}(0)$ is controllably invertible.

The following three lemmas gather the necessary information about the Jacobi fields of the hyperspheres and the generalized catenoids; here the ε-scaled catenoid $\varepsilon\Sigma$ in $\mathbf{R} \times \mathbf{R}^n$ is parametrized by

$$(s, \Theta) \in \mathbf{R} \times \mathbf{S}^{n-1} \mapsto \varepsilon(\psi(s), \phi(s)\Theta)$$

where $\phi(s) := (\cosh(n-1)s)^{1/(n-1)}$ and $\psi(s) := \int_0^s \phi^{2-n}(\sigma)d\sigma$.

Lemma 5 *The nontrivial Jacobi fields of the hypersphere S_α are generated by the restriction to S_α of the coordinate functions x^k, $k = 1, \cdots, n+1$.*

Lemma 6 ([3], Proposition 4) *Assume that $\delta < 0$ is fixed and $n \geq 3$. Then there is no non-trivial Jacobi field of the generalized catenoid $\varepsilon\Sigma$ which is bounded by a constant times $(\cosh s)^\delta$ and is invariant under the action of the symmetry $u(s, \Theta) = u(s, B(\Theta))$ for all $B \in O(n)$.*

Lemma 7 ([2], Lemma 4) *Assume that $\delta < 2$ is fixed and $n = 2$. The subspace of Jacobi fields of $\varepsilon\Sigma$ which are bounded by a constant times $(\cosh s)^\delta$ and are invariant under the action of symmetry $(s, \theta) \mapsto (s, \theta + \pi)$ is two-dimensional spanned by the functions $J^0(s, \theta) = s \tanh s - 1$ and $J^1(s, \theta) = s \tanh s$. Here the Jacobi fields J^0 arises from varying the parameter τ and J^1 arises from the rotation R_θ.*

2.6 Function Spaces and Norms

To obtain estimates (3), (4) and (5), one in addition needs to introduce an appropriate weighted Schauder norm to measure the "size" of functions $f \in C^{2,\beta}(\widetilde{\Lambda}_{\alpha,\tau})$. To define this norm, one must first define an appropriate weight function on $\widetilde{\Lambda}_{\alpha,\tau}$. Namely, let $\varepsilon = \varepsilon(\tau)$ be the scale parameter of $\widetilde{\Lambda}_{\alpha,\tau}$, and fix some ρ_0 independent of τ satisfying $\rho_0 \gg 2\rho_\varepsilon$ such that the balls of radii $2\rho_0$ centered on two different neck regions do not intersect. The weight function $\zeta_\varepsilon : \widetilde{\Lambda}_{\alpha,\tau}$ is defined by

$$\zeta_\varepsilon(q) = \begin{cases} \varepsilon \cosh s, & q \in R^{2k+1}_{\alpha+\tau/2} \circ K^{-1}(\varepsilon\psi(s), \varepsilon\phi(s)\Theta) \in \mathcal{N}^k_\varepsilon \\ \text{interpolation}, & q \in \mathcal{T}_\varepsilon, \\ \text{dist}\,(q, \gamma), & q \in \mathcal{E}_\varepsilon \cap [\cup_{k=0}^{N-1} B_{\rho_0}(R^{2k+1}_{\alpha+\tau/2}(p))] \\ \text{interpolation}, & q \in \mathcal{E}_\varepsilon \cap [\cup_{k=0}^{N-1} (B_{2\rho_0}(R^{2k+1}_{\alpha+\tau/2}(p)) \setminus B_{\rho_0}(R^{2k+1}_{\alpha+\tau/2}(p))] \\ 2\rho_0, & q \in \mathcal{E}_\varepsilon \setminus [\cup_{k=0}^{N-1} B_{2\rho_0}(R^{2k+1}_{\alpha+\tau/2}(p))]. \end{cases}$$

The interpolation is such that ζ_ε is smooth and monotone, and is such that ζ_ε is invariant under the symmetries of $\widetilde{\Lambda}_{\alpha,\tau}$.

Let T be any tensor on $\widetilde{\Lambda}_{\alpha,\tau}$, and let $\mathcal{U} \subseteq \Lambda_{\alpha,\tau}$ be any open subset. Recall the notation

$$|T|_{0,\mathcal{U}} = \sup_{q \in \mathcal{U}} |T(q)| \quad \text{and} \quad [T]_{\beta,\mathcal{U}} = \sup_{q,q' \in \mathcal{U}} \frac{|T(q') = \Xi_{q,q'}(T(q))|}{\text{dist}\,(q, q')^\beta},$$

where the norms and the distance function that appear are taken with respect to the induced metric of $\widetilde{\Lambda}_{\alpha,\tau}$, while $\Xi_{q,q'}$ is the corresponding parallel transport operator from q to q'. Now let $Tub_\rho(\gamma)$ be the tubular neighborhood of γ having width ρ, and for any $\arctan(\varepsilon/2) < \rho < \rho_0$, define the annular region $A_\rho = \widetilde{\Lambda}_{\alpha,\tau} \cap [Tub_\rho(\gamma) \setminus Tub_{\rho/2}(\gamma)]$. Then the norm on any $\mathcal{U} \subset A_\rho$ is

$$|f|_{\ell,\beta,\delta,\mathcal{U}\cap A_\rho} := \zeta_\varepsilon^{-\delta} |f|_{0,\mathcal{U}\cap A_\rho} + \cdots + \zeta_\varepsilon^{-\delta+\ell} |\nabla^\ell f|_{0,\mathcal{U}\cap A_\rho} + \zeta_\varepsilon^{-\delta+\ell+\beta} [\nabla^\ell f]_{\beta,\mathcal{U}\cap A_\rho}.$$

Let $\mathcal{U} \subset \widetilde{\Lambda}_{\alpha,\tau}$. Then $C^{\ell,\beta}_\delta$ norm of a function defined on $\mathcal{U}$ is given by

$$|f|_{C^{\ell,\beta}_\delta(\mathcal{U})} := \sum_{i=0}^{\ell} |\nabla^\ell f|_{0,\mathcal{U}\cap[\widetilde{\Lambda}_{\alpha,\tau} \setminus Tub_{\rho_0}(\gamma)]} + [\nabla^\ell f]_{\beta,\mathcal{U}\cap[\widetilde{\Lambda}_{\alpha,\tau} \setminus Tub_{\rho_0}(\gamma)]} + \sup_{\rho \in (2\varepsilon_0, \rho_0]} |f|_{\ell,\beta,\delta,\mathcal{U}\cap A_\rho}.$$

The Banach space $C^{\ell,\beta}_\delta(X)$ denotes the $C^{\ell,\beta}$ tensors fields in X measured with respect to this norm, where

$$X := \{f : \widetilde{\Lambda}_{\alpha,\tau} \to \mathbf{R} : f \circ R_{2\alpha+\tau} = f \circ T = f \text{ and } f \circ S^{01}_B = f, \ \forall B \in SO(n)\};$$

here $T \in O(n+2)$ is the reflection defined by $T(x^0, x^1, \cdots, x^{n+1}) = (x^0, -x^1, \cdots, x^{n+1})$ and $S_B^{01} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \in O(n+2)$. The operator $\Phi_{\alpha,\tau}$ can be symmetrized to yield a new operator (which is given the same name) $\Phi_{\alpha,\tau} : X \to X$. Then

$$\widetilde{\mathcal{L}}_{\alpha,\tau} := D\Phi_{\alpha,\tau}(f) : C^{2,\beta}_{\delta}(X) \to C^{0,\beta}_{\delta-2}(X)$$

is a bounded operator whose operator norm is bounded by a constant independent of τ.

2.7 The Linear and Nonlinear Estimates. The Proof of Theorem 1

For $n \geq 3$ and $n = 2$, choose $\delta \in (2-n, 0)$ and $\delta \in (-1, 0)$, respectively. Let $Y = C^{2,\beta}_{\delta}(X)$, $Z = C^{0,\beta}_{\delta-2}(X)$. The linear estimate (3) is given in Proposition 12 [1] for $n \geq 3$ and Proposition 13 [1] for $n = 2$. To obtain the nonlinear estimates (4) and (5) amounts to showing that $\Phi_{\alpha,\tau}(0) - H_\alpha$ is small in the $C^{0,\beta}_{\delta-2}$-norm and that $D\Phi_{\alpha,\tau}(f) - \widetilde{\mathcal{L}}_{\alpha,\tau}$ can be made to have small $C^{0,\beta}_{\delta-2}$-operator norm if f is chosen sufficiently small in the $C^{2,\beta}_{\delta}$-norm. These are given in Proposition 14 [1] and Proposition 15 [1].

To prove Theorem 1, observe that by Proposition 12 [1] and Proposition 13 [1], one has the linear estimate

$$|\widetilde{\mathcal{L}}^{-1}_{\alpha,\tau}(f)|_{C^{2,\beta}_{\delta}(X)} \leq C_L |f|_{C^{0,\beta}_{\delta-2}(X)},$$

where $C_L = O(\varepsilon^{\delta})$ in dimension 2 and $C_L = O(1)$ in higher dimensions. Therefore the linearization $\widetilde{\mathcal{L}}_{\alpha,\tau}$ is *injective* on $C^{2,\beta}_{\delta}(X)$. But $\widetilde{\mathcal{L}}_{\alpha,\tau} - \Delta$ is a compact operator so $\widetilde{\mathcal{L}}_{\alpha,\tau}$ has the same index as Δ on $C^{2,\beta}_{\delta}(X)$. By self-adjointness, this index is zero, so that $\widetilde{\mathcal{L}}_{\alpha,\tau}$ must be *surjective* as well.

Now in consideration of Proposition 15 [1], one makes the choice

$$|f|_{C^{2,\beta}_{\delta}(X)} \leq \rho, \quad \text{where } \rho = \begin{cases} O(\varepsilon^{1-2\delta}) & \text{in dimension } n = 2, \\ O(\varepsilon^{1-\delta}) & \text{in higher dimensions,} \end{cases}$$

to achieve the bound $|D\Phi_{\alpha,\tau}(f)(u) - \widetilde{\mathcal{L}}_{\alpha,\tau}(u)|_{C^{0,\beta}_{\delta-2}(X)} \leq \frac{1}{2C_L}|u|_{C^{2,\beta}_{\delta}(X)}$, for any $u \in C^{2,\beta}_{\delta}(\widetilde{\Lambda}_{\alpha,\tau})$. Moreover, Proposition 14 [1] asserts that

$$|\Phi_{\alpha,\tau}(0) - H_\alpha|_{C^{0,\beta}_{\delta-2}(X)} = O(\varepsilon^{(2-\delta)(3n-3)/(3n-2)}),$$

and therefore if ε is made sufficiently small by a small enough choice of τ and δ is chosen appropriately, then $|\Phi_{\alpha,\tau}(0) - H_\alpha|_{C^{0,\beta}_{\delta-2}} \leq \frac{\rho}{2C_L}$, and then by the inverse function theorem, a solution of $f := f_{\alpha,\tau}$ of the deformation problem can be found.

As a further consequence of these estimates, the Banach space inverse function theorem asserts that the solution of the equation $\Phi_{\alpha,\tau}(f_{\alpha,\tau}) = 0$ satisfies the estimate

$$|f_{\alpha,\tau}|_{C^{2,\beta}_{\delta}(X)} = O(C_L \varepsilon^{(2-\delta)(3n-3)/(3n-2)})$$

which is much smaller than ε. Therefore the size of the perturbation of $\widetilde{\Lambda}_{\alpha,\tau}$ created by the normal deformation of magnitude $f_{\alpha,\tau}$ is much smaller than the width of $\widetilde{\Lambda}_{\alpha,\tau}$ at its narrowest points, i.e., in the neck regions where the width is $O(\varepsilon)$. Thus $\widetilde{\Lambda}^0_{\alpha,\tau}$ remains embedded under this normal deformation.

3. PROOF OF MAIN THEOREM

The proof of Main Theorem follows broadly the same plan as the proof of Theorem 1 in [1], which is sketched in Section 2. The significant difference between this proof and that in [1] is the choice of the weight function and the weighted Schauder norm on the approximate solution, which is made in order to apply Banach space inverse function theorem (Theorem 2) to deform the approximate solution in the present setting.

3.1 Assembling the Approximate Solution

For small τ, let $\Lambda_{\alpha,\tau}$ be the Delaunay-like surface constructed in Theorem 1 and let

$$\Lambda_{\alpha,\tau} = \bigcup_{k=0}^{\infty} \Lambda_{\alpha,\tau}^{k}, \quad \text{where} \quad \Lambda_{\alpha,\tau}^{k} := R_{2\alpha+\tau}^{k}(\Lambda_{\alpha,\tau}^{0}),$$

where the central part of each summand $\Lambda_{\alpha,\tau}^{k}$, $k \geq 0$, is around the neck. Write $\Lambda_{\alpha,\tau}^{0} = \Lambda_{\alpha,\tau}^{0+} \cup \Lambda_{\alpha,\tau}^{0-}$ as the union of its upper and lower halves which are situated symmetrically on either side of the neck. Cut along the neck and separate the upper and lower halves of $\Lambda_{\alpha,\tau}^{0}$ by a small distance τ_*. The first step is to construct a smooth hypersurface with boundary in $\mathbf{R}^{n+1}$ that interpolates between the stereographic coordinate images of the separated upper and lower halves of $\Lambda_{\alpha,\tau}^{0}$. In order to preserve symmetry with respect to $R_{2\alpha+\tau+\tau_*}$, the same perturbation will be used for each summand.

Namely, for $\alpha_1 \in (0, \pi/2)$, choose τ_1 so small that $\Lambda_{\alpha_1,\tau_1}$ can be constructed as in Theorem 1. For a small positive number τ_*, we set $\widehat{\Lambda}_+ := R_{2\alpha_1+\frac{1}{2}\tau_*}(\Lambda_{\alpha_1,\tau_1}^{0+})$ and $\widehat{\Lambda}_- := R_{2\alpha_1-\frac{1}{2}\tau_*}(\Lambda_{\alpha_1,\tau_1}^{0-})$, and then let $\widehat{\Lambda}_{\alpha_1,\tau_1,\tau_*}^{0} = \widehat{\Lambda}_+ \cup \widehat{\Lambda}_-$. There exist $\tau_2 < \tau_1$, $\alpha_2 > \alpha_1$, $\tau_1 - \tau_2, \alpha_2 - \alpha_1$ being small, such that $\widehat{\Lambda}_{\alpha_1,\tau_1,\tau_*}^{0}$ and $\Lambda_{\alpha_2,\tau_2}^{0}$ have the same tangent spaces at the points where they intersect. Let $\widehat{\Lambda}_{\alpha_1,\tau_1,\tau_*}^{0}$ and $\Lambda_{\alpha_2,\tau_2}^{0}$ intersect at the set Γ and τ_{**} be the positive number such that

$$\operatorname{dist}(\Gamma, K^{-1}\{y_1 = 0\}) = \tau_{**}/2.$$

We have

$$\tau_{**} = O((\tau_*)^{s_*}), \quad \text{for some } s_* \in (0,1). \tag{6}$$

We notice that τ_2 and τ_{**} are completely determined by τ_1, τ_*, α_1 and α_2.

Let the stereographic coordinate image of $\widehat{\Lambda}_{\alpha_1,\tau_1,\tau_*}^{0}$ and $\Lambda_{\alpha_2,\tau_2}$ be the graphs of $\|\widehat{y}\| = \widehat{G}_{\alpha_1,\tau_1,\tau_*}(y_1)$ and $\|\widehat{y}\| = \widehat{G}_{\alpha_2,\tau_2}(y_1)$, respectively. Let $\eta : [0,\infty) \to \mathbf{R}$ be a smooth, monotone cut-off function satisfying $\eta(s) = 0$, for $s \in [0, 1/2]$ and $\eta(s) = 1$, for $s \in [2,\infty)$. Define the function $G_{\alpha_1,\tau_1,\alpha_2,\tau_*}$: $[-2\tan(\tau_{**}/8), 2\tan(\tau_{**}/8)] \subseteq \mathbf{R} \to \mathbf{R}$ by

$$G_{\alpha_1,\tau_1,\alpha_2,\tau_*}(y_1) = \left(1 - \eta\left(\frac{|y_1|}{\tan(\tau_{**}/8)}\right)\right)\widehat{G}_{\alpha_1,\tau_1,\tau_*}(y_1) + \eta\left(\frac{|y_1|}{\tan(\tau_{**}/8)}\right)\widehat{G}_{\alpha_2,\tau_2}(y_1). \tag{7}$$

Then define the hypersurfaces

$$\widehat{\Sigma}_{\tau_*}^{\pm} := \{(\pm y_1, G_{\alpha_1,\tau_1,\alpha_2,\tau_*}(\pm y_1)\Theta) : y_1 \in [0, 2\tan(\tau_{**}/8)]\} \subset \mathbf{R} \times \mathbf{S}^{n-1}.$$

Denote the stereographic coordinate image of $\widehat{\Lambda}_\pm$ by $\Lambda_\pm$. Then $\widehat{\Sigma}_{\tau_*} := \widehat{\Sigma}_{\tau_*}^{+} \cup \widehat{\Sigma}_{\tau_*}^{-}$ is a smooth hypersurface connecting $\Lambda_+ \setminus [(0, 2\tan(\tau_{**}/8)) \times \mathbf{R}^n]$ to $\Lambda_- \setminus [(-2\tan(\tau_{**}/8), 0) \times \mathbf{R}^n]$ through the stereographic coordinate image of $\Lambda_{\alpha_2,\tau_2}^{0}$. Now let

$$\widehat{S}(\tau) = \{q : \operatorname{dist}(q, K^{-1}(\{y_1 = 0\})) < \tau\},$$

be a strip on $\mathbf{S}^{n+1}$ centered at the neck $\{K^{-1}(\{y_1 = 0)\}\}$. Note that there exists a number $\widetilde{\tau}_{**}$ so that the boundary of

$$R_{2\alpha_1+\tau_1}^{k} R_{\alpha_1+\frac{1}{2}\tau_*}^{2k-1}\left(\widehat{\Lambda}_+ \setminus [\widehat{S}(\widetilde{\tau}_{**}) \setminus \widehat{S}(\tau_*/2)]\right) \bigcup R_{2\alpha_1+\tau_1}^{k+1} R_{\alpha_1+\frac{1}{2}\tau_*}^{2k-2}\left(\widehat{\Lambda}_- \setminus [\widehat{S}(\widetilde{\tau}_{**}) \setminus \widehat{S}(\tau_*/2)]\right)$$

under the stereographic projection $R^{2k+1}_{\alpha_1+\frac{1}{2}\tau_1} R^{4k-3}_{\frac{1}{2}\alpha_1+\frac{1}{4}\tau_*} \circ K$ coincides with $\partial(\widehat{\Sigma}^+_{\tau_*} \cup \widehat{\Sigma}^-_{\tau_*})$. Clearly $\widetilde{\tau}_{**} = O(\tau_{**})$.

The approximate solution with parameters α and τ_* is the hypersurface

$$\widehat{\Lambda}_{\alpha,\tau_1,\tau_2,\tau_*} := \Big[\bigcup_{k=0}^{\infty} R^k_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*}(\widehat{\Lambda}^0_{\alpha_1,\tau_1,\tau_*} \setminus \widehat{S}(\widetilde{\tau}_{**}))\Big] \bigcup \Big[\bigcup_{k=0}^{\infty} R^{2k+1}_{\alpha_1+\tau_1/2} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*} \circ K^{-1}(\widehat{\Sigma}_{\tau_*})\Big].$$

3.2 Deforming the Approximate Solution

Let

$$S(\tau) := \{(y^1, \widehat{y}) : \mathbf{R} \times \mathbf{R}^n : |y_1| < \tau\}$$

denote the strip of width τ centered at the subspace $\{y^1 = 0\}$ in $\mathbf{R} \times \mathbf{R}^n$. Analogously to [1] (see **2.5**), we divide the approximate solution $\widehat{\Lambda}_{\alpha,\tau_1,\tau_2,\tau_*}$ into the following three regions.

- Let $\widehat{\mathcal{N}}^k_{\tau_*} := R^{2k+1}_{\alpha_1+\frac{1}{2}\tau_1} R^{4k-3}_{\frac{1}{2}\alpha_1+\frac{1}{4}\tau_*} \circ K^{-1}(\widehat{\Sigma}_{\tau_*} \cap S\,(\frac{1}{2}\tan(\tau_{**}/8))$ and call $\widehat{\mathcal{N}}_{\tau_*} = \bigcup_{k=0}^{\infty} \widehat{\mathcal{N}}^k_{\tau_*}$ as the neck region of $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$.
- Let $\widehat{\mathcal{T}}^{k,\pm}_{\tau_*} := R^{2k+1}_{\alpha_1+\frac{1}{2}\tau_1} R^{4k-3}_{\frac{1}{2}\alpha_1+\frac{1}{4}\tau_*} \circ K^{-1}(\widehat{\Sigma}^{\pm}_{\tau_*} \cap (S\,(2\tan(\tau_{**}/8) \setminus S\,(\frac{1}{2}\tan(\tau_{**}/8))])$ and call $\widehat{\mathcal{T}}_{\tau_*} := \bigcup_{k=0}^{\infty}(\widehat{\mathcal{T}}^{k,+}_{\tau_*} \cup \widehat{\mathcal{T}}^{k,-}_{\tau_*})$ as the transition region of $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$.
- Let $\widehat{\mathcal{E}}^k_{\tau_*} := R^k_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*}(\Lambda^0_{\alpha_1,\tau_1,\tau_*} \setminus \widehat{S}(\widetilde{\tau}_{**}))$ and $\widehat{\mathcal{E}}_{\tau_*} := \bigcup_{k=0}^{\infty} \widehat{\mathcal{E}}^k_{\tau_*} = \widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*} \setminus [\,\widehat{\mathcal{N}}_{\tau_*} \cup \widehat{\mathcal{T}}_{\tau_*}]$ be the exterior region of $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$.

The approximate solution $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$ has mean curvature almost equal to H_α everywhere except in the neck and transition regions where the mean curvature transitions to H_α. We shall deform $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$ into an exactly CMC hypersurface by proceeding in an essentially parallel fashion to that adopted in [1], which is described in Section 2.

Namely, we proceed to find an exactly CMC normal graph near $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$ by considering a function $f : \widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*} \to \mathbf{R}$ and its corresponding deformation $\widehat{\Phi}_f : \widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*} \to \mathbf{S}^{n+1}$ given by $\widehat{\Phi}_f(q) := \exp_q(f(q)N(q))$ where $\exp_q$ is the exponential map at the point q and $N(q)$ is the outward unit normal vector field of $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$ at the point q. Let H_Λ denotes the mean curvature of a hypersurface Λ, and then let $\widehat{\Phi}_{\alpha,\tau_1,\alpha_2,\tau_*}$ be the operator

$$\widehat{\Phi}_{\alpha,\tau_1,\alpha_2\tau_*} : f \mapsto H_{\widehat{\Phi}_f(\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*})} - H_\alpha,$$

which is a quasi-linear, second order partial differential operator for the function f and whose zero gives the desired deformation of $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$. Finding a solution of the equation $\widehat{\Phi}_{\alpha,\tau_1,\alpha_2,\tau_*}(f) = 0$ when τ_* is small is accomplished by invoking Theorem 4 (the Banach space inverse function theorem).

3.3 Function Spaces and Norms

To obtain a linear estimate (3) and nonlinear estimates (4), (5) in Theorem 4, we need to introduce a suitable weighted Schauder norm to measure the "size" of functions $f \in C^{2,\beta}(\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*})$. To define this norm, we first need to define an appropriate weight function on $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$.

We first define a weight function on the generalized catenoid $\varepsilon\Sigma$. Recall that the ε-scaled catenoid $\varepsilon\Sigma$ in $\mathbf{R} \times \mathbf{R}^n$ is parametrized by

$$(s, \Theta) \in \mathbf{R} \times \mathbf{S}^{n-1} \mapsto \varepsilon(\psi(s), \phi(s)\Theta)$$

where $\phi(s) := (\cosh(n-1)s)^{1/(n-1)}$ and $\psi(s) := \int_0^s \phi^{2-n}(\sigma)d\sigma$. We observe that

$$\phi^{2-n} = \frac{\dot{\phi}}{\sinh s}.$$

Let $s_0 > 0$ be the value of s for which $|\sinh s_0| = 1$. We note that $\phi(s) - \varepsilon < |\psi(s)|$, when $|s| < |s_0|$, and $\phi(s) - \varepsilon > |\psi(s)|$, when $|s| > |s_0|$.

Suppose that $\Lambda_{\alpha_i,\tau_i}$ is of the neck width ε_i, $i = 1, 2$. We may henceforth assume that τ_* is so small that

$$\varepsilon_i\psi(s_0/2) > 2\tan(\tau_{**}/8), \quad i = 1, 2, \tag{8}$$

and that the region $K^{-1}(S(\tan(\tau_{**}/8)))$ is contained in the neck regions $\mathcal{N}^i_{\varepsilon_i}$ of $\Lambda_{\alpha_i,\tau_i}$. We define the function $\zeta_i(s, \Theta)$, $i = 1, 2$, on $\mathbf{R} \times \mathbf{S}^{n-1}$ as follows.

$$\zeta_i(s,\Theta) = \begin{cases} \frac{\tau_*}{2} & \text{if } 0 < \varepsilon_i\psi(s) < \frac{1}{2}\tan(\tau_*/8), \\ \text{interpolation} & \text{if } \frac{1}{2}\tan(\tau_*/8) < \varepsilon_i\psi(s) < \frac{1}{2}\tan(\tau_*/8), \\ \varepsilon_i\psi(s), & \text{if } \tan(\tau_{**}/8) < \varepsilon_i\psi(s) < 2\tan(\tau_{**}/8), \\ \text{interpolation} & \text{if } \varepsilon_i\psi(s) > 2\tan(\tau_{**}/8) \text{ and } s < s_0/2, \\ \varepsilon_i\phi(s), & s \geq s_0/2. \end{cases}$$

The interpolation is such that ζ_i is smooth and monotone, and is such that ζ_i is invariant under the symmetries of $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$.

Also we parametrize the neck and transient regions in $\widehat{\Lambda}^k_{\alpha_i,\tau_i}$, $i = 1, 2$, $k = 0, 1, \cdots, N-1$, in an obvious way by

$$(s,\Theta) \in (-s_i, s_i) \times \mathbf{S}^{n-1} \mapsto \varepsilon_i(\psi(s), \phi_{\tau_i}(s)\Theta), \tag{9}$$

for some $s_i \in \mathbf{R}$, $s_i > 0$, where ϕ_{τ_i} is some smooth function depending on α_i, τ_i.

Now fix some τ_0 satisfying $\tau_0 \gg 2\tau_*$ such that the strip $\widehat{S}(2\tau_0)$ of width $2\tau_0$ centered on two different neck regions do not intersect. And let $2\widehat{\rho}_0$ be the radius of $\partial\widehat{S}(2\tau_0) \cap \Lambda^0_{\alpha_1,\tau_1}$. Further, set

$$d_{1,k}(q) = \max\Big\{\mathrm{dist}\,(q,\gamma) - \varepsilon_1, \mathrm{dist}\,(q, R^{2k+1}_{2\alpha_1+\tau_1} R^{k+1}_{\alpha_1+\frac{1}{2}\tau_*} \circ K^{-1}\{y^1 = 0\}) - \frac{\tau_*}{2}\Big\}. \tag{10}$$

The weight function on $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ is defined as follows.

$$\zeta_{\tau_*}(q) = \begin{cases} \zeta_2, & q \in \widehat{\mathcal{N}}_{\tau_*}, \\ \text{interpolation}, & q \in [\bigcup_{k=0}^{N-1} R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*} \circ K^{-1}(S(\frac{3}{2}\tan(\frac{\tau_{**}}{8})))] \setminus \widehat{\mathcal{N}}_{\tau_*}, \\ \mathrm{dist}\,(q, K^{-1}\{y^1 = 0\})), & q \in \widehat{\mathcal{T}}_{\tau_*} \setminus [\bigcup_{k=0}^{N-1} R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*} \circ K^{-1}(S(\frac{3}{2}\tan(\frac{\tau_{**}}{8})))], \\ \text{interpolation}, & q \in \widehat{\mathcal{E}}^k_{\tau_*} \cap [\bigcup_{k=0}^{N-1} R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*} (\widehat{S}(\frac{1}{2}\tau_0))], \\ d_{1,k}(q), & q \in \widehat{\mathcal{E}}_{\tau_*} \cap [R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*} (\widehat{S}(\tau_0) \setminus \widehat{S}(\frac{1}{2}\tau_0))], \\ \text{interpolation}, & q \in \widehat{\mathcal{E}}_{\tau_*} \cap [\bigcup_{k=0}^{N-1} R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha+\frac{1}{2}\tau_*} (\widehat{S}(2\tau_0) \setminus \widehat{S}(\tau_0))], \\ 2\widehat{\rho}_0, & q \in \widehat{\mathcal{E}}_{\tau_*} \setminus [\bigcup_{k=0}^{N-1} R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha+\frac{1}{2}\tau_*} (\widehat{S}(2\tau_0))]. \end{cases}$$

The interpolation is such that ζ_{τ_*} is smooth and monotone, and is such that ζ_{τ_*} is invariant under the symmetries of $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$. Now for any $0 < \tau < \tau_0$, define the annular region $A_\tau = \widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*} \cap [\widehat{S}(\tau) \setminus \widehat{S}(\tau/2)]$. Then the norm on any $\mathcal{U} \subset A_\tau$ is

$$|f|_{\ell,\beta,\delta,\mathcal{U}\cap A_\tau} := \zeta_{\tau_*}^{-\delta}|f|_{0,\mathcal{U}\cap A_\tau} + \cdots + \zeta_{\tau_*}^{-\delta+\ell}|\nabla^\ell f|_{0,\mathcal{U}\cap A_\tau} + \zeta_{\tau_*}^{-\delta+\ell+\beta}[\nabla^\ell f]_{\beta,\mathcal{U}\cap A_\tau}.$$

Let $\mathcal{U} \subset \widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$. Then $\widehat{C}^{\ell,\beta}_{\delta}$ norm of a function defined on $\mathcal{U}$ is given by

$$|f|_{\widehat{C}^{\ell,\beta}_{\delta}(\mathcal{U})} := \sum_{i=0}^{\ell} |\nabla^{\ell} f|_{0,\mathcal{U}\cap[\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}\setminus\widehat{S}(\tau_0)]} + [\nabla^{\ell} f]_{\beta,\mathcal{U}\cap[\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}\setminus\widehat{S}(\tau_0)]} + \sup_{\tau\in(2\tau_*,\tau_0]} |f|_{\ell,\beta,\delta,\mathcal{U}\cap A_{\tau}}.$$

To meet the requirement that the functions under consideration be invariant with respect to all the symmetries of $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$, let us consider the space of tensor fields

$$X := \{f : \widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*} \to \mathbf{R} : f \circ R_{2\alpha+\tau} = f \circ T = f \text{ and } f \circ S^{01}_{B} = f, \ \forall B \in SO(n)\};$$

here $T \in O(n+2)$ is the reflection and $S^{01}_{B} \in O(n+2)$ for $B \in O(n)$ is the rotation of last n coordinates. as defined in **2.7**, Let $\widehat{C}^{\ell,\beta}_{\delta}(X)$ denote the $C^{\ell,\beta}$ tensor fields in X measured with respect to this norm.

Observe that the operator $\widehat{\Phi}_{\alpha,\tau_1,\alpha_2,\tau_*}$ can be symmetrized to yield a new operator (which is given the same name) $\widehat{\Phi}_{\alpha,\tau_1,\alpha_2,\tau_*} : X \to X$. Then it is easy to verify that

$$\widehat{\mathcal{L}}_{\alpha,\tau_1,\alpha_2,\tau_*} := D\widehat{\Phi}_{\alpha,\tau_1,\alpha_2,\tau_*}(f) : \widehat{C}^{2,\beta}_{\delta}(X) \to \widehat{C}^{0,\beta}_{\delta-2}(X)$$

is a bounded operator whose operator norm is bounded above by a constant independent of τ_*.

3.4 The Linear Estimates

The aim of this section is to derive the linear estimate (4) in the following form.

Poposition 8 *Suppose $n \geq 3$ and choose $\delta \in (2-n, 0)$ and $\tau_* > 0$ sufficiently small. The linearized operator $\widehat{\mathcal{L}}_{\alpha,\tau_1,\alpha_2,\tau_*} := D\Phi_{\alpha_1,\tau_1,\alpha_2,\tau_*}(f) : \widehat{C}^{2,\beta}_{\delta}(X) \to \widehat{C}^{0,\beta}_{\delta-2}(X)$ satisfies the estimate*

$$|\widehat{\mathcal{L}}_{\alpha,\tau_1,\alpha_2,\tau_*} u|_{\widehat{C}^{0,\beta}_{\delta-2}(X)} \geq C|u|_{\widehat{C}^{2,\beta}_{\delta}(X)},$$

where C is a constant independent of τ_*.

Poposition 9 *Let $\hat{s}_* = \min(2s_*, 1)$, where s_* is defined in (6). Suppose $n = 2$ and choose $\delta \in (-\hat{s}_*, 0)$ and $\tau_* > 0$ sufficiently small. The linearized operator $\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*} := D\Phi_{\alpha_1,\tau_1,\alpha_2,\tau_*}(f) : \widehat{C}^{2,\beta}_{\delta}(X) \to \widehat{C}^{0,\beta}_{\delta-2}(X)$ satisfies the estimate*

$$|\widehat{\mathcal{L}}_{\alpha,\tau_1,\alpha_2,\tau_*} u|_{\widehat{C}^{0,\beta}_{\delta-2}(X)} \geq C(\tau_*)^{-\delta}|u|_{\widehat{C}^{2,\beta}_{\delta}(X)}$$

where C is a constant independent of τ_*.

The method of proof of Proposition 8 and Proposition 9, analogous to that of Proposition 12 [1], is to construct an explicit solution of the equation $\widehat{\mathcal{L}}_{\alpha,\tau_1,\alpha_2,\tau_*} u = f$ by patching together local solutions on the neck region and away from the neck region.

3.4.1 Jacobi Fields on Delaunay-Like Hypersurfaces

It is essential to understand the Jacobi fields of the Delaunay-like hypersurfaces in some detail. For this purpose, let us parametrize the neck regions of $\Lambda_{\alpha_i,\tau_i}$, $i = 1, 2$, as in (9). Denoting $\widehat{\mathcal{N}}^{1}_{\varepsilon_1}$, $\widehat{\mathcal{T}}^{1}_{\varepsilon_1}$ and $\widehat{\mathcal{E}}^{1}_{\varepsilon_1}$ as the neck, transient and exterior regions of $\widehat{\Lambda}_{\alpha_1,\tau_1}$, we define the weight function $\widehat{\zeta_1}$ on $\widehat{\Lambda}_{\alpha_1,\tau_1}$ as follows.

$$\widehat{\zeta_1}(q) = \begin{cases} \zeta_1, & q \in \bigcup_{k=0}^{N-1} R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*} \circ K^{-1}(S(\frac{3}{2}\tan(\frac{\tau_{**}}{8}))), \\ \text{interpolation}, & q \in \widehat{\mathcal{T}}^1_{\varepsilon_1} \setminus \bigcup_{k=0}^{N-1} R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*} \circ K^{-1}(S(\frac{3}{2}\tan(\frac{\tau_{**}}{8}))), \\ d_{1,k}(q), & q \in \widehat{\mathcal{E}}^1_{\varepsilon_1} \cap [R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*}(\widehat{S}(\tau_0) \setminus \widehat{S}(2\widetilde{\tau}_{**}))], \\ \text{interpolation}, & q \in \widehat{\mathcal{E}}^1_{\varepsilon_1} \cap [\bigcup_{k=0}^{N-1} R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha+\frac{1}{2}\tau_*}(\widehat{S}(2\tau_0) \setminus \widehat{S}(\tau_0))], \\ 2\widehat{\rho_0}, & q \in \widehat{\mathcal{E}}^1_{\varepsilon_1} \setminus [\bigcup_{k=0}^{N-1} R^{k+1}_{2\alpha_1+\tau_1} R^{2k-1}_{\alpha_1+\frac{1}{2}\tau_*}(\widehat{S}(2\tau_0))], \end{cases}$$

where $d_{1,k}(q)$ is defined in (10).

Also, setting

$$d_2(q) = \max\{\mathrm{dist}\,(q,\gamma) - \varepsilon_2, \mathrm{dist}\,(q, K^{-1}\{y^1 = 0\})\},$$

and denoting $\mathcal{N}^2_{\varepsilon_2}$, $\mathcal{T}^2_{\varepsilon_2}$ and $\mathcal{E}^2_{\varepsilon_2}$ as the neck, transient and exterior regions of $\Lambda_{\alpha_2,\tau_2}$, we define the weight function on $\widehat{\zeta_2}$ on $\Lambda^0_{\alpha_2,\tau_2}$ as follows.

$$\widehat{\zeta_2}(q) = \begin{cases} \zeta_2, & q \in K^{-1}(S(\frac{3}{2}\tan(\frac{\tau_{**}}{8}))), \\ \text{interpolation}, & q \in \Lambda^0_{\alpha_2,\tau_2} \cap \mathcal{T}^2_{\varepsilon_2} \setminus K^{-1}(S(\frac{3}{2}\tan(\frac{\tau_{**}}{8}))), \\ d(q), & q \in \Lambda^0_{\alpha_2,\tau_2} \cap \mathcal{E}^2_{\varepsilon_2} \cap [(\widehat{S}(\tau_0) \setminus \widehat{S}(2\widetilde{\tau}_{**}))], \\ \text{interpolation}, & q \in \Lambda^0_{\alpha_2,\tau_2} \cap \mathcal{E}^2_{\varepsilon_2} \cap [\widehat{S}(2\tau_0) \setminus \widehat{S}(\tau_0)], \\ 2\widehat{\rho_0}, & q \in \Lambda^0_{\alpha_2,\tau_2} \cap \mathcal{E}^2_{\varepsilon_2} \setminus (\widehat{S}(2\tau_0)). \end{cases}$$

Let $|\cdot|_{\widehat{C}^{\ell,\beta}_\delta(\widehat{\Lambda}_{\alpha_1,\tau_1})}$ denote the weighted $\widehat{C}^{\ell+\beta}_\delta$-norm on $\widehat{\Lambda}_{\alpha_1,\tau_1}$ so that

$$|u|_{C^{\ell,\beta}_\delta(\widehat{\Lambda}_{\alpha_1,\tau_1})} := |(\widehat{\zeta_1})^{-\delta} u|_{0,\Lambda_{\widehat{\alpha}_1,\tau_1}} + \cdots + |(\widehat{\zeta_1})^{-\delta+\ell}\nabla^\ell u|_{0,\widehat{\Lambda}_{\alpha_1,\tau_1}} + |(\widehat{\zeta_1})^{-\delta+\ell+\beta}\nabla^\ell u|_{\beta,\widehat{\Lambda}_{\alpha_1,\tau_1}},$$

where the norms and derivatives correspond to the metric on $\Lambda_{\alpha_1,\tau_1}$. Also, let $|\cdot|_{\widehat{C}^{\ell,\beta}_\delta(\Lambda_{\alpha_2,\tau_2})}$ denote the weighted $\widehat{C}^{\ell+\beta}_\delta$-norm on $\Lambda_{\alpha_2,\tau_2}$ so that

$$|u|_{C^{\ell,\beta}_\delta(\Lambda_{\alpha_2,\tau_2})} := |(\widehat{\zeta_1})^{-\delta} u|_{0,\Lambda^0_{\alpha_2,\tau_2}} + \cdots + |(\widehat{\zeta_1})^{-\delta+\ell}\nabla^\ell u|_{0,\Lambda^0_{\alpha_2,\tau_2}} + |(\widehat{\zeta_1})^{-\delta+\ell+\beta}\nabla^\ell u|_{\beta,\Lambda^0_{\alpha_2,\tau_2}},$$

where the norms and derivatives correspond to the metric on $\Lambda_{\alpha_2,\tau_2}$. In this parametrization, the symmetries induced on functions of $\widehat{\Lambda}_{\alpha_1,\tau_1}$ and $\Lambda_{\alpha_2,\tau_2}$ are $u(s,\Theta) = u(-s,\Theta)$ and $u(s,\Theta) = u(s, B(\Theta))$ for all $B \in O(n)$. The corresponding spaces of functions invariant under these symmetries will be denoted by $\widehat{C}^{\ell,\beta}_{\delta,sym}(\widehat{\Lambda}_{\alpha_1,\tau_1})$ and $\widehat{C}^{\ell,\beta}_{\delta,sym}(\Lambda_{\alpha_2,\tau_2})$, respectively.

The proof of Lemma 10 and Lemma 16 below is adapted from that of Proposition 7 in [3].

Lemma 10 *Assume that $\delta \in (2-n, 0)$ is fixed and $n \geq 3$ and parametrize the neck regions of $\Lambda_{\alpha_i,\tau_i}$, $i = 1, 2$, as in (9). Then there is no non-trivial Jacobi field of the surface $\widehat{\Lambda}_{\alpha_1,\tau_1}$, which is bounded by a constant times $(\widehat{\zeta_1})^\delta$, and is invariant under the action of the symmetry $u(s,\Theta) = u(s, B(\Theta))$ for all $B \in O(n)$.*

Also, there is no non-trivial Jacobi field of the surface $\Lambda^0_{\alpha_2,\tau_2}$, which is bounded by a constant times $|\widehat{\zeta_2}|^\delta$, and is invariant under the action of the symmetry $u(s,\Theta) = u(s, B(\Theta))$ for all $B \in O(n)$.

Proof We only prove the second statement, as the first can be prove analogously. The proof is by contradiction. First we show that for sufficiently small τ_2^j the statement of Lemma 10 is true. Assume that for some sequence τ_2^j tending to 0, the Jacobi operator $\widehat{\mathcal{L}}_{\alpha_2,\tau_2^j} = \Delta_{\Lambda_{\alpha_2,\tau_2^j}} + |B_{\Lambda_{\alpha_2,\tau_2^j}}|^2 + n$ of $\Lambda_{\alpha_i,\tau_2^j}$ has non-trivial element w_2^j in the kernel which is invariant under the symmetries $w_2^j(s,\Theta) = w_2^j(s, B(\Theta))$ for all $B \in O(n)$, and is bounded by a constant times $(\widehat{\zeta_2})^\delta$. Let $q_2^j = \varepsilon'(\psi(s_2^j), \phi(s_2^j)\Theta_2^j)$, $q_2^j \in \Lambda^0_{\alpha_2,\tau_2}$ be the point where $(\zeta_0(q_2^j))^{-\delta} w_i(q_2^j) = 1$.

One can assume that $\Lambda_{\alpha_2,\tau_2^j}$ converges in a smooth enough sense to copies of the hyperspheres S_{α_2} with gluing points $R^k_{\alpha_2}(p)$ removed and the operators $\widehat{\mathcal{L}}_{\alpha_2,\tau_2^j}$ converge to the Jacobi operator on S_{α_2}.

Case 1. First assume that q_2^j converges, up to a subsequence, to some point in the hypersphere $S_{\alpha_2} \setminus \{p\}$. The use of elliptic estimates together with Arzela-Ascoli's theorem is enough to prove that, up to a subsequence, w_2^j converges to a limit function w_2^∞ uniformly on compact subsets of the hypersphere $S_{\alpha_2} \setminus \{p\}$. The limit function w_2^∞ satisfies $\mathcal{L}_{\alpha_i} w_2^\infty = 0$ on $S_{\alpha_2} \setminus \{p\}$ and

$$|(\mathrm{dist}\,(\cdot,\gamma))^{-\delta} w_2^\infty|_{L^\infty(S_{\alpha_2})} = 1. \tag{11}$$

Since we have assumed that $\delta > n-2$, the singularity is removable and hence w_2^∞ is smooth. Finally, since w_2^j is invariant under the action of the symmetry $w_2^\infty(s,\Theta) = w_2^\infty(s, B(\Theta))$ for all $B \in O(n)$. The limit function w_2^∞ is also invariant under the action of the symmetry $w_2^\infty(s,\Theta) = w_2^\infty(s, B(\Theta))$ for all $B \in O(n)$. This implies by Lemma 5 that $w_2^\infty = 0$, which is in contradiction with (11) and rules out this first case.

Case 2. Next assume that q_2^j in the neck region converges, up to a subsequence, to some point in the generalized catenoid $\varepsilon_2\Sigma$. The use of elliptic estimates together with Arzela-Ascoli's theorem is enough to prove that, up to a subsequence, w_2^j converges to a limit function w_2^∞ uniformly on compact subsets of the catenoid $\varepsilon_2\Sigma$. The limit function w_2^∞ satisfies $\mathcal{L}_{\varepsilon_2\Sigma} w_2^\infty = 0$ on $\varepsilon_2\Sigma$ and

$$|(\zeta_i)^{-\delta} w_\infty|_{L^\infty(\varepsilon_2\Sigma)} = 1. \tag{12}$$

Finally, since w_2^j is invariant under the action of the symmetry $w_2^\infty(s,\Theta) = w_2^\infty(s, B(\Theta))$ for all $B \in O(n)$. the limit function w_2^∞ is also invariant under the action of the symmetry $w_2^\infty(s,\Theta) = w_2^\infty(s, B(\Theta))$ for all $B \in O(n)$. Since we have assumed $\delta < 0$, Lemma 6 implies that $w_2^\infty = 0$, which is in contradiction with (12). This rules out this case.

Case 3. Next assume that q_2^j converges, up to a subsequence, to the point p. The use of elliptic estimates together with Arzela-Ascoli's theorem is enough to prove that, up to a subsequence, w_2^j converges to a limit function w_2^∞ uniformly on compact subsets of $S_{\alpha_2} \setminus \{p\}$. The limit function w_i^∞ satisfies $\Delta_{\mathbf{S}^n} w_2^\infty = 0$ on $S_{\alpha_2} \setminus \{p\}$ and

$$|(\mathrm{dist}\,(\cdot,\gamma))^{-\delta} w_2^\infty|_{L^\infty} \leq 1. \tag{13}$$

Since we have assumed that $\delta > n-2$, the singularity is removable and hence w_2^∞ is smooth. This implies that $w_2^\infty = 0$, which is in contradiction with (13) and rule out this second case.

Having ruled out all the possible cases, the proof for small τ_2 is complete.

Next let T be the set of τ for which the second statement of this lemma. We observe that, since $\Lambda_{\alpha_2,\tau_2^j} \to \Lambda_{\alpha_2,\tau_2}$ as $\tau_2^j \to \tau_2$, in a smooth enough sense, analogous arguments show that the T is an open subset of $\mathbf{R}$ and complete the proof. $\diamond$

On the other hand, for the cases $n = 2$ and future purposes, recall that any one-parameter family of isometries R_t of the ambient space X in which a CMC hypersurface Λ is situated gives rise to the element $\langle \frac{d}{dt} R_t\Big|_{t=0}, N_\Lambda \rangle$ in the kernel of the linearized mean curvature operator, where N_Λ is the unit normal vector field of Λ. To describe these Jacobi fields, parametrize the neck and transient regions of $\Lambda_{\alpha,\tau}$ by

$$(s,\Theta) \in \mathbf{R} \times \mathbf{S}^{n-1} \mapsto \varepsilon(\psi(s), \phi_\tau(s)\Theta),$$

where ϕ_τ is some smooth function depending on α, τ.

Lemma 11 *The Jacobi field $\widehat{J_1}$ of the hypersurface $\Lambda_{\alpha,\tau}$ arising from the rotation R_θ has odd symmetry and is bounded in $|s|$, which can be normalized to have the asymptotic expansion*

$$\widehat{J_1}(y^1) = 1 + \widetilde{J_1}(y^1), \quad \text{where} \lim_{|y^1|\to\tau^*/2} \widetilde{J_1}(y^1) = 0. \tag{14}$$

This follows since the Jacobi field J_1 is calculated by projecting a constant vector field along the normal vector field $N_{\Lambda_{\alpha,\tau}}$ of $\Lambda_{\alpha,\tau}$.

Lemma 12 *The Jacobi field $\widehat{J_0}$ of the hypersurface $\Lambda_{\alpha,\tau}$ arising from varying the parameter τ is bounded in s in dimension $n \geq 3$ and has linear growth in $|s| = |y^1|$ when the dimension $n = 2$. When $n = 2$, $\widehat{J_0}$ can be normalized to have the asymptotic expansion*

$$\widehat{J_0}(y^1) = -\gamma_* + |y^1 - \frac{\tau_*}{2}| + \widetilde{J_0}(y^1), \quad \text{where} \lim_{|y^1|\to\tau_*/2} \widetilde{J_0}(y^1) = 0, \tag{15}$$

where γ_1 is a positive constant.

Proof Recall that $\Lambda_{\alpha,\tau}$ is constructed via normal deformations of $\widetilde{\Lambda}_{\alpha,\tau}$. Namely, choose a function $f : \widetilde{\Lambda}_{\alpha,\tau} \to \mathbf{R}$ and then consider the deformation $\Phi_f : \widetilde{\Lambda}_{\alpha,\tau} \to \mathbf{S}^{n+1}$ given by $\Phi_f(q) := \exp_q(f(q)N(q))$ where $\exp_q$ is the exponential map at the point q and $N(q)$ is the outward unit normal vector field of $\widetilde{\Lambda}_{\alpha,\tau}$ at the point q. We have

$$\exp_q(f(q)N(q)) = q + f(q)N(q) + O(|f|^2), \quad q \in \widetilde{\Lambda}_{\alpha,\tau}.$$

Denote V^{dil} as the vector field generating dilation of the catenoid, namely,

$$V^{\text{dil}} := \sum_{k=1}^{n+1} y^k \frac{\partial}{\partial y^k} = \psi \frac{\partial}{\partial y^1} - \phi P_\Theta,$$

where P_Θ is the position vector field of the $\mathbf{R}^n$ factor evaluated at the point $\Theta \in \mathbf{S}^{n-1}$. We obtain in the neck region

$$\begin{aligned} J_0 &= \langle V^{\text{dil}}, N_{\Lambda_{\alpha,\tau}}\rangle + O\Big(|\nabla f|\Big|\tfrac{\partial|\widetilde{y}|}{\partial\tau}\Big|\Big)\langle N_{\varepsilon\Sigma}, N_{\Lambda_{\alpha,\tau}}\rangle + O(|f|)\langle N_{\varepsilon\Sigma}, \tfrac{\partial}{\partial\tau}N_{\Lambda_{\alpha,\tau}}\rangle \\ &= \langle V^{\text{dil}}, N_{\Lambda_{\alpha,\tau}}\rangle + O\Big(|\nabla f|\Big|\tfrac{\partial|\widetilde{y}|}{\partial\tau}\Big|\Big)\langle N_{\varepsilon\Sigma}, N_{\Lambda_{\alpha,\tau}}\rangle + O(|f|)\langle N_{\varepsilon\Sigma} - N_{\Lambda_{\alpha,\tau}}, \tfrac{\partial}{\partial\tau}N_{\Lambda_{\alpha,\tau}}\rangle, \end{aligned}$$

since $\langle N_{\Lambda_{\alpha,\tau}}, \frac{\partial}{\partial\tau}N_{\Lambda_{\alpha,\tau}}\rangle = \frac{\partial}{\partial\tau}\langle N_{\Lambda_{\alpha,\tau}}, N_{\Lambda_{\alpha,\tau}}\rangle = 0$.

Recall that

$$\langle V^{\text{dil}}, N_{\Lambda_{\alpha,\tau}}\rangle = s\tanh s - 1.$$

Therefore the first term on the right hand side has linear growth in $|s|$ when the dimension $n = 2$ and is bounded in $|s|$ in higher dimensions. Since the last two terms on the right hand side are clearly bounded, and since $\psi = y^1$ when $n = 2$ and ψ is bounded in higher dimensions, the statement of Lemma 12 follows. $\diamond$

Define a smooth, odd function $\chi : \Lambda_{\alpha,\tau} \to \mathbf{R}$ with the property

$$\chi(s) = 1 \text{ for } s \geq 1 \quad \text{and} \quad \chi(s) = -1 \text{ for } s \leq -1.$$

Now set $\widehat{K_1} := \chi\widehat{J_1}$ and define the linear subspace

$$\mathcal{D} := \text{span}_{\mathbf{R}}\{\widehat{J_0}, \widehat{K_1}\}. \tag{16}$$

From Lemma 7 and an argument analogous to that used to prove Lemma 10, we obtain the following.

Lemma 13 *Assume $\delta < 2$ is fixed and $n = 2$. The subspace of Jacobi fields of $\Lambda_{\alpha_2,\tau_2}$ which are bounded by a constant times $(\zeta_2)^\delta$ and are invariant under the action of symmetry $(s,\theta) \mapsto (s,\theta+\pi)$ is two-dimensional and spanned by the functions $\widehat{J_0}$ and $\widehat{K_1}$.*

The following two results are obvious and will be needed in **3.4.4.** to treat the two dimensional case.

Lemma 14 *For $n \geq 2$, one of the following cases must occur.*

Case 1. *There is a nontrivial Θ-independent Jacobi fields on Λ_{α,τ_0}.*

Case 2. *There is no nontrivial Θ-independent Jacobi fields on Λ_{α,τ_0} and there exists a singular solution $\widehat{G}$ of $\widehat{\mathcal{L}}_{\alpha,\tau_0}$ on $\Lambda_{\alpha,\tau_0} \setminus K^{-1}(\{y^1 = 0\})$.*

Lemma 15 *When $n = 2$, if there exists a Θ-independent singular solution $\widehat{G}(\|\widehat{y}(y^1)\|)$ of $\widehat{\mathcal{L}}_{\alpha,\tau_0}$ on $\Lambda_{\alpha,\tau_0} \setminus K^{-1}(\{y^1 = 0\})$ whose neck width is ε_0, then it has the asymptotic expansion*

$$\widehat{G}(\|\widehat{y}\|) = \gamma_0 + \gamma_1 \Big| \|\widehat{y}\| - \varepsilon_0 \Big| + \widetilde{G}(\|\widehat{y}\|^2), \quad \text{where } \widetilde{G}(\|\widehat{y}(y^1)\|) = O(\|\widehat{y}\|^2) \text{ for large } \|\widehat{y}\|, \tag{17}$$

γ_0, γ_1 are a non-zero constants and

$$\gamma_0 = O(\log |\|\widehat{y}\| - \varepsilon_0|), \quad \text{for small } y^1.$$

3.4.2 Linear Estimates for Jacobi Operators on $\Lambda_{\alpha_i,\tau_i}$, $i = 1, 2$

We will need the following linear estimates for Jacobi operators on the hypersurfaces $\widehat{\Lambda}_{\alpha_1,\tau_1}$ and $\Lambda_{\alpha_2,\tau_2}$.

Lemma 16 *Assume that $\delta \in (2-n, 0)$ is fixed and $n \geq 3$. Then each function $f \in \widehat{C}^{2,\beta}_\delta(\widehat{\Lambda}_{\alpha_1,\tau_1})$, satisfies the estimate*

$$|\widehat{\mathcal{L}}_{\alpha_1,\tau_1} u|_{\widehat{C}^{0,\beta}_{\delta-2}(\widehat{\Lambda}_{\alpha_1,\tau_1})} \geq C|u|_{\widehat{C}^{2,\beta}_{\delta}(\widehat{\Lambda}_{\alpha_1,\tau_1})},$$

and each function $f \in \widehat{C}^{2,\beta}_\delta(\Lambda_{\alpha_2,\tau_2})$ satisfies the estimate

$$|\widehat{\mathcal{L}}_{\alpha_i,\tau_i} u|_{\widehat{C}^{0,\beta}_{\delta-2}(\Lambda_{\alpha_2,\tau_2})} \geq C|u|_{\widehat{C}^{2,\beta}_{\delta}(\Lambda_{\alpha_2,\tau_2})},$$

where C is a constant (certainly independent of τ_*).

Proof We prove only the second statement, as the first can be proved by an analogous argument. Observe that Schauder's elliptic estimates imply that it is enough to prove that

$$|(\widehat{\zeta_2})^{2-\delta} \widehat{\mathcal{L}}_{\alpha_2,\tau_2} u|_{L^\infty(\Lambda_{\alpha_2,\tau_2})} \geq C|(\widehat{\zeta_2})^{-\delta} u|_{L^\infty(\Lambda_{\alpha_2,\tau_2})}, \tag{18}$$

where C is a constant independent of τ_*.

The proof is by contradiction. First we show that for sufficiently small τ_2, (18) is true. Assume that for some sequence τ_2^j tending to 0, there is a sequence of functions u_2^j defined on $\Lambda_{\alpha_2,\tau_2}$ each of which is invariant under the symmetries $u_2^j(s,\Theta) = u_2^j(s, B(\Theta))$ for all $B \in O(n)$, along with a sequence of linear operators satisfying the following estimates:

$$\lim_{j\to\infty} |(\widehat{\zeta_2})^{2-\delta} \widehat{\mathcal{L}}_{\alpha_2,\tau_2} u_2^j|_{L^\infty(\Lambda_{\alpha_2,\tau_2})} = 0 \quad \text{and} \quad |(\widehat{\zeta_2})^{-\delta} u_2^j|_{L^\infty(\Lambda_{\alpha_2,\tau_2})} = 1.$$

Let $q_i^j = \varepsilon_2(\psi(s_2^j), \phi(s_2^j)\Theta_2^j)$ be the point in $\Lambda^0_{\alpha_2,\tau_2}$ where $(\widehat{\zeta}_2(q_2^j))^{-\delta}|u_i^j(q_2^j)| = 1$.

Case 1. First assume that q_2^j converges, up to a subsequence, to some point in the hypersphere $S_{\alpha_2}\setminus\{p\}$. The use of elliptic estimates together with Arzela-Ascoli's theorem is enough to prove that, up to a subsequence, u_2^j converges to a limit function u_2^∞ uniformly on compact subsets of the catenoid $S_{\alpha_2}\setminus\{p\}$. The limit function u_2^∞ satisfies $\mathcal{L}_{\alpha_2}u_2^\infty = 0$ on $S_{\alpha_2}\setminus\{0\}$ and $|(\mathrm{dist}\,(\cdot,\gamma))^{-\delta}u_2^\infty|_{L^\infty(S_{\alpha_2})} = 1$. The reasoning used to rule out Case 1 in the proof of Lemma 10 also rules out this case.

Case 2. Next assume that q_2^j in the neck region converges, up to a subsequence, to some point in the generalized catenoid $\varepsilon_2\Sigma$. The use of elliptic estimates together with Arzela-Ascoli's theorem is enough to prove that, up to a subsequence, u_2^j converges to a limit function u_2^∞ uniformly on compact subsets of the catenoid $\varepsilon_2\Sigma$. The limit function u_2^∞ satisfies $\mathcal{L}_{\varepsilon_2\Sigma}u_2^\infty = 0$ on $\varepsilon_2\Sigma$ and $|(\widehat{\zeta}_2)^{-\delta}u_2^\infty|_{L^\infty(\varepsilon_2\Sigma)} = 1$. The reasoning used to rule out Case 2 in the proof of Lemma 10 also rules out this case.

Case 3. Next assume that q_2^j converges, up to a subsequence, to the point p. The use of elliptic estimates together with Arzela-Ascoli's theorem is enough to prove that, up to a subsequence, u_2^j converges to a limit function u_2^∞ uniformly on compact subsets of $S_{\alpha_2}\setminus\{p\}$. The limit function u_2^∞ satisfies $\Delta_{\mathbf{S}^n}u_2^\infty = 0$ on $S_{\alpha_2}\setminus\{p\}$ and $|(\mathrm{dist}\,(\cdot,\gamma))^{-\delta}u_2^\infty|_{L^\infty} \leq 1$. The reasoning used to rule out Case 3 in the proof of Lemma 10 also rules out this case.

Having ruled out all the possible cases, the proof for small τ_2 is complete.

Next let T be the set of τ for which the second statement of this lemma is true. We observe that analogous arguments show that the T is an open subset of $\mathbf{R}$ and complete the proof. $\diamond$

When $\delta \in (2-n, 0)$, in view of Lemma 10, the operator $\widehat{\mathcal{L}}_{\alpha_2,\tau_2} : \widehat{C}^{2,\beta}_{\delta,sym}(\Lambda_{\alpha_2,\tau_2}) \to \widehat{C}^{0,\beta}_{\delta-2,sym}(\Lambda_{\alpha_2,\tau_2})$ is injective for $\delta \in (2-n,0)$ and by duality it is surjective for $\delta \in (0, n-2)$.

Let $f \in \widehat{C}^{0,\beta}_{\delta-2}(\Lambda_{\alpha_2,\tau_2})$, with $\delta \in (2-n,0)$. Then $f \in \widehat{C}^{0,\beta}_{-\delta-2}(\Lambda_{\alpha_2,\tau_2})$ and therefore there exists $u \in \widehat{C}^{2,\beta}_{-\delta}(\Lambda_{\alpha_2,\tau_2})$ such that $\widehat{\mathcal{L}}_{\alpha_2,\tau_2}u = f$. Lemma 16 shows that in fact $u \in \widehat{C}^{2,\beta}_{\delta}(\Lambda_{\alpha_2,\tau_2})$ and leads to the first statement in the following lemma, while the second statement follows analogously.

Lemma 17 *The operator* $\widehat{\mathcal{L}}_{\alpha_2,\tau_2} : \widehat{C}^{2,\beta}_{\delta,sym}(\Lambda_{\alpha_2,\tau_2}) \to \widehat{C}^{0,\beta}_{\delta-2,sym}(\Lambda_{\alpha_2,\tau_2})$ *is bijective for* $\delta \in (2-n,0)$. *Also the operator* $\widehat{\mathcal{L}}_{\alpha_2,\tau_2} : \widehat{C}^{2,\beta}_{\delta,sym}(\widehat{\Lambda}_{\alpha_1,\tau_1}) \to \widehat{C}^{0,\beta}_{\delta-2,sym}(\widehat{\Lambda}_{\alpha_1,\tau_1})$ *is bijective for* $\delta \in (2-n,0)$.

In an analogous manner we obtain the following result from Lemma 13 and the argument used to prove Lemma 16.

Lemma 18 *When* $n = 2$, *the operator*

$$\widehat{\mathcal{L}}_{\alpha,\tau_2} : \widehat{C}^{2,\beta}_{\delta,sym}(\Lambda_{\alpha_2,\tau_2}) \oplus \mathcal{D} \to \widehat{C}^{0,\beta}_{\delta-2,sym}(\Lambda_{\alpha_2,\tau_2})$$

is surjective in the range $\delta \in (-1,0)$ *with one-dimensional kernel spanned by* $\widehat{J}_0$, *where* $\mathcal{D}$ *is defined in (18). Furthermore, there is a bounded right inverse mapping into* $C^{2,\beta}_{\delta,sym}(\Lambda_{\alpha_2,\tau_2}) \oplus \mathcal{D}_0$ *where* $\mathcal{D}_0 := \mathrm{span}_{\mathbf{R}}\{K_1\}$.

3.4.3 Proof of Proposition 8

As in [1], the patching argument requires partitions of unity for the various pieces of $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$. First, for any $\tau \in (\widetilde{\tau}_{**}, \tau_0)$, define $\widehat{\mathcal{E}}^k_{\tau_*}(\tau) := \widehat{\mathcal{E}}^k_{\tau_*} \setminus \widehat{S}(\tau)$, and define $\widehat{\mathcal{N}}^k_{\tau_*}(\tau)$ to be the disjoint component of $\widehat{S}(\tau)$ containing $\widehat{\mathcal{N}}^k_{\tau_*}$. Next, define the smooth, monotone cut-off functions

$$\chi^k_{ext,\tau}(q) := \begin{cases} 1, & q \in \widehat{\mathcal{E}}^k_{\tau_*}(2\tau) \\ 0, & q \in \widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*} \cap \widehat{S}_\tau \end{cases}$$

$$\chi^k_{neck,\tau}(q) := \begin{cases} 1, & q \in \widehat{\mathcal{N}}^k_{\tau_*}(\tau) \\ 0, & q \in \widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*} \setminus \widehat{S}_{2\tau} \end{cases},$$

so that $\sum_{k=0}^{N-1} \chi^k_{ext,\tau} + \sum_{k=0}^{N-1} \chi^k_{neck,\tau} = 1$. Secondly, define another set of cut-off functions

$$\eta^k_{ext}(q) := \begin{cases} 1, & q \in \widehat{\mathcal{E}}^k_{\tau_*} \\ 0, & q \in \widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*} \setminus [\widehat{\mathcal{T}}^{k-1,+}_{\tau'} \cup \widehat{\mathcal{E}}^k_{\tau_*} \cup \widehat{\mathcal{T}}^{k,-}_{\tau_*}] \end{cases}$$

$$\eta^k_{neck}(q) := \begin{cases} 1, & q \in \widehat{\mathcal{N}}^k_{\tau_*} \\ 0, & q \in \widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*} \setminus [\widehat{\mathcal{T}}^{k,-}_{\tau_*} \cup \widehat{\mathcal{N}}^k_{\tau_*} \cup \widehat{\mathcal{T}}^{k,+}_{\tau_*}] \end{cases}$$

so that $\sum_{k=0}^{N-1} \eta^k_{ext} + \sum_{k=0}^{N-1} \eta^k_{neck} = 1$. In addition, one can assume that these cut-off functions are invariant under all the desired symmetries. To begin the process, fix a small $\tau \in (\widetilde{\tau}_{**}, \tau_0)$ and write

$$f = \sum_{k=0}^{N-1} f^k_{ext} + \sum_{k=0}^{N-1} f^k_{neck}$$

where

$$f^k_{ext} := f \cdot \chi^k_{ext,\tau} \text{ and } f^k_{neck} := f \cdot \chi^k_{neck,\tau}.$$

Step 1. Local Solutions on the Neck Regions. Choose $\widehat{\tau} \in (\widetilde{\tau}_{**}, \tau_0)$. The set $\widehat{\mathcal{N}}^k_{\tau_*}(\widehat{\tau})$ is a perturbation of a compact subset of the surface $\Lambda_{\alpha_2,\tau_2}$. Consequently, the function f_{neck} and the equation $\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(u) = f_{neck}$ can be pulled back to a compact subset of the surface $\Lambda_{\alpha_2,\tau_2}$. View f_{neck} as a function of compact support on $\Lambda_{\alpha_2,\tau_2}$ carrying the metric induced from $\mathbf{R}^{n+1}$, and the equation that will be solved in this step is

$$\widehat{\mathcal{L}}_{\alpha_2,\tau_2}(u) = f_{neck},$$

where we recall that $\widehat{\mathcal{L}}_{\alpha_2,\tau_2}$ is the linearized mean curvature operator of $\Lambda_{\alpha_2,\tau_2}$ with this metric. In addition, $f_{neck,\widehat{\tau}}$ is invariant under the symmetries T and S^{01}_B for all $B \in O(n)$.

Lemma 17 provides us with the unique solution $u_{neck} \in \widehat{C}^{2,\beta}_\delta(\Lambda_{\alpha_2,\tau_2})$ of the equation

$$\widehat{\mathcal{L}}_{\alpha_2,\tau_2}(u_{neck}) = f_{neck},$$

which satisfies the estimate $|u_{neck}|_{\widehat{C}^{2,\beta}_\delta(\Lambda_{\alpha_2,\tau_2})} \le C|f_{neck}|_{\widehat{C}^{0,\beta}_{\delta-2}(\Lambda_{\alpha_2,\tau_2})}$, where C is a constant (certainly independent of τ_*). With slight abuse of notation, extend this function to all of $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$ by defining

$$\overline{u}_{neck} = \sum_{k=0}^{N-1} \chi^k_{neck,\widehat{\tau}} \cdot u_{neck}.$$

One has the estimate

$$|\overline{u}_{neck}|_{\widehat{C}^{2,\beta}_\delta(X)} \le C|f|_{\widehat{C}^{0,\beta}_{\delta-2}(X)}.$$

Step 2. Local Solutions on the Exterior Regions. Once a local solution $\overline{u}_{neck}$ is constructed in the previous step, we choose a small $\kappa \in (0,1)$ and define

$$\widehat{f}^k_{ext} := \chi^k_{ext,\kappa\widehat{\tau}}(f - \widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(\overline{u}_{neck})).$$

By symmetry, we can say that $\widehat{f}^k_{ext} := \widehat{f}_{ext}$ for each k. In addition, $\widehat{f}_{ext}$ is invariant under the symmetries T and S^{01}_B for all $B \in O(n)$.

The function $\widehat{f}_{ext}$ can be viewed as a function of compact support on $\widehat{\Lambda}_{\alpha_1,\tau_1}$. The equation that will be solved here is

$$\eta_{ext}\widehat{\mathcal{L}}_{\alpha_1,\tau_1}(u) = \widehat{f}_{ext},$$

where we recall $\widehat{\mathcal{L}}_{\alpha_1,\tau_1}$ is the linearized mean curvature operator of $\widehat{\Lambda}_{\alpha_1,\tau_1}$ when it carries the metric induced from $\mathbf{R}^{n+1}$.

For $\delta \in (2-n, 0)$, Lemma 16 and Lemma 17 provide us with the unique solution $u_{ext} \in C^{2,\beta}_{sym}(\widehat{\Lambda}_{\alpha_1,\tau_1})$ of the equation

$$\eta_{ext}\widehat{\mathcal{L}}_{\alpha,\tau}(u_{ext}) = \widehat{f}_{ext}.$$

which satisfies the estimate

$$|u_{ext}|_{\widehat{C}^{2,\beta}_\delta(\widehat{\Lambda}_{\alpha_1,\tau_1})} \le C_\kappa |f|_{\widehat{C}^{0,\beta}_{\delta-2}(\widehat{\Lambda}_{\alpha_1,\tau_1})}.$$

The point $\tau = \tau_*/2$ is at the neck region and therefore $u_{ext}(\tau_*/2) = 0$. By examining the Taylor expansion of u_{ext} at the points $\tau = \tau_*/2$, and invoking its symmetries, one finds that

$$|u_{ext}(\tau)| \le C_\kappa \left|\tau - \frac{\tau_*}{2}\right|^2 |f|_{C^{0,\beta}_{\delta-2}(\widehat{\Lambda}_{\alpha_1,\tau_1})}. \tag{19}$$

One can now extend u_{ext} to all of $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$, again with slight abuse of notation, as the function

$$\overline{u}_{ext} := \sum_{k=0}^{N-1} \eta^k_{ext} \cdot u_{ext}.$$

Step 3. Estimates and Convergence. Define $\overline{u} := \overline{u}_{neck} + \overline{u}_{ext}$. Then

$$\begin{aligned} \widehat{\mathcal{L}}_{\alpha_1,\tau_1,\tau_2,\tau_*}(\overline{u}) - f \quad &= \quad \sum_{k=0}^{N-1}\Big[\eta^k_{ext}(\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*} - \widehat{\mathcal{L}}_{\alpha_1,\tau_1})u_{ext} \\ &\quad + \chi^k_{neck,\kappa\widehat{\tau}}(\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*} - \widehat{\mathcal{L}}_{\alpha_2,\tau_2})u_{neck} + [\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}, \eta^k_{ext}]u_{ext}\Big], \end{aligned} \tag{20}$$

where $[\mathcal{L}, \eta](u) := \mathcal{L}(\eta u) - \eta\mathcal{L}(u)$. Each term in (20) will be shown to be small in the $C^{0,\beta}_{\delta-2}$ norm.

Begin with the first term in (20). Note that this term is supported in the transition region $\mathcal{T}_{\tau_*}$. Because the surfaces $\widehat{\Lambda}_{\alpha,\tau_1}$ and Λ_{α,τ_2} meet tangentially, we have

$$|\widehat{G}_{\alpha_2,\tau_2} - \widehat{G}_{\alpha_1,\tau_1}| = O(|y^1 - \tan(\tau_{**}/8)|^2) = O(|y^1|^2).$$

Hence, at the point $(y^1, G_{\alpha_1,\tau_1,\alpha_2,\tau_*}(y^1)\Theta) \in \widehat{\Sigma}_{\tau_*} \cap \{(y^1, \widehat{y}) : \tan(\tau_{**}/8) \le y^1 \le 2\tan(\tau_{**}/8)\}$, we have

$$\begin{aligned} G_{\alpha_1,\tau_1,\alpha_2,\tau_*}(y) \quad &= \widehat{G}_{\alpha_1,\tau_1,\tau_*} + \eta(O(|\tan(\tau_{**}/8) - y^1|^2)) = \widehat{G}_{\alpha_1,\tau_1,,\tau_*} + \eta(O(|y^1|^2)) \\ &= \widehat{G}_{\alpha_2,\tau_2} + (\eta-1)(O(|\tan(\tau_{**}/8) - y^1|^2)) = \widehat{G}_{\alpha_2,\tau_2} + (\eta-1)(O(|y^1|^2)). \end{aligned}$$

Set

$$E = \widehat{G}_{\alpha_2,\tau_2} - \widehat{G}_{\alpha_1,\tau_1,\tau_*},$$

and, for $i = 1, 2$, let Δ_i, B_i and $\mathcal{L}_{\alpha_i,\tau_i}$ be respectively the Laplacian, the second fundamental form and the linearized mean curvature operator of Λ_{α,τ_i} with respect to the metric coming from stereographic projection, and let Δ_*, B_* and $\mathcal{L}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ be respectively the Laplacian, the second fundamental form and the linearized mean curvature operator of $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ with respect to the metric coming from stereographic projection. One finds

$$\begin{aligned}&\mathcal{L}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(u) - \mathcal{L}_{\alpha_i,\tau_i}(u) = (\Delta_* - \Delta_i)(u) + (\|B_*\|^2 - \|B_i\|^2)u \\ &= O\Big((1-\eta(y^1/\tan(\tau_{**}/8)))E\Big)\cdot\|\nabla^2 u\| + O\Big(\Big\|\nabla[(1-\eta(y^1/\tan(\tau_{**}/8)))E]\Big\|\Big)\cdot\|\nabla u\| \\ &\quad + O\Big(\Big\|\nabla^2[(1-\eta(y^1/\tan(\tau_{**}/8)))E]\Big\|\Big)u,\end{aligned}$$

evaluated on an arbitrary function u supported in $\mathcal{T}_{\tau_*}$ and $i = 1, 2$. Then, by using Lemma 4 in [1], one obtains

$$\begin{aligned}&\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(u) - \widehat{\mathcal{L}}_{\alpha_i,\tau_i}(u) \\ &= (\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(u) - \mathcal{L}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(u)) - (\widehat{\mathcal{L}}_{\alpha_i,\tau_i}(u) - \mathcal{L}_{\alpha_i,\tau_i}(u)) + (\mathcal{L}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(u) - \mathcal{L}_{\alpha_i,\tau_i}(u)) \\ &= O\Big((1-\eta(y^1/\tan(\tau_{**}/8)))E\Big)\cdot\|\nabla^2 u\| + O\Big(\Big\|\nabla[(1-\eta(y^1/\tan(\tau_{**}/8)))E]\Big\|\Big)\cdot\|\nabla u\| \\ &\quad + O\Big(\Big\|\nabla^2[(1-\eta(y^1/\tan(\tau_{**}/8)))E]\Big\|\Big)u \\ &= O((\tau_{**})^2)\|\nabla^2 u\| + O(\tau_{**})\cdot\|\nabla u\| + O(1)u;\end{aligned}$$

therefore, for an arbitrary function u supported in $\mathcal{T}_{\tau_*}$ and $i = 1, 2$,

$$\begin{aligned}&|(\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*} - \widehat{\mathcal{L}}_{\alpha_i,\tau_i})(u)|_{\widehat{C}^0_{\delta-2}(\mathcal{T}^k_{\tau_*})} \\ &\quad = O((\tau_{**})^{4-\delta})\|\nabla^2 u\| + O((\tau_{**})^{3-\delta})\cdot\|\nabla u\| + O((\tau_{**})^{2-\delta})u \\ &\quad \le C(\tau_{**})^2|u|_{\widehat{C}^{2,\beta}_\delta(\mathcal{T}_{\tau_*})}.\end{aligned}$$

Analogous Hölder coefficient estimates can be found in the same way. In the end we find

$$|\eta^k_{ext}(\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*} - \widehat{\mathcal{L}}_{\alpha_1,\tau_1})(u_{ext})|_{\widehat{C}^{0,\beta}_{\delta-2}(\mathcal{T}^k_{\tau_*})(X)} \le C(\tau_{**})^2|f|_{\widehat{C}^{0,\beta}_{\delta-2}}$$

and

$$|\chi^k_{neck,\kappa\rho}(\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*} - \widehat{\mathcal{L}}_{\alpha_2,\tau_2})(u_{neck})|_{\widehat{C}^{0,\beta}_{\delta-2}(\mathcal{T}^k_{\tau_*})(X)} \le C(\tau_{**})^2|f|_{\widehat{C}^{0,\beta}_{\delta-2}}.$$

Finally, the estimate of the last term, namely

$$[\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}, \eta^k_{ext}]u_{ext}|_{\widehat{C}^{0,\beta}_{\delta-2}(\mathcal{T}^k_{\tau_*})} \le C(\tau_{**})^2,$$

follows from (19), together with the fact that the $C^{0,\beta}_{\delta-2}$ norm of the coefficients of $[\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}, \eta_{neck}]$ and $[\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}, \eta_{ext}]$ are $O(1)$ by definition of the cut-off functions.

From the above analysis one obtains a function $\overline{u}$ satisfying the estimate $|\overline{u}|_{C^{2,\beta}_\delta(X)} \le C_\kappa|f|_{C^{0,\beta}_{\delta-2}(X)}$ as well as

$$|\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(\overline{u}) - f| \le C(\tau_{**})^2|f|_{C^{0,\beta}_{\delta-2}(X)}.$$

where C is bounded independent of τ_* and κ. Since the constant in front of $|f|_{C^{0,\beta}_{\delta-2}(X)}$ can be made as small as desired by choosing sufficiently small τ_*, a straightforward iteration argument now proves the existence of a solution $u \in C^{2,\beta}_\delta(\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*})$ of the equation $\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*} = f$ satisfying the estimate $|u|_{C^{2,\beta}_\delta(X)} \le C|f|_{C^{0,\beta}_{\delta-2}(X)}$.

3.4.4 Proof of Proposition 9

The significant difference between this proof and the preceeding one is how the local solutions are found on the neck.

Lemma 17 provides us with a solution $u_{neck} \in \widehat{C}^{2,\beta}_{\delta,sym}(\Lambda_{\alpha_2,\tau_2}) \oplus \mathcal{D}_0$ of the equation $\widehat{\mathcal{L}}_{\Lambda_{\alpha_2,\tau_2}}(u_{neck}) = f_{neck}$, which can be decomposed as

$$u_{neck} = v_{neck} + a_1 K_1 \quad \text{where } v_{neck} \in \widehat{C}^{2,\beta}_{\delta,sym}(\Lambda_{\alpha_2,\tau_2}) \text{ and } a_1 \in \mathbf{R}$$

and satisfies

$$|v_{neck}|_{\widehat{C}^{2,\beta}_{\delta}(\Lambda_{\alpha_2,\tau_2})} + (\tau_*)^{-\delta}|a_1| \le C|f|_{\widehat{C}^{0,\beta}_{\delta-2}(\Lambda_{\alpha_2,\tau_2})} \le C|f|_{\widehat{C}^{0,\beta}_{\delta-2}(X)},$$

for some constant C independent of τ_*.

Now if Case 1 in Lemma 14 occurs, let $\widehat{G}$ be the nontrivial Θ-independent Jacobi field on $\widehat{\Lambda}_{\alpha_1,\tau_1}$, while if Case 2 in Lemma 14 occurs, let $\widehat{G}$ be the singular solution $\widehat{G}(y^1)$ of $\widehat{\mathcal{L}}_{\alpha_1,\tau_1}$ on $\widehat{\Lambda}_{\alpha_1,\tau_1} \setminus \partial\widehat{S}(\tau_*/2)$ with even symmetry, which has the asymptotic expansion (17), where γ_1 is a non-zero constant and $\gamma_0 = O(\log||y^1| - \tan(\tau_*/8)|)$, for $|y^1|$ close to $\tan(\tau_*/8)$. The function u_{neck} can thus be extended to $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_{**}}$ by prescribing

$$\overline{u}_{neck} := \sum_{k=0}^{N-1} \chi^k_{neck,\rho} v_{neck} + \sum_{k=0}^{N-1} (\eta^k_{neck}(b_1 J_0 + a_1 K_1) + \eta^k_{ext} c_1 \widehat{G})$$

where the constants b_1 and c_1 are chosen to ensure matching in the constant and linear terms of asymptotic expansions of u_{neck} and $\widehat{G}$, i.e., $b_1 = a_1\gamma_1/(\gamma_0 + \gamma_*\gamma_1)$ and $c_1 = a_1/(\gamma_0 + \gamma_*\gamma_1)$. In the end, this solution satisfies the estimates

$$|\overline{u}_{neck}|_{\widehat{C}^{0,\beta}_{\delta-2}(\widehat{\Lambda}_{\alpha_1,\tau_1})} \le (\tau_*)^{\delta}|f|_{\widehat{C}^{0,\beta}_{\delta-2}(\widehat{\Lambda}_{\alpha_1,\tau_1})}$$

as well as

$$|\chi^k_{neck,\kappa\rho}(\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*} - f_{neck})|_{\widehat{C}^{0,\beta}_{\delta-2}(\widehat{\Lambda}_{\alpha_1,\tau_1})} \le C(\tau_{**})^2(\tau_*)^{\delta}|\log\tau_*||f|_{\widehat{C}^{0,\beta}_{\delta-2}(\widehat{\Lambda}_{\alpha_1,\tau_1})}.$$

Note that $\delta > -2s_*$ is required to ensure that the quantity in front of $|f|_{\widehat{C}^{0,\beta}_{\delta-2}(\widehat{\Lambda}_{\alpha_1,\tau_1})}$ small as $\tau_* \to 0$.

The next step is to define $\widehat{f_{ext}}$ in the same way as before and find the local solution on the exterior region of $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$. Then define the extended function

$$\overline{u}_{ext} := \sum_{k=0}^{N-1} \eta^k_{ext} \cdot u_{ext}.$$

The remainder of the analysis is the same as before, and leads to a good approximation solution of the equation $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*} = f$. The solution procedure can be iterated to yield an exact solution. The result is a solution satisfying the bound $|u|_{\widehat{C}^{0,\beta}_{\delta-2}(\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*})} \le (\tau_*)^{\delta}|f|_{\widehat{C}^{0,\beta}_{\delta-2}(\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*})}$. $\diamond$

3.5 Nonlinear Estimates

The proof of Main Theorem requires two more estimates in addition to the linear estimates: it is necessary to show that $\Phi_{\alpha_1,\tau_1,\alpha_2,\tau_*}(0)$ is small in the $C^{0,\beta}_{\delta-2}$-norm; and it is necessary to show that $D\Phi_{\alpha_1,\tau_1,\alpha_2,\tau_*}(f) - \widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ can be made to have small $C^{2,\beta}_{\delta}$-operator norm if f is chosen sufficiently small in the $C^{2,\beta}_{\delta}$ norm.

Poposition 19 *If $\tau_* > 0$ is sufficiently small, then there exists a constant C independent of τ_* so that*

$$|\Phi_{\alpha_1,\tau_1,\alpha_2,\tau_*}(0) - H_\alpha|_{\widehat{C}^{0,\beta}_{\delta-2}(X)} \le C(\tau_*)^{(2-\delta)s_*}.$$

Poposition 20 *If $\tau_* > 0$ is sufficiently small and $f \in \widehat{C}^{2,\beta}_\delta(\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*})$ has sufficiently small $C^{2,\beta}_\delta$ norm, then there exists a constant C independent of τ_* so that*

$$|D\Phi_{\alpha_1,\tau_1,\alpha_2,\tau_*}(f)(u) - \widehat{\mathcal{L}}_{\alpha,\tau_1,\tau_2,\tau_*}(u)|_{\widehat{C}^{0,\beta}_{\delta-2}(X)} \le C(\tau_*)^{\delta-1}|f|_{C^{2,\beta}_\delta(X)}|u|_{\widehat{C}^{2,\beta}_\delta(X)}$$

for any function $u \in \widehat{C}^{2,\beta}_\delta(\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*})$.

As in Proposition 15 in [1], Proposition 20 follows from the scaling property of the mean curvature operator.

Proof of Proposition 19. We only need to estimate the deviation of the mean curvature from H_α in the neck and transition regions. Since $|H_{\alpha_1} - H_{\alpha_2}| = O(\tau_*)$, it is clear that we have

$$|H(\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}) - H_{\alpha_1}|_{\widehat{C}^{0,\beta}_{\delta-2}(\widehat{N}_{\tau_*})} \le C(\tau_*)^{2-\delta}.$$

It remains to estimate the mean curvature in the transition region $\mathcal{T}_{\tau_*}$ whose image under the stereographic projection is the graph of the function $G := \widehat{G}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ defined in (7).

Let $y = \|\widehat{y}\|$. For a smooth surface $\Lambda = \{(F(y), \widehat{y}) : \tau_{**}/2 \le F(y) \le 2\tau_{**}\}$ defined by a smooth function $F(y)$, its mean curvature at the point $(F(y), \widehat{y})$ is given by

$$H(\Lambda)(y) = \frac{1 + y^2 + F(y)^2}{2}\left(\frac{\ddot{F}(y)}{(1 + \dot{F}(y)^2)^{3/2}} + \frac{(n-1)\dot{F}(y)}{(1 + \dot{F}(y)^2)^{1/2}}\right) + \frac{n(F(y) - y\dot{F}(y))}{(1 + \dot{F}(y)^2)^{1/2}}. \tag{21}$$

Let $y(y^1) = \|\widehat{y}(y^1)\| = \widehat{G}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(y^1)$ and $\widehat{F} : [(y(\tau_{**}/2), y(2\tau_{**})] \to (\tau_{**}/2, 2\tau_{**})$, $\widehat{F} = \widehat{G}^{-1}_{\alpha,\tau_1,\tau_2,\tau_*}$. Then

$$\widehat{F}(y) = c_n + C_n L_n(y) + \eta(O(|y - y(\tau_{**})|^2)),$$

where $L_n(y)$ defines a surface $\Lambda_n = \{(L_n(y), \widehat{y}) : \tau_{**}/2 \le F(y) \le 2\tau_{**}\}$ such that taking $F = L_n$ in (21) we obtain $H(\Lambda_n)(y) = H_{\alpha_1}$ and c_n, C_n are constants depending on n. Using this, we see that

$$|H(\widehat{\Lambda}_{\alpha_1,\tau_1,\tau_2,\tau_*}) - H_{\alpha_1}|_{\widehat{C}^{0}_{\delta-2}(\widehat{\mathcal{T}}_{\tau_*})} \le C(\tau_{**})^{2-\delta}$$

in the region $y \in (y(\tau_{**}/2), y(2\tau_{**}))$.

The Hölder coeffient estimate in this region follows similarly and takes the same form, which yields Proposition 9. $\diamond$

3.6 Proof of Main Theorem (Theorem 2)

Observe that by Proposition 8 and Proposition 9 the linearization $\widehat{\mathcal{L}}_{\alpha,\tau_1,\alpha_2,\tau_*}$ is *injective* on $C^{2,\beta}_\delta(X)$. But $\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*} - \Delta$ is a compact operator so $\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ has the same index as Δ on $C^{2,\beta}_\delta(X)$. By self-adjointness, this index is zero, so that $\widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ must be *surjective* as well. One has the estimate

$$|\widehat{\mathcal{L}}^{-1}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(f)|_{\widehat{C}^{2,\beta}_\delta(X)} \le C_{\tau_*}|f|_{\widehat{C}^{0,\beta}_{\delta-2}(X)},$$

where $C_{\tau_*} = O((\tau_*)^\delta)$ in dimension 2 and $C_{\tau_*} = O(1)$ in higher dimensions.

Now in order to apply the Banach space inverse function theorem (Theorem 2), we have to choose t so that whenever $|f|_{\widehat{C}^{2,\beta}_\delta(X)} \le t$ there holds

$$|D\Phi_{\alpha,\tau_*}(f)(u) - \widehat{\mathcal{L}}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(u)|_{\widehat{C}^{0,\beta}_{\delta-2}(X)} \le \frac{1}{2C_{\tau_*}}|u|_{\widehat{C}^{2,\beta}_\delta(X)}, \tag{22}$$

for any $u \in C^{2,\beta}_{\delta}(\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*})$; therefore, in consideration of Proposition 20, we choose

$$|f|_{\widehat{C}^{2,\beta}_{\delta}(X)} \leq t, \quad \text{where } t = \begin{cases} O((\tau_*)^{1-2\delta}) & \text{in dimension } n = 2 \\ O((\tau_*)^{1-\delta}) & \text{in higher dimensions,} \end{cases}$$

so that (22) is true. Then by the inverse function theorem, a solution of $f := f_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ of the deformation problem can be found if $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ has been so constructed that

$$|\widehat{\Phi}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(0) - H_{\alpha_1}|_{\widehat{C}^{0,\beta}_{\delta-2}(X)} \leq \frac{t}{2C_{\tau_*}}. \tag{23}$$

Since Proposition 19 asserts that

$$|\widehat{\Phi}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(0) - H_{\alpha_1}|_{\widehat{C}^{0,\beta}_{\delta-2}(X)} = O((\tau_{**})^{(2-\delta)}),$$

we see by (6) that the bound (23) is true as long as τ_* is made sufficiently small and δ is chosen appropriately.

As a further consequence of these estimates, the Banach space inverse function theorem asserts that the solution of the equation $\widehat{\Phi}_{\alpha_1,\tau_1,\alpha_2,\tau_*}(f_{\alpha_1,\tau_1,\alpha_2,\tau_*}) = 0$ satisfies the estimate

$$|f_{\alpha_1,\tau_1,\alpha_2,\tau_*}|_{\widehat{C}^{2,\beta}_{\delta}(X)} = O(C_{\tau_*}(\tau_{**})^{(2-\delta)})$$

which is much smaller than ε_2. Therefore the size of the perturbation of $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ created by the normal deformation of magnitude $f_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ is much smaller than the width of $\widehat{\Lambda}_{\alpha,\tau_1,\alpha_2,\tau_*}$ at its narrowest points, i.e., in the neck regions where the width is $O(\varepsilon)$. Thus $\widehat{\Lambda}_{\alpha_1,\tau_1,\alpha_2,\tau_*}$ remains embedded under this normal deformation. This completes the proof of Theorem 2.

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