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The Application of Bifurcation Method to Klein-Gordon-Zakharov Equations

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Abstract: Bifurcation method of dynamical systems is employed to study the Klein-Gordon-Zakharov equations. Under some parameter conditions, some explicit expressions of solutions for the equation are obtained. These solutions contain solitary wave solutions, blow-up solutions, periodic solutions, periodic blow-up solutions and kink-shaped solutions.

Key Words: Klein-Gordon-Zakharov equations; Exact solutions; Bifurcation method

1. INTRODUCTION

In recent years, solving nonlinear evolution equations has become a valuable task in many scientific areas including applied mathematics as well as the physical sciences and engineering. For this purpose, some accurate methods have been presented, such as Inverse scattering transform method[1], Bäcklund transformation method[2], Jacobi elliptic function method[3], F-expansion method[4], Hirota's bilinear method[5], the extended hyperbolic functions method[6, 7], Homotopy perturbation method[8], Bifurcation method[9–11] and so on.

In this paper, we consider the following Klein-Gordon-Zakharov equations (KGZ-equations in short):

$$u_{tt} - u_{xx} + u + \alpha nu = 0, \tag{1}$$

$$n_{tt} - n_{xx} = \beta(|u|^2)_{xx}, \tag{2}$$

with u a complex function and n a real function, where α, β are two nonzero real parameters. Recently, there have been many works on the qualitative research of the global solutions for the KGZ-equations (1)-(2)[12–15]. Chen Lin considered orbital stability of solitary waves for the KGZ-equations in [16]. Based on the extended hyperbolic functions method, Shang Yadong et al. obtain the multiple exact explicit solutions of the KGZ-equations in[7].

In this paper, we employ the bifurcation method and qualitative theory of dynamical systems[9–11] to investigate the KGZ-equations (1)-(2), and we obtain some explicit expressions of solutions for the KGZ-equations (1)-(2). These solutions contain solitary wave solutions, blow-up solutions, periodic solutions, periodic blow-up solutions and kink-shaped solutions.

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This paper is organized as follows: In Section 2, we present our main results. In Section 3, we give the theoretical derivation for our main results. A short conclusion will be given in Section 4.

2. MAIN RESULTS

In this section, we will give the explicit exact solutions of the KGZ-equations (1)-(2). To begin with, let us consider the following transformation

$$u(x,t) = \varphi(x,t)e^{i(kx+\omega t)}, \tag{3}$$

where $\varphi(x,t)$ is a real-valued function, k and ω are two real constants. Substituting (3) into (1)-(2), we get

$$\varphi_{tt} - \varphi_{xx} + (k^2 - \omega^2 + 1)\varphi + \alpha n\varphi = 0, \tag{4}$$

$$\omega\varphi_t - k\varphi_x = 0, \tag{5}$$

$$n_{tt} - n_{xx} = \beta(\varphi^2)_{xx}. \tag{6}$$

In view of (5), we suppose

$$\varphi(x,t) = \varphi(\xi) = \varphi(\omega x + kt). \tag{7}$$

Substituting (7) into (4), we get

$$n(x,t) = \frac{(\omega^2 - k^2)\varphi''(\xi)}{\alpha\varphi(\xi)} + \frac{\omega^2 - k^2 - 1}{\alpha}. \tag{8}$$

Therefore, we can also suppose

$$n(x,t) = \psi(\xi) = \psi(\omega x + kt). \tag{9}$$

Substituting (9) into (6) and integrating twice with respect to ξ, we get

$$n(x,t) = \psi(\xi) = \frac{\beta\omega^2\varphi^2(\xi)}{k^2 - \omega^2} + C, \tag{10}$$

where C is an integration constant. Substituting (10) into (8), we get

$$\varphi''(\xi) - l\varphi(\xi) - m\varphi^3(\xi) = 0, \tag{11}$$

where $l = -\frac{k^2-\omega^2+1+\alpha C}{k^2-\omega^2}$, $m = -\frac{\alpha\beta\omega^2}{(k^2-\omega^2)^2}$.

Now we state our main results. To relate conveniently, let

$$\varphi_* = \sqrt{\frac{-l + \sqrt{l^2 - 4mh}}{m}}, \tag{12}$$

$$\varphi^* = \sqrt{\frac{l + \sqrt{l^2 - 4mh}}{m}}, \tag{13}$$

$$\bar{\varphi}_* = \sqrt{\frac{-l - \sqrt{l^2 - 4mh}}{m}}, \tag{14}$$

$$\bar{\varphi}^* = \sqrt{\frac{l - \sqrt{l^2 - 4mh}}{m}}, \tag{15}$$

$$\varphi_r = \sqrt{-\frac{l}{m}}, \tag{16}$$

$$\bar{\varphi}_r = \sqrt{-\frac{2l}{m}}, \tag{17}$$

$$\bar{\varphi}_0 = \sqrt{-\frac{2l}{m} - \varphi_0^2}, \tag{18}$$

where h and φ_0 will be given later. Using the notations above, our main results are stated in Propositions 1 and 2.

For easy of exposition, we only give the solution $\varphi(x,t)$. By (3) and (10), one can obtain the solutions $u(x,t)$ and $n(x,t)$ of the KGZ-equations (1)-(2) easily.

Proposition 1. For given constants l, m and h, when $lm > 0$ the KGZ-equations (1)-(2) has the following exact solutions:

(1) If $l > 0$, $m > 0$ and $h = 0$, we get two blow-up solutions

$$\varphi_{1^\pm}(x,t) = \pm\sqrt{\frac{2l}{m}}\mathrm{csch}[\sqrt{l}(\omega x + kt)]. \tag{19}$$

(2) If $l > 0$, $m > 0$ and $h < 0$, we get two blow-up solutions

$$\varphi_{2^\pm}(x,t) = \pm\sqrt{\frac{(\varphi_*)^2 + (\varphi^*)^2}{\mathrm{sn}^2[\sqrt{\frac{m}{2}}\sqrt{(\varphi_*)^2 + (\varphi^*)^2}(\omega x + kt)]} - (\varphi^*)^2}. \tag{20}$$

(3) If $l < 0$, $m < 0$ and $h > 0$, we get one periodic solution

$$\varphi_3(x,t) = \bar{\varphi}_*\mathrm{cn}\left[\sqrt{-\frac{m}{2}}\sqrt{(\bar{\varphi}_*)^2 + (\bar{\varphi^*})^2}(\omega x + kt)\right]. \tag{21}$$

Proposition 2. For given constants l, m and h, when $lm < 0$ the KGZ-equations (1)-(2) has the following exact solutions:

(1) If $l > 0$, $m < 0$ and $h = 0$, we get two solitary wave solutions

$$\varphi_{4^\pm}(x,t) = \pm\sqrt{-\frac{2l}{m}}\mathrm{sech}[\sqrt{l}(\omega x + kt)]. \tag{22}$$

(2) If $l > 0$, $m < 0$ and $h < 0$, we get two periodic wave solutions

$$\varphi_{5^\pm}(x,t) = \pm\frac{\bar{\varphi}_0\varphi_0}{\sqrt{(\bar{\varphi}_0)^2 - [(\bar{\varphi}_0)^2 - (\varphi_0)^2]\mathrm{sn}\left[\sqrt{-\frac{2l}{m}}\bar{\varphi}_0|\omega x + kt|\right]}}. \tag{23}$$

(3) If $l > 0$, $m < 0$ and $h > 0$, we get one periodic solution

$$\varphi_6(x,t) = \bar{\varphi}_*\mathrm{cn}\left[\sqrt{-\frac{m}{2}}\sqrt{(\bar{\varphi}_*)^2 + (\bar{\varphi^*})^2}(\omega x + kt)\right]. \tag{24}$$

(4) If $l < 0$, $m > 0$ and $h = H(\varphi_r, 0)$, we get two kink-shaped solutions

$$\varphi_{7^\pm}(x,t) = \pm\varphi_r\tanh\left[\varphi_r\sqrt{\frac{m}{2}}(\omega x + kt)\right], \tag{25}$$

and two blow-up solutions

$$\varphi_{8^{\pm}}(x,t) = \pm\varphi_r \coth\left[\varphi_r\sqrt{\frac{m}{2}}(\omega x + kt)\right]. \tag{26}$$

(5) If $l < 0$, $m > 0$ and $h < H(\varphi_r, 0)$, we get one periodic solution

$$\varphi_9(x,t) = \varphi_0 \mathrm{sn}\left[\bar{\varphi_0}\sqrt{\frac{m}{2}}(\omega x + kt)\right], \tag{27}$$

and two blow-up solutions

$$\varphi_{10^{\pm}}(x,t) = \pm\frac{\bar{\varphi_0}}{\mathrm{sn}\left[\bar{\varphi_0}\sqrt{\frac{m}{2}}(\omega x + kt)\right]}. \tag{28}$$

3. THE THEORETIC DERIVATIONS OF MAIN RESULTS

In this section, we will give the theoretic derivations of our main results. Via (11), we establish the following planar system:

$$\begin{cases} \dfrac{d\varphi}{d\xi} = y, \\ \dfrac{dy}{d\xi} = l\varphi + m\varphi^3, \end{cases} \tag{29}$$

which admits the following first integral

$$H(\varphi, y) = \frac{1}{2}y^2 - \frac{1}{2}\varphi^2(l + \frac{m}{2}\varphi^2) = h, \tag{30}$$

where h is the constant of integration. Thus from (30), it follows

$$y = \pm\sqrt{\varphi^2(l + \frac{m}{2}\varphi^2) + 2h}. \tag{31}$$

Substituting (31) into $\frac{d\varphi}{d\xi} = y$, we obtain

$$\frac{d\varphi}{\sqrt{\varphi^2(l + \frac{m}{2}\varphi^2) + 2h}} = \pm d\xi. \tag{32}$$

Under general conditions it is difficult to integrate the left of (32). But under some conditions the integral can be finished. Thus the solutions of the KGZ-equations (1)-(2) can be obtained.

Now we discuss the bifurcation phase portraits of system (29), suppose

$$f(\varphi) = l\varphi + m\varphi^3. \tag{33}$$

According to system (29), we know that in $\varphi - y$ plane all the singular points of system (29) are on φ-axis. Thus to study the distribution of singular points of system (29), we only need to investigate the fixed points of (33). About (33), we know

(1) When $lm > 0$, then $f(\varphi)$ only one fixed point.

(2) When $lm < 0$, then $f(\varphi)$ has two fixed points.

Thus, we can easily draw the graphics of function $f(\varphi)$ in Figure 1 (when $lm > 0$) and Figure 2 (when $lm < 0$).

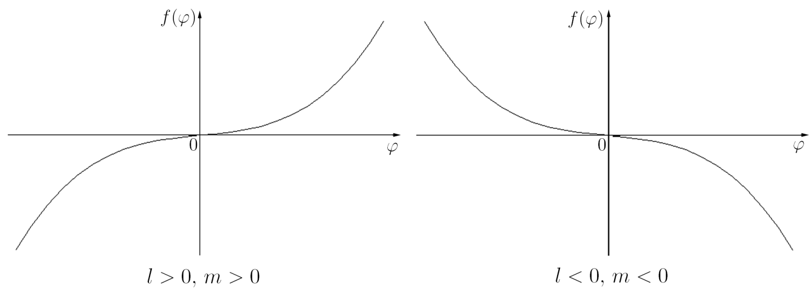

Figure 1: The graphics of function $f(\varphi)$ when $lm > 0$

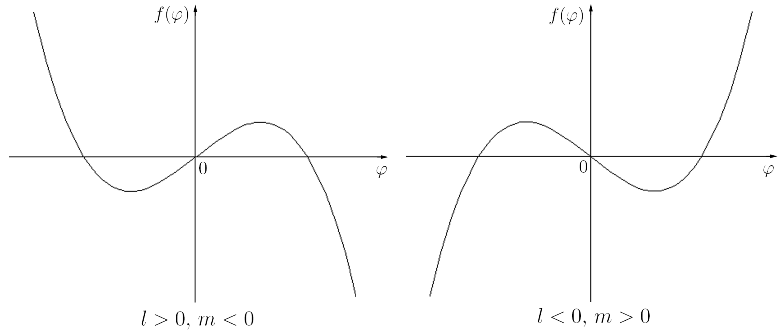

Figure 2: The graphics of function $f(\varphi)$ when $lm < 0$

Suppose that $(\varphi, 0)$ is a singular point of system (29), then at the singular point $(\varphi, 0)$ the characteristic values of the linearized system of system (29) is

$$\lambda_{1,2} = \pm\sqrt{f'(\varphi)}. \tag{34}$$

According to the qualitative theory of dynamical systems, we obtain the following conclusions

(1) If $f'(\varphi) < 0$, $(\varphi, 0)$ is a center point.

(2) If $f'(\varphi) > 0$, $(\varphi, 0)$ is a saddle point.

(3) If $f'(\varphi) = 0$, $(\varphi, 0)$ is a degenerate saddle point.

Therefore, we obtain the phase portraits of system (29) in Figure 3 (when $lm > 0$) and Figure 4 (when $lm < 0$).

Now we will derive the Proposition 1.

(1) If $l > 0$, $m > 0$ and $h = 0$, from the phase portrait in Figure 3, the orbits connected at the saddle point $(0, 0)$ are given as

$$y = \pm\varphi\sqrt{l + \frac{m}{2}\varphi^2}. \tag{35}$$

Substituting (35) into $\frac{d\varphi}{d\xi} = y$ and integrating them along the orbits, it follows that

$$\int_{\varphi}^{+\infty} \frac{1}{s\sqrt{l + \frac{m}{2}s^2}} ds = |\xi|, \quad (\varphi \geq 0), \tag{36}$$

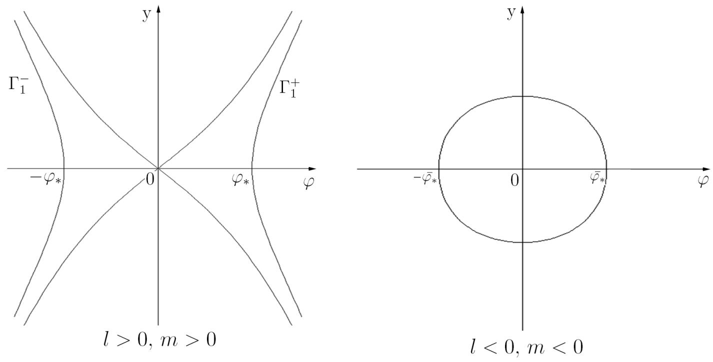

Figure 3: The phase portraits of system (29) when $lm > 0$

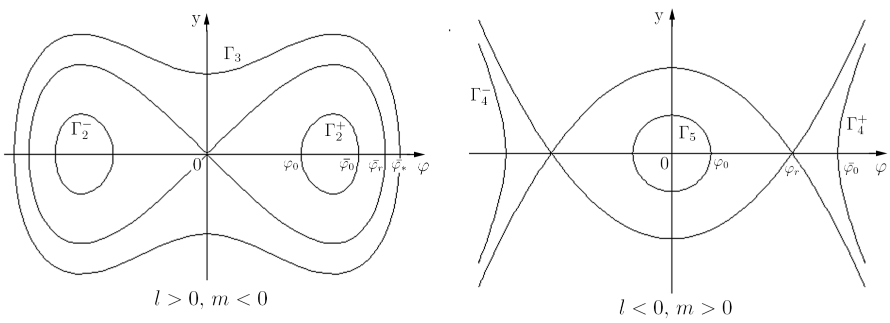

Figure 4: The phase portraits of system (29) when $lm < 0$

and

$$\int_{\varphi}^{-\infty} \frac{1}{s\sqrt{l+\frac{m}{2}s^2}}\mathrm{d}s = |\xi|, \quad (\varphi \leq 0). \tag{37}$$

From (36) and (37), we have

$$\varphi = \pm\sqrt{\frac{2l}{m}}\operatorname{csch}\sqrt{l}\xi. \tag{38}$$

Noting that $\varphi = \varphi(\xi)$ and $\xi = \omega x + kt$, we get two blow-up solutions $\varphi_{1^\pm}(x, t)$ as (19).

(2) If $l > 0$, $m > 0$ and $h < 0$, from the phase portrait in Figure 3, the orbits $\Gamma_1^\pm$ are given as

$$\mathrm{y} = \pm\sqrt{\frac{m}{2}(\varphi^2 - (\varphi_*)^2)(\varphi^2 + (\varphi^*)^2)}. \tag{39}$$

Substituting (39) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \mathrm{y}$ and integrating them along the orbits $\Gamma_1^\pm$, it follows that

$$\int_{\varphi}^{+\infty} \frac{1}{\sqrt{(s^2 - (\varphi_*)^2)(s^2 + (\varphi^*)^2)}}\mathrm{d}s = \sqrt{\frac{m}{2}}|\xi|, \quad (\varphi \geq 0), \tag{40}$$

and

$$\int_{\varphi}^{-\infty} \frac{1}{\sqrt{(s^2 - (\varphi_*)^2)(s^2 + (\varphi^*)^2)}}\mathrm{d}s = \sqrt{\frac{m}{2}}|\xi|, \quad (\varphi \leq 0). \tag{41}$$

From (40) and (41), we have

$$\varphi = \pm\sqrt{\frac{(\varphi_*)^2 + (\varphi^*)^2}{\mathrm{sn}^2[\sqrt{\frac{m}{2}}\sqrt{(\varphi_*)^2 + (\varphi^*)^2}\xi]} - (\varphi^*)^2}. \tag{42}$$

Noting that $\varphi = \varphi(\xi)$ and $\xi = \omega x + kt$, we get two blow-up solutions $\varphi_{2^\pm}(x, t)$ as (20).

(3) If $l < 0$, $m < 0$ and $h > 0$, from the phase portrait in Figure 3, the periodic orbit is given as

$$y = \pm\sqrt{\frac{m}{2}(\varphi^2 - (\bar{\varphi}_*)^2)(\varphi^2 + (\bar{\varphi}^*)^2)}. \tag{43}$$

Substituting (43) into $\frac{d\varphi}{d\xi} = y$ and integrating them along the periodic orbit , it follows that

$$\int_{\varphi}^{\bar{\varphi}_*} \frac{1}{\sqrt{((\bar{\varphi}_*)^2 - s^2)(s^2 + (\bar{\varphi}^*)^2)}} ds = \sqrt{-\frac{m}{2}}|\xi|. \tag{44}$$

From (44), we have

$$\varphi = \bar{\varphi}_* \mathrm{cn}\left[\sqrt{-\frac{m}{2}}\sqrt{(\bar{\varphi}_*)^2 + (\bar{\varphi}^*)^2}\xi\right]. \tag{45}$$

Noting that $\varphi = \varphi(\xi)$ and $\xi = \omega x + kt$, we get one periodic solution $\varphi_3(x, t)$ as (21).

Thus, the derivation of Proposition 1 has finished. Now we will derive the Proposition 2.

(1) If $l > 0$, $m < 0$ and $h = 0$, from the phase portrait in Figure 4, the two symmetric homoclinic orbits connected at the saddle point are given as

$$y = \pm\varphi\sqrt{l + \frac{m}{2}\varphi^2}. \tag{46}$$

Substituting (46) into $\frac{d\varphi}{d\xi} = y$ and integrating them along the homoclinic orbits , it follows that

$$\int_{\varphi}^{\bar{\varphi}_r} \frac{1}{s\sqrt{l + \frac{m}{2}s^2}} ds = |\xi| \quad (\varphi \geq 0), \tag{47}$$

and

$$\int_{\varphi}^{-\bar{\varphi}_r} \frac{1}{s\sqrt{l + \frac{m}{2}s^2}} ds = |\xi| \quad (\varphi \leq 0). \tag{48}$$

From (47) and (48), we have

$$\varphi = \pm\sqrt{-\frac{2l}{m}}\mathrm{sech}[\sqrt{l}\xi]. \tag{49}$$

Noting that $\varphi = \varphi(\xi)$ and $\xi = \omega x + kt$, we get two solitary wave solutions $\varphi_{4^\pm}(x, t)$ as (22).

(2) If $l > 0$, $m < 0$ and $h < 0$, from the phase portrait in Figure 4, the two symmetric periodic orbits $\Gamma_2^\pm$ are given as

$$y = \pm\sqrt{-\frac{m}{2}(\varphi^2 - \varphi_0^2)((\bar{\varphi}_0)^2 - \varphi^2)}. \tag{50}$$

Substituting (50) into $\frac{d\varphi}{d\xi} = y$ and integrating them along the periodic orbits, it follows that

$$\int_{\varphi_0}^{\varphi} \frac{1}{\sqrt{(s^2 - \varphi_0^2)((\bar{\varphi}_0)^2 - s^2)}} ds = \sqrt{-\frac{m}{2}}|\xi|, \tag{51}$$

and

$$\int_{\varphi}^{-\varphi_0} \frac{1}{\sqrt{(s^2-\varphi_0^2)((\bar{\varphi}_0)^2-s^2)}} ds = \sqrt{-\frac{m}{2}}|\xi|. \tag{52}$$

From (51) and (52), we have

$$\varphi = \pm \frac{\bar{\varphi}_0\varphi_0}{\sqrt{(\bar{\varphi}_0)^2 - [(\bar{\varphi}_0)^2-(\varphi_0)^2]\text{sn}\left[\sqrt{-\frac{2l}{m}}\bar{\varphi}_0|\xi|\right]}}. \tag{53}$$

Noting that $\varphi = \varphi(\xi)$ and $\xi = \omega x + kt$, we get two periodic wave solutions $\varphi_{5^\pm}(x,t)$ as (23).

(3) If $l > 0$, $m < 0$ and $h > 0$, from the phase portrait in Figure 4, the periodic orbit Γ_3 is given as

$$\text{y} = \pm\sqrt{\frac{m}{2}(\varphi^2-(\bar{\varphi}_*)^2)(\varphi^2+(\bar{\varphi}^*)^2)}. \tag{54}$$

Substituting (54) into $\frac{d\varphi}{d\xi} = \text{y}$ and integrating them along the periodic orbit, it follows that

$$\int_{\varphi}^{\bar{\varphi}_*} \frac{1}{\sqrt{((\bar{\varphi}_*)^2-s^2)(s^2+(\bar{\varphi}^*)^2)}} ds = \sqrt{-\frac{m}{2}}|\xi|. \tag{55}$$

From (55), we have

$$\varphi = \bar{\varphi}_*\text{cn}\left[\sqrt{-\frac{m}{2}}\sqrt{(\bar{\varphi}_*)^2+(\bar{\varphi}^*)^2}\xi\right]. \tag{56}$$

Noting that $\varphi = \varphi(\xi)$ and $\xi = \omega x + kt$, we get one periodic solution $\varphi_6(x,t)$ as (24).

(4) If $l < 0$, $m > 0$ and $h = H(\varphi_r, 0)$, from the phase portrait in Figure 4, the heterclinic orbits connected at saddle points $(-\varphi_r, 0)$ and $(\varphi_r, 0)$ are given as

$$\text{y} = \pm\sqrt{\frac{m}{2}}(\varphi_r-\varphi)(\varphi_r+\varphi). \tag{57}$$

Substituting (57) into $\frac{d\varphi}{d\xi} = \text{y}$ and integrating them along heterclinic orbits, it follows that

$$\int_0^{\varphi} \frac{1}{(\varphi_r-s)(\varphi_r+s)} ds = \sqrt{\frac{m}{2}}|\xi|, \tag{58}$$

and

$$\int_{\varphi}^{+\infty} \frac{1}{(\varphi_r-s)(\varphi_r+s)} ds = \sqrt{\frac{m}{2}}|\xi|. \tag{59}$$

From (58) and (59), we have

$$\varphi = \pm\varphi_r \tanh\left[\varphi_r\sqrt{\frac{m}{2}}\xi\right], \tag{60}$$

and

$$\varphi = \pm\varphi_r \coth\left[\varphi_r\sqrt{\frac{m}{2}}\xi\right]. \tag{61}$$

Noting that $\varphi = \varphi(\xi)$ and $\xi = \omega x + kt$, we get two kink-shaped solutions $\varphi_{7^\pm}(x,t)$ as (25) and two blow-up solutions $\varphi_{8^\pm}(x,t)$ as (26).

(5) If $l < 0$, $m > 0$ and $h < H(\varphi_r, 0)$, from the phase portrait in Figure 4, the orbits $\Gamma_4^{\pm}$ and periodic orbit points Γ_5 are given as

$$y = \pm\sqrt{\frac{m}{2}(\varphi_0^2 - \varphi^2)(\bar{\varphi_0} - \varphi^2)}. \tag{62}$$

Substituting (62) into $\frac{d\varphi}{d\xi} = y$ and integrating them along the orbits $\Gamma_4^{\pm}$ and periodic orbit points Γ_5, it follows that

$$\int_0^{\varphi} \frac{1}{\sqrt{(\varphi_0^2 - s^2)(\bar{\varphi_0} - s^2)}} ds = \sqrt{\frac{m}{2}}|\xi|, \tag{63}$$

and

$$\int_{\varphi}^{+\infty} \frac{1}{\sqrt{(\varphi_0^2 - s^2)(\bar{\varphi_0} - s^2)}} ds = \sqrt{\frac{m}{2}}|\xi|. \tag{64}$$

From (63) and (64), we have

$$\varphi = \varphi_0 \mathrm{sn}\left[\bar{\varphi_0}\sqrt{\frac{m}{2}}\xi\right], \tag{65}$$

and

$$\varphi(x,t) = \pm\frac{\bar{\varphi_0}}{\mathrm{sn}\left[\bar{\varphi_0}\sqrt{\frac{m}{2}}\xi\right]}. \tag{66}$$

Noting that $\varphi = \varphi(\xi)$ and $\xi = \omega x + kt$, we get one periodic solution $\varphi_9(x,t)$ as (27) and two blow-up solutions $\varphi_{10^{\pm}}(x,t)$ as (28). Thus, the derivation of Proposition 2 has finished.

4. CONCLUSIONS

In this paper, we have employed bifurcation method of dynamical systems to study the traveling solutions of the KGZ-equations (1)-(2). We have obtained many solutions (19)-(28) for the KGZ-equations (1)-(2). Compared with [7], we also find that the solutions (19), (22), (25) and (26) are the same as in [7]. But the other solutions we obtained can not be found in other papers, so we believe that they are new solutions for the KGZ-equations (1)-(2). It is valuable to point out that the method used in this paper can be widely applied to other nonlinear equations.

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