

Asymptotic Solutions of Fifth Order Over-Damped Nonlinear Systems with Cubic Nonlinearity

Sk. Tanzer Ahmed Siddique^{1,*}; M. Ali Akbar¹

¹Department of Applied Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh

*Corresponding author. Email: ali_math74@yahoo.com

Received 30 June 2011; accepted 18 August 2011

Abstract: Many engineering problems and physical systems of fifth degrees of freedom are governed by the fifth order nonlinear differential equations which are over-damped. In this article a fifth order nonlinear differential equation modeling an over-damped symmetrical system is considered. A perturbation technique based on the KBM method and the work of Akbar *et al.* is developed for obtaining the transient response when the eigenvalues are in integral multiple. The results obtained by the presented technique agree with those results obtained by the numerical method nicely. An example is solved to illustrate the method.

Keywords: Nonlinearity; Over-damped systems; Perturbation; Eigenvalues

Sk. Tanzer Ahmed Siddique, M. Ali Akbar (2011). Asymptotic Solutions of Fifth Order Over-Damped Nonlinear Systems with Cubic Nonlinearity. *Studies in Mathematical Sciences*, 3(1), 24-34. Available from: URL: <http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220110301.170>. DOI: <http://dx.doi.org/10.3968/j.sms.1923845220110301.170>.

INTRODUCTION

The asymptotic method of Krylov–Bogoliubov–Mitropolskii (KBM)^[6,8] is particularly convenient, and one of the widely-used tools to obtain solutions of nonlinear systems with small nonlinearities. The method originally developed by Krylov and Bogoliubov^[8], for systems with periodic solutions with small nonlinearities, was later amplified and justified by Bogoliubov and Mitropolskii^[6]. Popov^[15] extended the method to nonlinear systems affected by strong linear damping forces. Owing to physical importance Popov's results were rediscovered by Mendelson^[9]. Later, Murty and Deekshatulu^[11] extended the method to over-damped nonlinear systems. Sattar^[16] has studied second order critically-damped nonlinear systems by making use of the KBM method. Murty^[13] presented a unified KBM method for second order nonlinear systems which covers the undamped, the over-damped and the damped oscillatory cases. First, Osiniskii^[14] extended the KBM method to solve third-order nonlinear differential systems using some restrictions, which make the solution over-simplified. Mulholland^[10] had removed these restrictions and found desired solutions of third-order nonlinear systems. Sattar^[17] investigated solutions of three-dimensional over-damped nonlinear systems. Shamsul^[18] presented an asymptotic method for second-order over-damped and critically-damped nonlinear systems. Then Shamsul^[20] extended the method presented in^[18] to third-order over-damped nonlinear systems under some special conditions.

In article^[12] Murty *et al.* also extended the KBM method for solving fourth-order over-damped nonlinear

systems which was a simple (particular) problem. Akbar *et al.*^[1] generalized the method presented in^[12]. Akbar *et al.*^[1] also show that their method is easier than the method of Murty *et al.*^[12]. Then Akbar *et al.*^[2] extended the method presented in^[1] to damped oscillatory nonlinear systems. Akbar *et al.*^[3] also presented a simple technique for solving fourth order over-damped nonlinear systems. Akbar and Uddin^[5] found solutions of an over-damped system in the case of special damping forces. Habibur *et al.*^[7] investigated solutions of certain forth order damped oscillatory systems.

Very recently, Akbar and Tanzer^[4] extended the KBM method for solving the fifth order oscillatory nonlinear systems with small nonlinearities.

In this article, an asymptotic solution of fifth-order over-damped symmetrical nonlinear system is investigated, based on the KBM method and the work of Akbar *et al.*^[3]. The results obtained by the presented technique show good coincidence with numerical results obtained by the fourth-order Runge-Kutta method.

1. THE METHOD

Consider a nonlinear symmetrical over-damped system governed by the fifth order differential equation:

$$\frac{d^5 x}{dt^5} + k_1 \frac{d^4 x}{dt^4} + k_2 \frac{d^3 x}{dt^3} + k_3 \frac{d^2 x}{dt^2} + k_4 \frac{dx}{dt} + k_5 x = -\varepsilon f(x) \quad (1)$$

where k_1, k_2, k_3, k_4, k_5 are characteristics parameter, f is such a nonlinear function that the system (1) becomes symmetrical and ε is a small parameter. When $\varepsilon = 0$, the equation (1) becomes linear, let us consider the five real and negative eigenvalues of the linear equation are $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, -\lambda_5$. Here, the over-damping forces are represented by the real and negative eigenvalues. In this case, the solution of the linear equation is:

$$x(t, 0) = a_{1,0} e^{-\lambda_1 t} + a_{2,0} e^{-\lambda_2 t} + a_{3,0} e^{-\lambda_3 t} + a_{4,0} e^{-\lambda_4 t} + a_{5,0} e^{-\lambda_5 t} \quad (2)$$

where $a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}$ and $a_{5,0}$ are arbitrary constants.

When $\varepsilon \neq 0$, following Murty and Deekshatulu^[11] and Shamsul^[19], we seek a solution of equation (1) in an asymptotic expansion of the form:

$$x(t, \varepsilon) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} + a_5 e^{-\lambda_5 t} + \varepsilon u_1(a_1, a_2, a_3, a_4, a_5, t) + \varepsilon^2 \dots \quad (3)$$

where $\dot{a}_j, j = 1, 2, \dots, 5$ satisfy the first-order differential equations

$$\dot{a}_j = \varepsilon A_1(a_1, a_2, a_3, a_4, a_5, t) + \varepsilon^2 \dots \quad (4)$$

Differentiating (3) five times with respect to t , substituting (3) and the derivatives in the original equation (1), using relations of (4), and finally extracting the coefficients of ε , we obtain:

$$\begin{aligned} & e^{-\lambda_1 t} \left(\frac{d}{dt} + \lambda_2 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_1 \right) A_1 \\ & + e^{-\lambda_2 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_2 \right) A_2 \\ & + e^{-\lambda_3 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_3 \right) A_3 \\ & + e^{-\lambda_4 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_4 \right) A_4 \\ & + e^{-\lambda_5 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_5 \right) A_5 \\ & + \left(\frac{d}{dt} + \lambda_1 \right) \left(\frac{d}{dt} + \lambda_2 \right) \left(\frac{d}{dt} + \lambda_3 \right) \left(\frac{d}{dt} + \lambda_4 \right) \left(\frac{d}{dt} + \lambda_5 \right) u_1 = -f^{(0)}(a_1, a_2, a_3, a_4, a_5, t) \end{aligned} \quad (5)$$

where $f^{(0)} = f(x_0)$ and $x_0 = \sum_{j=1}^5 a_j(t) e^{-\lambda_j t}$.

In general, the functional $f^{(0)}$ can be expanded in the Taylor series (see also^[11,20] for details) as:

$$f^{(0)} = \sum_{i_1^-, \dots, i_5^- = 0}^{\infty, \dots, \infty} F_{i_1^-} \cdots i_5^-(a_1, a_2, a_3, a_4, a_5) e^{-(i_1 \lambda_1 + \dots + i_5 \lambda_5) t} \quad (6)$$

Substituting the value of $f^{(0)}$ from (6) into (5), we obtain:

$$\begin{aligned} & e^{-\lambda_1 t} \left(\frac{d}{dt} + \lambda_2 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_1 \right) A_1 \\ & + e^{-\lambda_2 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_2 \right) A_2 \\ & + e^{-\lambda_3 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_3 \right) A_3 \\ & + e^{-\lambda_4 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_4 \right) A_4 \\ & + e^{-\lambda_5 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_5 \right) A_5 \\ & + \left(\frac{d}{dt} + \lambda_1 \right) \left(\frac{d}{dt} + \lambda_2 \right) \left(\frac{d}{dt} + \lambda_3 \right) \left(\frac{d}{dt} + \lambda_4 \right) \left(\frac{d}{dt} + \lambda_5 \right) u_1 \\ & = - \sum_{i_1^-, \dots, i_5^- = 0}^{\infty, \dots, \infty} F_{i_1^-} \cdots i_5^-(a_1, a_2, a_3, a_4, a_5) e^{-(i_1 \lambda_1 + \dots + i_5 \lambda_5) t} \end{aligned} \quad (7)$$

Since the order of the equation (1) is finite, therefore, it is possible to choose $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5$. Therefore, in order to solve equation (7) for the unknown functions A_1, A_2, A_3, A_4, A_5 and u_1 , it is assumed that u_1 does not contain terms $e^{-(i_1 \lambda_1 + i_2 \lambda_2 + i_3 \lambda_3 + i_4 \lambda_4 + i_5 \lambda_5) t}$, where $i_1 \leq 1$ (see also^[3] for details). This is a significant assumption, since, under this assumption the coefficients of the terms of u_1 do not become large as well as u_1 does not contain secular type terms $t e^{-t}$. Thus, in accordance with this assumptions, we obtain:

$$\begin{aligned} & e^{-\lambda_1 t} \left(\frac{d}{dt} + \lambda_2 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_1 \right) A_1 \\ & + e^{-\lambda_2 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_2 \right) A_2 \\ & + e^{-\lambda_3 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_3 \right) A_3 \\ & + e^{-\lambda_4 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_4 \right) A_4 \\ & + e^{-\lambda_5 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_5 \right) A_5 \\ & = - \sum_{i_1^-, \dots, i_5^- = 0}^{\infty, \dots, \infty} F_{i_1^-} \cdots i_5^-(a_1, a_2, a_3, a_4, a_5) e^{-(i_1 \lambda_1 + \dots + i_5 \lambda_5) t} \end{aligned} \quad (8)$$

where $i_1 \leq 1$.

And:

$$\begin{aligned} & \left(\frac{d}{dt} + \lambda_1 \right) \left(\frac{d}{dt} + \lambda_2 \right) \left(\frac{d}{dt} + \lambda_3 \right) \left(\frac{d}{dt} + \lambda_4 \right) \left(\frac{d}{dt} + \lambda_5 \right) u_1 \\ & = - \sum_{i_1^-, \dots, i_5^- = 0}^{\infty, \dots, \infty} F_{i_1^-} \cdots i_5^-(a_1, a_2, a_3, a_4, a_5) e^{-(i_1 \lambda_1 + \dots + i_5 \lambda_5) t} \end{aligned} \quad (9)$$

where:

$$i_1 > 1. \quad (10)$$

Now, we have to resolve the equation (8) for obtaining the unknown functions A_1, A_2, A_3, A_4 and A_5 . However, the unknown functions A_1, A_2, A_3, A_4 and A_5 cannot be found easily by solving equation (8) if the nonlinear function f , and the eigenvalues, $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$ and $-\lambda_5$ are not specified. Therefore, to resolve the equation (8) for the unknown functions A_1, A_2, A_3, A_4, A_5 and u_1 , in this article, we have imposed the condition $\lambda_{i_1} \approx 3\lambda_{i_1+1}$ ($i_1 = 1, 2, 3, 4$) among the eigenvalues. *i.e.* the eigenvalues are multiple to one another. This condition is important, since such type of relation ($\lambda_{i_1} \approx 3\lambda_{i_1+1}$), among the eigenvalues comes out naturally in the nonlinear equation with cubic nonlinearity. If the nonlinearity is the form other than cubic, in that case the condition $\lambda_{i_1} \approx 3\lambda_{i_1+1}$ is not applicable and in that case different conditions should be used, and it is our work in the future. Using the above condition, we obtain:

$$\begin{aligned} A_1 &= \sum_{i_1=0, \dots, i_5=0}^{\infty, \dots, \infty} l_{i_1, \dots, i_5} e^{-(\lambda_1+i_1 \lambda_1+i_2 \lambda_2+i_3 \lambda_3+i_4 \lambda_4+i_5 \lambda_5) t}, \\ A_2 &= \sum_{i_1=0, \dots, i_5=0}^{\infty, \dots, \infty} m_{i_1, \dots, i_5} e^{-(\lambda_2+i_1 \lambda_1+i_2 \lambda_2+i_3 \lambda_3+i_4 \lambda_4+i_5 \lambda_5) t}, \\ A_3 &= \sum_{i_1=0, \dots, i_5=0}^{\infty, \dots, \infty} n_{i_1, \dots, i_5} e^{-(\lambda_3+i_1 \lambda_1+i_2 \lambda_2+i_3 \lambda_3+i_4 \lambda_4+i_5 \lambda_5) t} \\ A_4 &= \sum_{i_1=0, \dots, i_5=0}^{\infty, \dots, \infty} p_{i_1, \dots, i_5} e^{-(\lambda_4+i_1 \lambda_1+i_2 \lambda_2+i_3 \lambda_3+i_4 \lambda_4+i_5 \lambda_5) t}, \\ A_5 &= \sum_{i_1=0, \dots, i_5=0}^{\infty, \dots, \infty} q_{i_1, \dots, i_5} e^{-(\lambda_5+i_1 \lambda_1+i_2 \lambda_2+i_3 \lambda_3+i_4 \lambda_4+i_5 \lambda_5) t}, \end{aligned} \quad (11)$$

where the coefficients $l_{i_1, \dots, i_5}, m_{i_1, \dots, i_5}, n_{i_1, \dots, i_5}, p_{i_1, \dots, i_5}$ and $q_{i_1, \dots, i_5}$ do not become large, as well as A_1, A_2, A_3, A_4 and A_5 do not become large, for any time t . Substituting the values of A_1, A_2, A_3, A_4 and A_5 from equation (11) into equation (4), and integrating, we shall obtain the values of a_1, a_2, a_3, a_4 and a_5 . Again, solving equation (9), we shall obtain the value of u_1 .

Thus, the determination of the solution is completed.

2. EXAMPLE

As an illustration of the above method, we have considered the Duffing equation type fifth order nonlinear differential systems:

$$\frac{d^5 x}{dt^5} + k_1 \frac{d^4 x}{dt^4} + k_2 \frac{d^3 x}{dt^3} + k_3 \frac{d^2 x}{dt^2} + k_4 \frac{dx}{dt} + k_5 x = -\varepsilon x^3 \quad (12)$$

Here $f = x^3$, and thus, the system (12) is symmetrical (equation (12) remain unchanged if x is replaced by $-x$).

Therefore,

$$\begin{aligned} f^{(0)} &= a_1^3 e^{-3\lambda_1 t} + 3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} + a_2^3 e^{-3\lambda_2 t} + 3a_1^2 a_3 e^{-(2\lambda_1+\lambda_3)t} \\ &+ 3a_1^2 a_4 e^{-(2\lambda_1+\lambda_4)t} + 3a_1^2 a_5 e^{-(2\lambda_1+\lambda_5)t} + 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 6a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t} \\ &+ 6a_1 a_2 a_5 e^{-(\lambda_1+\lambda_2+\lambda_5)t} + 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 3a_2^2 a_4 e^{-(2\lambda_2+\lambda_4)t} + 3a_2^2 a_5 e^{-(2\lambda_2+\lambda_5)t} \\ &+ 3a_1 a_3^2 e^{-(\lambda_1+2\lambda_3)t} + 3a_1 a_4^2 e^{-(\lambda_1+2\lambda_4)t} + 3a_1 a_5^2 e^{-(\lambda_1+2\lambda_5)t} + 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} \\ &+ 6a_1 a_3 a_5 e^{-(\lambda_1+\lambda_3+\lambda_5)t} + 6a_1 a_4 a_5 e^{-(\lambda_1+\lambda_4+\lambda_5)t} + 3a_2 a_3^2 e^{-(\lambda_2+2\lambda_3)t} + 3a_2 a_4^2 e^{-(\lambda_2+2\lambda_4)t} \\ &+ 3a_2 a_5^2 e^{-(\lambda_2+2\lambda_5)t} + 6a_2 a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} + 6a_2 a_4 a_5 e^{-(\lambda_2+\lambda_4+\lambda_5)t} + 6a_3 a_4 a_5 e^{-(\lambda_3+\lambda_4+\lambda_5)t} \\ &+ a_3^3 e^{-3\lambda_3 t} + 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t} + 3a_3^2 a_5 e^{-(2\lambda_3+\lambda_5)t} \\ &+ 6a_3 a_4 a_5 e^{-(\lambda_3+\lambda_4+\lambda_5)t} + 3a_4^2 a_5 e^{-(2\lambda_4+\lambda_5)t} + 3a_3 a_5^2 e^{-(\lambda_3+2\lambda_5)t} + 3a_4 a_5^2 e^{-(\lambda_4+2\lambda_5)t} + a_5^3 e^{-3\lambda_5 t} \end{aligned} \quad (13)$$

Therefore, for equation (12), equations (8)-(9) respectively become:

$$\begin{aligned}
 & e^{-\lambda_1 t} \left(\frac{d}{dt} + \lambda_2 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_1 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_1 \right) A_1 \\
 & + e^{-\lambda_2 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_2 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_2 \right) A_2 \\
 & + e^{-\lambda_3 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_3 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_3 \right) A_3 \\
 & + e^{-\lambda_4 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_4 \right) \left(\frac{d}{dt} + \lambda_5 - \lambda_4 \right) A_4 \\
 & + e^{-\lambda_5 t} \left(\frac{d}{dt} + \lambda_1 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_2 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_3 - \lambda_5 \right) \left(\frac{d}{dt} + \lambda_4 - \lambda_5 \right) A_5 \\
 & = -[3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} + 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3a_1 a_3^2 e^{-(\lambda_1+2\lambda_3)t} + 3a_1 a_4^2 e^{-(\lambda_1+2\lambda_4)t} + 3a_1 a_5^2 e^{-(\lambda_1+2\lambda_5)t} \\
 & + 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} + 6a_1 a_4 a_5 e^{-(\lambda_1+\lambda_4+\lambda_5)t} + 6a_1 a_3 a_5 e^{-(\lambda_1+\lambda_3+\lambda_5)t} + 6a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t} \\
 & + 6a_1 a_2 a_5 e^{-(\lambda_1+\lambda_2+\lambda_5)t} + a_2^3 e^{-3\lambda_2 t} + 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 3a_2^2 a_4 e^{-(2\lambda_2+\lambda_4)t} + 3a_2^2 a_5 e^{-(2\lambda_2+\lambda_5)t} \\
 & + 3a_2 a_3^2 e^{-(\lambda_2+2\lambda_3)t} + 3a_2 a_4^2 e^{-(\lambda_2+2\lambda_4)t} + 3a_2 a_5^2 e^{-(\lambda_2+2\lambda_5)t} + 6a_2 a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} + 6a_2 a_4 a_5 e^{-(\lambda_2+\lambda_4+\lambda_5)t} \\
 & + 6a_2 a_3 a_5 e^{-(\lambda_2+\lambda_3+\lambda_5)t} + a_3^3 e^{-3\lambda_3 t} + a_4^3 e^{-3\lambda_4 t} + 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + 3a_3^2 a_5 e^{-(2\lambda_3+\lambda_5)t} \\
 & + 3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + 6a_3 a_4 a_5 e^{-(\lambda_3+\lambda_4+\lambda_5)t} + a_5^3 e^{-3\lambda_5 t} + 3a_4^2 a_5 e^{-(2\lambda_4+\lambda_5)t} + 3a_4 a_5^2 e^{-(\lambda_4+2\lambda_5)t}] \quad (14)
 \end{aligned}$$

And:

$$\begin{aligned}
 & \left(\frac{d}{dt} + \lambda_1 \right) \left(\frac{d}{dt} + \lambda_2 \right) \left(\frac{d}{dt} + \lambda_3 \right) \left(\frac{d}{dt} + \lambda_4 \right) \left(\frac{d}{dt} + \lambda_5 \right) u_1 \\
 & = -(a_1^3 e^{-3\lambda_1 t} + 3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1^2 a_3 e^{-(2\lambda_1+\lambda_3)t} + 3a_1^2 a_4 e^{-(2\lambda_1+\lambda_4)t} + 3a_1^2 a_5 e^{-(2\lambda_1+\lambda_5)t}) \quad (15)
 \end{aligned}$$

where $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5$. Solving equation (15), we obtain:

$$\begin{aligned}
 u_1 = & r_1 a_1^3 e^{-3\lambda_1 t} + r_2 a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + r_3 a_1^2 a_3 e^{-(2\lambda_1+\lambda_3)t} \\
 & + r_4 a_1^2 a_4 e^{-(2\lambda_1+\lambda_4)t} + r_5 a_1^2 a_5 e^{-(2\lambda_1+\lambda_5)t} \quad (16)
 \end{aligned}$$

where:

$$\begin{aligned}
 r_1 &= 1/\{2\lambda_1(\lambda_2 - 3\lambda_1)(\lambda_3 - 3\lambda_1)(\lambda_4 - 3\lambda_1)(\lambda_5 - 3\lambda_1)\} \\
 r_2 &= 3/\{(\lambda_1 + \lambda_2)(2\lambda_1)(2\lambda_1 + \lambda_2 - \lambda_3)(2\lambda_1 + \lambda_2 - \lambda_4)(2\lambda_1 + \lambda_2 - \lambda_5)\} \\
 r_3 &= 3/\{(\lambda_1 + \lambda_3)(2\lambda_1)(2\lambda_1 + \lambda_3 - \lambda_2)(2\lambda_1 + \lambda_3 - \lambda_4)(2\lambda_1 + \lambda_3 - \lambda_5)\} \\
 r_4 &= 3/\{(\lambda_1 + \lambda_4)(2\lambda_1)(2\lambda_1 + \lambda_4 - \lambda_2)(2\lambda_1 + \lambda_4 - \lambda_3)(2\lambda_1 + \lambda_4 - \lambda_5)\} \\
 r_5 &= 3/\{(\lambda_1 + \lambda_5)(2\lambda_1)(2\lambda_1 + \lambda_5 - \lambda_2)(2\lambda_1 + \lambda_5 - \lambda_3)(2\lambda_1 + \lambda_5 - \lambda_4)\} \quad (17)
 \end{aligned}$$

For obtaining the unknown functions A_1, A_2, A_3, A_4 and A_5 , we have to separate the equation (14). Under the imposed condition (in this paper, we have imposed the condition $\lambda_{i_1} \approx 3\lambda_{i_1+1}, i_1 = 1, 2, 3, 4$), we obtain:

$$\begin{aligned}
 & e^{-\lambda_1 t} \left(\frac{d}{dt} - \lambda_1 + \lambda_2 \right) \left(\frac{d}{dt} - \lambda_1 + \lambda_3 \right) \left(\frac{d}{dt} - \lambda_1 + \lambda_4 \right) \left(\frac{d}{dt} - \lambda_1 + \lambda_5 \right) A_1 \\
 & = -(3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} + 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3a_1 a_3^2 e^{-(\lambda_1+2\lambda_3)t} + 3a_1 a_4^2 e^{-(\lambda_1+2\lambda_4)t} \\
 & + 3a_1 a_5^2 e^{-(\lambda_1+2\lambda_5)t} + 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} + 6a_1 a_4 a_5 e^{-(\lambda_1+\lambda_4+\lambda_5)t} + 6a_1 a_3 a_5 e^{-(\lambda_1+\lambda_3+\lambda_5)t} \\
 & + 6a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t} + 6a_1 a_2 a_5 e^{-(\lambda_1+\lambda_2+\lambda_5)t} + a_2^3 e^{-3\lambda_2 t}) \quad (18)
 \end{aligned}$$

$$\begin{aligned}
 & e^{-\lambda_2 t} \left(\frac{d}{dt} - \lambda_2 + \lambda_1 \right) \left(\frac{d}{dt} - \lambda_2 + \lambda_3 \right) \left(\frac{d}{dt} - \lambda_2 + \lambda_4 \right) \left(\frac{d}{dt} - \lambda_2 + \lambda_5 \right) A_2 \\
 & = -(3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 3a_2^2 a_4 e^{-(2\lambda_2+\lambda_4)t} + 3a_2^2 a_5 e^{-(2\lambda_2+\lambda_5)t} \\
 & + 3a_2 a_3^2 e^{-(\lambda_2+2\lambda_3)t} + 3a_2 a_4^2 e^{-(\lambda_2+2\lambda_4)t} + 3a_2 a_5^2 e^{-(\lambda_2+2\lambda_5)t} + 6a_2 a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} \\
 & + 6a_2 a_4 a_5 e^{-(\lambda_2+\lambda_4+\lambda_5)t} + 6a_2 a_3 a_5 e^{-(\lambda_2+\lambda_3+\lambda_5)t} + a_3^3 e^{-3\lambda_3 t}) \quad (19)
 \end{aligned}$$

$$e^{-\lambda_3 t} \left(\frac{d}{dt} - \lambda_3 + \lambda_1 \right) \left(\frac{d}{dt} - \lambda_3 + \lambda_2 \right) \left(\frac{d}{dt} - \lambda_3 + \lambda_4 \right) \left(\frac{d}{dt} - \lambda_3 + \lambda_5 \right) A_3$$

$$= -(a_4^3 e^{-3\lambda_4 t} + 3 a_3^2 a_4 e^{-(2\lambda_3 + \lambda_4)t} + 3 a_3 a_4^2 e^{-(\lambda_3 + 2\lambda_4)t} + 3 a_3^2 a_5 e^{-(2\lambda_3 + \lambda_5)t}$$

$$+ 3 a_3 a_5^2 e^{-(\lambda_3 + 2\lambda_5)t} + 6 a_3 a_4 a_5 e^{-(\lambda_3 + \lambda_4 + \lambda_5)t}) \quad (20)$$

$$e^{-\lambda_4 t} \left(\frac{d}{dt} - \lambda_4 + \lambda_1 \right) \left(\frac{d}{dt} - \lambda_4 + \lambda_2 \right) \left(\frac{d}{dt} - \lambda_4 + \lambda_3 \right) \left(\frac{d}{dt} - \lambda_4 + \lambda_5 \right) A_4$$

$$= -(a_5^3 e^{-3\lambda_5 t} + 3 a_4^2 a_5 e^{-(2\lambda_4 + \lambda_5)t} + 3 a_4 a_5^2 e^{-(\lambda_4 + 2\lambda_5)t}) \quad (21)$$

$$e^{-\lambda_5 t} \left(\frac{d}{dt} - \lambda_5 + \lambda_1 \right) \left(\frac{d}{dt} - \lambda_5 + \lambda_2 \right) \left(\frac{d}{dt} - \lambda_5 + \lambda_3 \right) \left(\frac{d}{dt} - \lambda_5 + \lambda_4 \right) A_5 = 0 \quad (22)$$

Now we have to solve equations (18)-(22). Using the symbolic computation software, like, Maple, Mathematica, Matlab etc. solutions of equations (18)-(22) are utter simple. In this article, to solve equations (18)-(22), we have used Maple 13.

Therefore, solving equations (18)-(22), we obtain:

$$A_1 = l_1 a_1 a_2^2 e^{-2\lambda_2 t} + l_2 a_1 a_2 a_3 e^{-(\lambda_2 + \lambda_3)t} + l_3 a_1 a_3^2 e^{-2\lambda_3 t} + l_4 a_1 a_4^2 e^{-2\lambda_4 t} + l_5 a_1 a_5^2 e^{-2\lambda_5 t}$$

$$+ l_6 a_1 a_3 a_4 e^{-(\lambda_3 + \lambda_4)t} + l_7 a_1 a_4 a_5 e^{-(\lambda_4 + \lambda_5)t} + l_8 a_1 a_3 a_5 e^{-(\lambda_3 + \lambda_5)t} + l_9 a_1 a_2 a_4 e^{-(\lambda_2 + \lambda_4)t}$$

$$+ l_{10} a_1 a_2 a_5 e^{-(\lambda_2 + \lambda_5)t} + l_{11} a_2^3 e^{(\lambda_1 - 3\lambda_2)t} \quad (23)$$

$$A_2 = m_1 a_2^2 a_3 e^{-(\lambda_2 + \lambda_3)t} + m_2 a_2^2 a_4 e^{-(\lambda_2 + \lambda_4)t} + m_3 a_2^2 a_5 e^{-(\lambda_2 + \lambda_5)t} + m_4 a_2 a_3^2 e^{-2\lambda_3 t}$$

$$+ m_5 a_2 a_4^2 e^{-2\lambda_4 t} + m_6 a_2 a_5^2 e^{-2\lambda_5 t} + m_7 a_2 a_3 a_4 e^{-(\lambda_3 + \lambda_4)t} + m_8 a_2 a_4 a_5 e^{-(\lambda_4 + \lambda_5)t}$$

$$+ m_9 a_2 a_3 a_5 e^{-(\lambda_3 + \lambda_5)t} + m_{10} a_3^3 e^{(\lambda_2 - 3\lambda_3)t} \quad (24)$$

$$A_3 = n_1 a_4^3 e^{(\lambda_3 - 3\lambda_4)t} + n_2 a_3^2 a_4 e^{-(\lambda_3 + \lambda_4)t} + n_3 a_3 a_4^2 e^{-2\lambda_4 t} + n_4 a_3^2 a_5 e^{-(\lambda_3 + \lambda_5)t}$$

$$+ n_5 a_3^4 a_4 a_5 e^{-(\lambda_4 + \lambda_5)t} + n_6 a_3 a_5^2 e^{-2\lambda_5 t} \quad (25)$$

$$A_4 = p_1 a_5^3 e^{(\lambda_4 - 3\lambda_5)t} + p_2 a_4 a_5^2 e^{-2\lambda_5 t} + p_3 a_4^2 a_5 e^{-(\lambda_4 + \lambda_5)t} \quad (26)$$

$$A_5 = 0 \quad (27)$$

where:

$$l_1 = 3/\{(\lambda_1 + \lambda_2)(\lambda_3 - \lambda_2 - 2\lambda_2)(\lambda_4 - \lambda_2 - 2\lambda_2)(\lambda_5 - \lambda_2 - 2\lambda_2)\}$$

$$l_2 = -6/\{(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_2)(\lambda_4 - \lambda_1 - \lambda_2 - \lambda_3)(\lambda_5 - \lambda_1 - \lambda_2 - \lambda_3)\}$$

$$l_3 = 3/\{(\lambda_1 + \lambda_3)(\lambda_2 - \lambda_1 - 2\lambda_3)(\lambda_4 - \lambda_1 - 2\lambda_3)(\lambda_5 - \lambda_1 - 2\lambda_3)\}$$

$$l_4 = 3/\{(\lambda_1 + \lambda_4)(\lambda_2 - \lambda_1 - 2\lambda_4)(\lambda_3 - \lambda_1 - 2\lambda_4)(\lambda_5 - \lambda_1 - 2\lambda_4)\}$$

$$l_5 = 3/\{(\lambda_1 + \lambda_5)(\lambda_2 - \lambda_1 - 2\lambda_5)(\lambda_3 - \lambda_1 - 2\lambda_5)(\lambda_4 - \lambda_1 - 2\lambda_5)\}$$

$$l_6 = -6/\{(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)(\lambda_2 - \lambda_1 - \lambda_3 - \lambda_4)(\lambda_5 - \lambda_1 - \lambda_3 - \lambda_4)\}$$

$$l_7 = -6/\{(\lambda_1 + \lambda_4)(\lambda_1 + \lambda_5)(\lambda_2 - \lambda_1 - \lambda_4 - \lambda_5)(\lambda_3 - \lambda_1 - \lambda_4 - \lambda_5)\}$$

$$l_8 = -6/\{(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_5)(\lambda_2 - \lambda_1 - \lambda_3 - \lambda_5)(\lambda_4 - \lambda_1 - \lambda_3 - \lambda_5)\}$$

$$l_9 = -6/\{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_4)(\lambda_3 - \lambda_1 - \lambda_2 - \lambda_4)(\lambda_5 - \lambda_1 - \lambda_2 - \lambda_4)\}$$

$$l_{10} = -6/\{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_5)(\lambda_3 - \lambda_1 - \lambda_2 - \lambda_5)(\lambda_4 - \lambda_1 - \lambda_2 - \lambda_5)\}$$

$$l_{11} = 1/\{2\lambda_2(\lambda_3 - 3\lambda_2)(\lambda_4 - 3\lambda_2)(\lambda_5 - 3\lambda_2)\}$$

$$m_1 = 3/\{(2\lambda_2)(\lambda_1 - 2\lambda_2 - \lambda_3)(\lambda_4 - 2\lambda_2 - \lambda_3)(\lambda_5 - 2\lambda_2 - \lambda_3)\}$$

$$m_2 = 3/\{(2\lambda_2)(\lambda_1 - 2\lambda_2 - \lambda_4)(\lambda_3 - 2\lambda_2 - \lambda_4)(\lambda_5 - 2\lambda_2 - \lambda_3)\}$$

$$m_3 = 3/\{(2\lambda_2)(\lambda_1 - 2\lambda_2 - \lambda_5)(\lambda_3 - 2\lambda_2 - \lambda_5)(\lambda_4 - 2\lambda_2 - \lambda_5)\}$$

$$\begin{aligned}
 m_4 &= 3/[(\lambda_2 + \lambda_3)(\lambda_1 - \lambda_2 - 2\lambda_3)(\lambda_4 - \lambda_2 - 2\lambda_3)(\lambda_5 - \lambda_2 - 2\lambda_3)] \\
 m_5 &= 3/[(\lambda_2 + \lambda_4)(\lambda_1 - \lambda_2 - 2\lambda_4)(\lambda_3 - \lambda_2 - 2\lambda_4)(\lambda_5 - \lambda_2 - 2\lambda_4)] \\
 m_6 &= 3/[(\lambda_2 + \lambda_5)(\lambda_1 - \lambda_2 - 2\lambda_5)(\lambda_3 - \lambda_2 - 2\lambda_5)(\lambda_4 - \lambda_2 - 2\lambda_5)] \\
 m_7 &= -6/[(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)(\lambda_5 - \lambda_2 - \lambda_3 - \lambda_4)] \\
 m_8 &= -6/[(\lambda_2 + \lambda_4)(\lambda_2 + \lambda_5)(\lambda_1 - \lambda_2 - \lambda_4 - \lambda_5)(\lambda_3 - \lambda_2 - \lambda_4 - \lambda_5)] \\
 m_9 &= -6/[(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_5)(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_5)(\lambda_4 - \lambda_2 - \lambda_3 - \lambda_5)] \\
 m_{10} &= 1/[(2\lambda_3)(\lambda_1 - 3\lambda_3)(\lambda_4 - 3\lambda_3)(\lambda_5 - 3\lambda_3)] \\
 n_1 &= 1/[(2\lambda_4)(\lambda_1 - 3\lambda_4)(\lambda_2 - 3\lambda_4)(\lambda_5 - 3\lambda_4)] \\
 n_2 &= 3/[(2\lambda_3)(\lambda_1 - 2\lambda_3 - \lambda_4)(\lambda_2 - 2\lambda_3 - \lambda_4)(\lambda_5 - 2\lambda_3 - \lambda_4)] \\
 n_3 &= 3/[(\lambda_3 + \lambda_4)(\lambda_1 - \lambda_3 - 2\lambda_4)(\lambda_2 - \lambda_3 - 2\lambda_4)(\lambda_5 - \lambda_3 - 2\lambda_4)] \\
 n_4 &= 3/[(2\lambda_3)(\lambda_1 - 2\lambda_3 - \lambda_5)(\lambda_2 - 2\lambda_3 - \lambda_5)(\lambda_4 - 2\lambda_3 - \lambda_5)] \\
 n_5 &= -6/[(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_5)(\lambda_1 - \lambda_3 - \lambda_4 - \lambda_5)(\lambda_2 - \lambda_3 - \lambda_4 - \lambda_5)] \\
 n_6 &= 3/[(\lambda_3 + \lambda_5)(\lambda_1 - \lambda_3 - 2\lambda_5)(\lambda_2 - \lambda_3 - 2\lambda_5)(\lambda_4 - \lambda_3 - 2\lambda_5)] \\
 p_1 &= 1/[(\lambda_1 - 3\lambda_5)(\lambda_2 - 3\lambda_5)(\lambda_3 - 3\lambda_5)(\lambda_4 - 3\lambda_5)] \\
 p_2 &= 3/[(\lambda_4 + \lambda_5)(\lambda_1 - \lambda_4 - 2\lambda_5)(\lambda_2 - \lambda_4 - 2\lambda_5)(\lambda_3 - \lambda_4 - 2\lambda_5)] \\
 p_3 &= 3/[(2\lambda_4)(\lambda_1 - 2\lambda_4 - \lambda_5)(\lambda_2 - 2\lambda_4 - \lambda_5)(\lambda_3 - 2\lambda_4 - \lambda_5)]
 \end{aligned}$$

Substituting the values of A_1, A_2, A_3, A_4 and A_5 from equations (23)-(27) into equation (4), we obtain:

$$\begin{aligned}
 \frac{da_1}{dt} &= \varepsilon (l_1 a_1 a_2^2 e^{-2\lambda_2 t} + l_2 a_1 a_2 a_3 e^{-(\lambda_2 + \lambda_3)t} + l_3 a_1 a_3^2 e^{-2\lambda_3 t} + l_4 a_1 a_4^2 e^{-2\lambda_4 t} + l_5 a_1 a_5^2 e^{-2\lambda_5 t} \\
 &\quad + l_6 a_1 a_3 a_4 e^{-(\lambda_3 + \lambda_4)t} + l_7 a_1 a_4 a_5 e^{-(\lambda_4 + \lambda_5)t} + l_8 a_1 a_3 a_5 e^{-(\lambda_3 + \lambda_5)t} + l_9 a_1 a_2 a_4 e^{-(\lambda_2 + \lambda_4)t} \\
 &\quad + l_{10} a_1 a_2 a_5 e^{-(\lambda_2 + \lambda_5)t} + l_{11} a_2^3 e^{(\lambda_1 - 3\lambda_2)t}) \\
 \frac{da_2}{dt} &= \varepsilon (m_1 a_2^2 a_3 e^{-(\lambda_2 + \lambda_3)t} + m_2 a_2^2 a_4 e^{-(\lambda_2 + \lambda_4)t} + m_3 a_2^2 a_5 e^{-(\lambda_2 + \lambda_5)t} + m_4 a_2 a_3^2 e^{-2\lambda_3 t} \\
 &\quad + m_5 a_2 a_4^2 e^{-2\lambda_4 t} + m_6 a_2 a_5^2 e^{-2\lambda_5 t} + m_7 a_2 a_3 a_4 e^{-(\lambda_3 + \lambda_4)t} + m_8 a_2 a_4 a_5 e^{-(\lambda_4 + \lambda_5)t} \\
 &\quad + m_9 a_2 a_3 a_5 e^{-(\lambda_3 + \lambda_5)t} + m_{10} a_3^3 e^{(\lambda_2 - 3\lambda_3)t}) \\
 \frac{da_3}{dt} &= \varepsilon (n_1 a_4^3 e^{(\lambda_3 - 3\lambda_4)t} + n_2 a_3^2 a_4 e^{-(\lambda_3 + \lambda_4)t} + n_3 a_3 a_4^2 e^{-2\lambda_4 t} + n_4 a_3^2 a_5 e^{-(\lambda_3 + \lambda_5)t} \\
 &\quad + n_5 a_3^4 a_4 a_5 e^{-(\lambda_4 + \lambda_5)t} + n_6 a_3 a_5^2 e^{-2\lambda_5 t}) \\
 \frac{da_4}{dt} &= \varepsilon (p_1 a_5^3 e^{(\lambda_4 - 3\lambda_5)t} + p_2 a_4 a_5^2 e^{-2\lambda_5 t} + p_3 a_4^2 a_5 e^{-(\lambda_4 + \lambda_5)t}) \\
 \frac{da_5}{dt} &= 0
 \end{aligned} \tag{28}$$

Since ε is a small quantity, we can solve (28) by assuming that a_1, a_2, a_3, a_4 and a_5 are constants in the right-hand side of (28). This assumption was first made by Murty et al^[11,12] to solve similar type of nonlinear equations. Thus the solutions of the equations of (28) are:

$$\begin{aligned}
 a_1(t) &= a_{1,0} + \varepsilon \left[l_1 a_{1,0} a_{2,0}^2 \frac{1 - e^{-2\lambda_2 t}}{2\lambda_2} + l_2 a_{1,0} a_{2,0} a_{3,0} \frac{1 - e^{-(\lambda_2 + \lambda_3)t}}{(\lambda_2 + \lambda_3)} + l_3 a_{1,0} a_{3,0}^2 \frac{1 - e^{-2\lambda_3 t}}{2\lambda_3} \right. \\
 &\quad + l_4 a_{1,0} a_{4,0}^2 \frac{1 - e^{-2\lambda_4 t}}{2\lambda_4} + l_5 a_{1,0} a_{5,0}^2 \frac{1 - e^{-2\lambda_5 t}}{2\lambda_5} + l_6 a_{1,0} a_{3,0} a_{4,0} \frac{1 - e^{-(\lambda_3 + \lambda_4)t}}{(\lambda_3 + \lambda_4)} \\
 &\quad + l_7 a_{1,0} a_{4,0} a_{5,0} \frac{1 - e^{-(\lambda_4 + \lambda_5)t}}{(\lambda_4 + \lambda_5)} + l_8 a_{1,0} a_{3,0} a_{5,0} \frac{1 - e^{-(\lambda_3 + \lambda_5)t}}{(\lambda_3 + \lambda_5)} + l_9 a_{1,0} a_{2,0} a_{4,0} \frac{1 - e^{-(\lambda_2 + \lambda_4)t}}{(\lambda_2 + \lambda_4)} \\
 &\quad \left. + l_{10} a_{1,0} a_{2,0} a_{5,0} \frac{1 - e^{-(\lambda_2 + \lambda_5)t}}{(\lambda_2 + \lambda_5)} + l_{11} a_{2,0}^3 \frac{1 - e^{(\lambda_1 - 3\lambda_2)t}}{(3\lambda_2 - \lambda_1)} \right]
 \end{aligned}$$

$$\begin{aligned}
 a_2(t) &= a_{2,0} + \varepsilon \left[m_1 a_{2,0}^2 a_{3,0} \frac{1 - e^{-(\lambda_2 + \lambda_3)t}}{(\lambda_2 + \lambda_3)} + m_2 a_{2,0}^2 a_{4,0} \frac{1 - e^{-(\lambda_2 + \lambda_4)t}}{(\lambda_2 + \lambda_4)} + m_3 a_{2,0}^2 a_{5,0} \frac{1 - e^{-(\lambda_2 + \lambda_5)t}}{(\lambda_2 + \lambda_5)} \right. \\
 &\quad + m_4 a_{2,0} a_{3,0}^2 \frac{1 - e^{-2\lambda_3 t}}{2\lambda_3} + m_5 a_{2,0} a_{4,0}^2 \frac{1 - e^{-2\lambda_4 t}}{2\lambda_4} + m_6 a_{2,0} a_{5,0}^2 \frac{1 - e^{-2\lambda_5 t}}{2\lambda_5} + m_7 a_{2,0} a_{3,0} a_{4,0} \frac{1 - e^{-(\lambda_3 + \lambda_4)t}}{(\lambda_3 + \lambda_4)} \\
 &\quad \left. + m_8 a_{2,0} a_{4,0} a_{5,0} \frac{1 - e^{-(\lambda_4 + \lambda_5)t}}{(\lambda_4 + \lambda_5)} + m_9 a_{2,0} a_{3,0} a_{5,0} \frac{1 - e^{-(\lambda_3 + \lambda_5)t}}{(\lambda_3 + \lambda_5)} + m_{10} a_{3,0}^3 \frac{1 - e^{(\lambda_2 - 3\lambda_3)t}}{(3\lambda_3 - \lambda_2)} \right] \\
 a_3(t) &= a_{3,0} + \varepsilon \left[n_1 a_{4,0}^3 \frac{1 - e^{(\lambda_3 - 3\lambda_4)t}}{3\lambda_4 - \lambda_3} + n_2 a_{3,0}^2 a_{4,0} \frac{1 - e^{-(\lambda_3 + \lambda_4)t}}{(\lambda_3 + \lambda_4)} + n_3 a_{3,0} a_{4,0}^2 \frac{1 - e^{-2\lambda_4 t}}{2\lambda_4} \right. \\
 &\quad \left. + n_4 a_{3,0}^2 a_{5,0} \frac{1 - e^{-(\lambda_3 + \lambda_5)t}}{(\lambda_3 + \lambda_5)} + n_5 a_{3,0}^4 a_{4,0} a_{5,0} \frac{1 - e^{-(\lambda_4 + \lambda_5)t}}{(\lambda_4 + \lambda_5)} + n_6 a_{3,0} a_{5,0}^2 \frac{1 - e^{-2\lambda_5 t}}{2\lambda_5} \right] \\
 a_4(t) &= a_{4,0} + \varepsilon \left[p_1 a_{5,0}^3 \frac{1 - e^{(\lambda_4 - 3\lambda_5)t}}{(3\lambda_5 - \lambda_4)} + p_2 a_{4,0} a_{5,0}^2 \frac{1 - e^{-2\lambda_5 t}}{2\lambda_5} + p_3 a_{4,0}^2 a_{5,0} \frac{1 - e^{-(\lambda_4 + \lambda_5)t}}{(\lambda_4 + \lambda_5)} \right] \\
 a_5(t) &= a_{5,0}
 \end{aligned} \tag{29}$$

when $\lambda_{i_1} \approx 3\lambda_{i_1+1}$, $i_1 = 1, 2, 3, 4$. But when $\lambda_{i_1} = 3\lambda_{i_1+1}$, $i = 1, 2, 3, 4$, the terms $l_{11} a_{2,0}^3 \frac{1 - e^{(\lambda_1 - 3\lambda_2)t}}{(3\lambda_2 - \lambda_1)}$, $m_{10} a_{3,0}^3 \frac{1 - e^{(\lambda_2 - 3\lambda_3)t}}{(3\lambda_3 - \lambda_2)}$, $n_1 a_{4,0}^3 \frac{1 - e^{(\lambda_3 - 3\lambda_4)t}}{3\lambda_4 - \lambda_3}$ and $p_1 a_{5,0}^3 \frac{1 - e^{(\lambda_4 - 3\lambda_5)t}}{(3\lambda_5 - \lambda_4)}$ of the first, second, third and fourth equation of (29) will be replaced by $l_{11} a_{2,0}^3 t$, $m_{10} a_{3,0}^3 t$, $n_1 a_{4,0}^3 t$ and $p_1 a_{5,0}^3 t$ respectively.

Therefore, the first-order approximate solution of the equation (12) is:

$$x(t, \varepsilon) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} + a_5 e^{-\lambda_5 t} + \varepsilon u_1 \tag{30}$$

where a_1, a_2, a_3, a_4 and a_5 are given by equation (29) and u_1 is given by equation (16).

3. RESULTS AND DISCUSSION

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we compare the approximate solution to the numerical solution. With regard to such a comparison concerning the presented technique of this article, we refer to the work of Murty *et al.*^[12].

First of all, we have considered the eigenvalues $\lambda_1 = 42.0$, $\lambda_2 = 13.60$, $\lambda_3 = 4.55$, $\lambda_4 = 1.6$, $\lambda_5 = 0.5$. Therefore $\lambda_{i_1} \approx 3\lambda_{i_1+1}$, $i_1 = 1, 2, 3, 4$. We have computed $x(t, \varepsilon)$ by equation (30) in which a_1, a_2, a_3, a_4 and a_5 are computed by equation (29) together with initial conditions $a_{1,0} = 0.25$, $a_{2,0} = 0.25$, $a_{3,0} = 0.25$, $a_{4,0} = 0.25$ and $a_{5,0} = 0.25$ [or $x(0) = 1.250000$, $\frac{dx(0)}{dt} = -15.562494$, $\frac{d^2x(0)}{dt^2} = 493.118114$, $\frac{d^3x(0)}{dt^3} = -19175.469518$ and $\frac{d^4x(0)}{dt^4} = 786588.807443$] for various values of t when $\varepsilon = 0.1$. The perturbation results are presented in **Figure 1**, plotted by the dot line. The corresponding numerical solution has been computed by a fourth-order Runge-Kutta method, and the results are presented in **Figure 1**, plotted by the continuous line. The correlation between these two results has also been calculated which is 0.999505.

Secondly, we have considered the eigenvalues $\lambda_1 = 52.0$, $\lambda_2 = 17.30$, $\lambda_3 = 5.70$, $\lambda_4 = 1.92$, $\lambda_5 = 0.63$. Therefore $\lambda_{i_1} \approx 3\lambda_{i_1+1}$, $i_1 = 1, 2, 3, 4$. We have computed $x(t, \varepsilon)$ using equation (30) in which a_1, a_2, a_3, a_4 and a_5 are computed by equation (29), together with initial conditions $a_{1,0} = 0.28$, $a_{2,0} = 0.32$, $a_{3,0} = 0.32$, $a_{4,0} = 0.40$ and $a_{5,0} = 0.40$ [or $x(0) = 1.720000$, $\frac{dx(0)}{dt} = -22.939976$, $\frac{d^2x(0)}{dt^2} = 864.922842$, $\frac{d^3x(0)}{dt^3} = -41089.306748$ and $\frac{d^4x(0)}{dt^4} = 2076284.765397$] for various values of t when $\varepsilon = 0.25$ and the perturbation

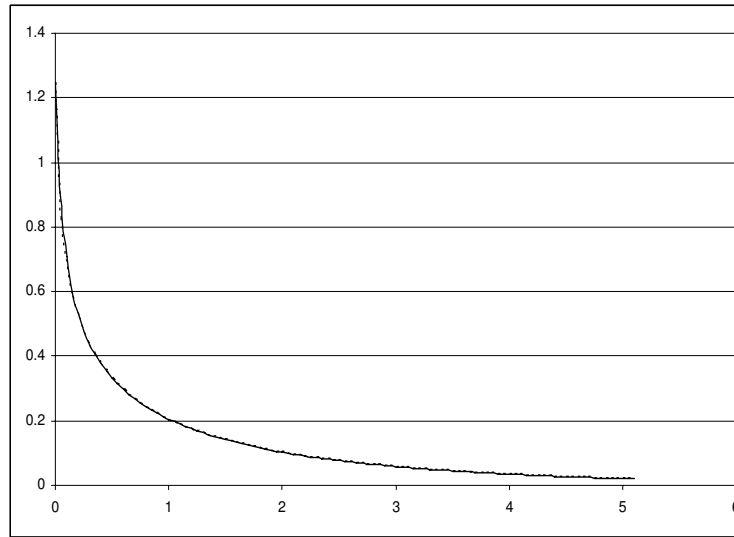


Figure 1
Perturbation Results are Plotted by Dot Line and Numerical Results are Plotted by Continuous Line

results are presented in **Figure 2**, plotted by the dot line. The corresponding numerical solution has been computed by a fourth-order Runge-Kutta method, and the results are presented in **Figure 2**, plotted by the continuous line. The correlation between these two results have also been calculated which is 0.997105.

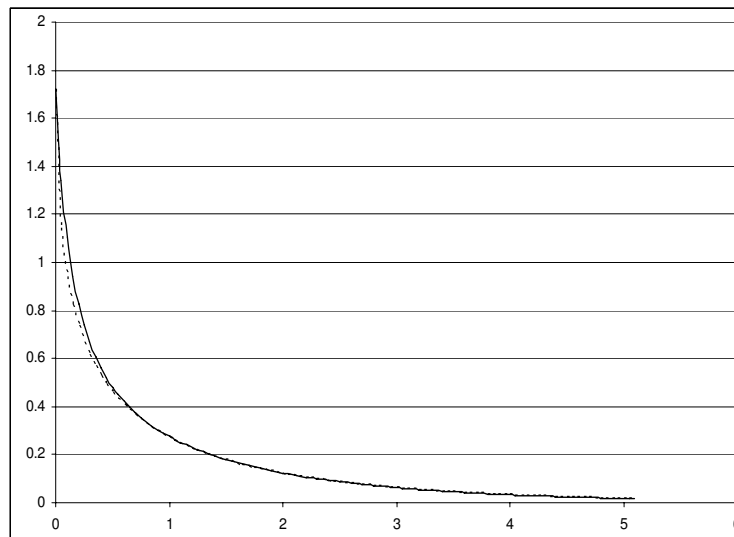


Figure 2
Perturbation Results are Plotted by Dot Line and Numerical Results are Plotted by Continuous Line

Finally, we have considered the eigenvalues $\lambda_1 = 21.30$, $\lambda_2 = 7.09$, $\lambda_3 = 2.35$, $\lambda_4 = 0.77$, $\lambda_5 = 0.250$. Therefore $\lambda_{i_1} \approx 3\lambda_{i_1+1}$, $i_1 = 1, 2, 3, 4$. We have again computed $x(t, \varepsilon)$ by equation (30) in which a_1, a_2, a_3, a_4 and a_5 are computed by equation (29), together with initial conditions $a_{1,0} = 0.25$, $a_{2,0} = 0.25$, $a_{3,0} = 0.25$,

$a_{4,0} = 0.25$ and $a_{5,0} = 0.25$ [or $x(0) = 2.100000$, $\frac{dx(0)}{dt} = -13.568513$, $\frac{d^2x(0)}{dt^2} = 222.005787$, $\frac{d^3x(0)}{dt^3} = -4445.7000619$ and $\frac{d^4x(0)}{dt^4} = 94031.172125$] for various values of t when $\varepsilon = 0.1$. The perturbation results are presented in **Figure 3**, plotted by the dot line. The corresponding numerical solution has been computed by a fourth-order Runge-Kutta method, and the results are presented in **Figure 3**, plotted by continuous line. The correlation between these two results have also been calculated which is 0.999995.

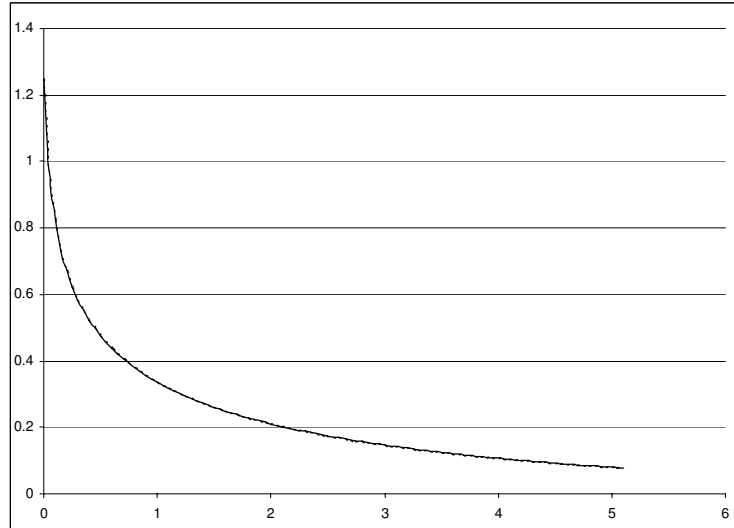


Figure 3
Perturbation Results are Plotted by Dot Line and Numerical Results are Plotted by Continuous Line

CONCLUSION

A perturbation technique, based on the work of Akbar *et al.*^[3] is developed to obtain the transient response of fifth-order over-damped symmetrical nonlinear systems. We calculated the correlation between the results obtained by the presented perturbation technique and the corresponding numerical results obtained by the fourth-order Runge-Kutta method. These two results are strongly-correlated. The results obtained for different sets of initial conditions, as well as for different damping forces, show good coincidence with those results obtained by numerical method.

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