CSCanada
Studies in Mathematical Sciences
Vol. 3, No. 1, 2011, pp. 64-74
DOI: 10.3968/j.sms.1923845220110301.110

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
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Some Fixed Point Theorems in Fuzzy Metric Spaces

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Received 5 July 2011; accepted 18 August 2011

Abstract: In present paper we prove fixed point theorems in fuzzy metric space and then extend the result to fuzzy 2-metric and fuzzy 3-metric spaces. Our results generalize some well known existing results.
Keywords: Fixed point theorem; Fuzzy metric Space

Zaheer K. Ansari, Rajesh Shrivastava, Gunjan Ansari and Manish Sharma (2011). Some Fixed Point Theorems in Fuzzy Metric Spaces. *Studies in Mathematical Sciences, 3*(1), 64-74. Available from: URL: http://www.cscanada.net/index.php/sms/article/view/j.sms.1923845220110301.110. DOI: http://dx.doi.org/10.3968/j.sms.1923845220110301.110.

1. INTRODUCTION AND PRELIMINARIES

As an extension of classical notion of set, Zadeh[22] introduced the concept of fuzzy set that led to a rich growth of fuzzy mathematics[20]. This theory got further developed as a result of lot of research done by different mathematicians in various disciplines. In the sequel the concept of fuzzy metric space was introduced by Kramsoil and Michalek[17].Afterward the most celebrated fixed point theorems due to Banach and Eldestien in metric space were further extended by Grabiec[11] in fuzzy metric space. George and Veeramani[10] initiated the concept of induced fuzzy metric space and adapted the notion of fuzzy metric space with the help of t-norm defined by Schweizer and Sklar[19]. The study of fixed point theorems has been followed by many authors, see[1−5,12−14,18]. There are many view points of the notion of metric space in fuzzy topology. We are interested in the results in which the distance between the objects is fuzzy, the objects themselves may be fuzzy or not .The most interesting references in this direction are[4,15,16]. Gähler in a series of papers[7,8,9] investigated 2-metric spaces. Many authors investigated contraction mappings in 2-metric spaces. Later on the notion of 3-metric space were introduced. Using the idea of fuzzy 2-metric space and fuzzy 3-metric space Sharma[21] extended the result of Fisher[6] and established fixed point theorems in fuzzy metric space.

The purpose of this paper is to generalize the result of Sharma[21] and obtain fixed point theorem in fuzzy metric space and then extend to fuzzy 2-metric and fuzzy 3-metric space.

Now we recall some known definitions and preliminary concepts.

Definition 1 [19]: A binary operation $*: [0,1]\times[0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ i.e $\forall a, b, c, d \in [0, 1]$. Examples of t- norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 2 [17]: The 3-tuple $(X, M, *)$ is called fuzzy metric space if X is an arbitrary set, M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM^1-1) $M(x, y, 0) = 0$,
(FM^1-2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
(FM^1-3) $M(x, y, t) = M(y, x, t)$,
(FM^1-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$,
(FM^1-5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous $\forall x, y, z \in X$ and $t, s > 0$.

Then M is called a fuzzy metric on X and $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Exampleb1[10]**(Induced fuzzy metric)**: Let (X, d) be a metric space, define $a*b = ab$ (or $a*b = \min\{a, b\}$) for all $a, b \in [0, 1]$ and Let M_d be fuzzy set on $X^2 \times [0, \infty)$ defined as.

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \tag{1}$$

$$\forall\, x, y \in X \text{ and } t > 0.$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric M_d induced by the metric d as the standard intuitionstic fuzzy metric.

Lemma 1[11]: For all $x, y \in X$, $M(x, y, \cdot)$ is nondecreasing.

Definition 3 [11]: Let $(X, M, *)$ be a fuzzy metric space. Then

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, denoted by

$$\lim_{n\to\infty} x_n = n \text{ if } \lim_{n\to\infty} M(x_{n,} x_n, t) = 1 \quad \forall\, t > 0.$$

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p,} x_n, t) = 1 \quad \forall t > 0 \text{ and } p > 0.$$

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark 1: Since * is continuous, it follows from (FM^1-4), i.e.

$M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$, that the limit of the sequence in fuzzy metric space is uniquely determined.

Let $(X, M, *)$ is a fuzzy metric space with the following condition.

(FM^1-6) $\lim_{t\to\infty} M(x, y, t) = 1 \quad \forall x, y \in X$

Lemma 2[3]: Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with condition (FM^1-6). If there exists a number $q \in (0,\ 1)$ such that

$$M(x_{n+2}, x_{n+1}, qt) \geq M(x_{n+1}, x_n, t) \tag{2}$$

for all $t > 0$ and $n = 1, 2,$ ....., then $\{x_n\}$ is a Cauchy sequence in X.

Proof: For $t > 0$ and $q \in (0,\ 1)$, we have $M(x_2, x_3, qt) \geq M(x_1, x_2, t) \geq M(x_0, x_1, t/q)$

or $M(x_2, x_3, qt) \geq M(x_0, x_1, t/q^2)$

by simple induction with the condition (2)[17], we have for all $t > 0$ and $n = 1, 2, ...$

$$M(x_{n+1}, x_{n+2}, t) \geq M(x_1, x_2, t/q^n) \tag{3}$$

thus by (3) and condition $(FM^1 - 4)$ for any positive integer p and real number $t > 0$, we have

$$\begin{aligned} M(x_n, x_{n+p}, t) &\geq M(x_n, x_{n+1}, t/p) * ......p - \ times * M(x_{n+p-1}, x_{n+p}, t/p) \\ &\geq M(x_1, x_2, t\big/pq^{n-1}) * ......p \ - times * M(x_1, x_2, t\big/pq^{n+p-2}) \end{aligned}$$

therefore by $(FM^1 - 6)$ we have,

$$\lim_{n\to\infty} M(x_n, x_{n+p}, t) \geq 1 * ....p \ - times.... * 1 \geq 1.$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. This completes the proof.

Lemma 3[18]: If for all $x, y \in X$, $t > 0$ and for a number $q \in (0, 1)$, $M(x, y, qt) \geq M(x, y, t)$, then $x = y$.

Lemmas 1, 2, 3 and Remark (1) hold for fuzzy 2 – metric spaces and fuzzy 3 – metric spaces also.

Definition 4: A function M is continuous in fuzzy metric space iff whenever $x_n \to x$, $y_n \to y$ then

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t).$$

for each $t > 0$.

Definition 5: Two mappings $A$and S on fuzzy metric space X are weakly commuting iff

$$M(ASu, SAu, t) \geq M(Au, Su, t).$$

for all $u \in X$ and $t > 0$.

Definition 6: A binary operation $* : [0, 1] * [0, 1] * [0, 1] \to [0, 1]$ is called a continuous t-norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that

$a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ for a_1, a_2, b_1, b_2 and c_1, c_2 are in $[0, 1]$.

Definition 7: The 3- tuple $(X, M, *)$ is a called a fuzzy 2-metric space if X is an arbitary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^3 \times [0, \infty)$ satisfying the following conditions; for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$.

$(FM^2 - 1)$ $M(x, y, z, 0) = 0$,
$(FM^2 - 2)$ $M(x, y, z, t) = 1$, $t > 0$ and when at least two of the three points are equal,
$(FM^2 - 3)$ $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$,
(Symmetry about three variables)
$(FM^2 - 4)$ $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$,
(This corresponds to tetrahedron inequality in 2-metric space)

The function value $M(x, y, z, t)$ may be interpreted as the probability the area of triangle is less than 1.

$(FM^2 - 5)$ $M(x, y, z, \cdot) : [0, 1) \to [0, 1]$ is left continuous.

Definition 8: Let $(X, M, *)$ be a fuzzy 2-metric space, then

(i) A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, a, t) = 1 \forall a \in X$ and $t > 0$.

(ii) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if $\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$, $\forall a \in X$ $t > 0$, $p > 0$.

(iii) A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 9: A function M is continuous in fuzzy 2-metric space iff whenever

$x_n \to x, y_n \to y$ then

$\lim_{n\to\infty} M(x_n, y_n, a, t) = M(x, y, a, t)$ for all $a \in X$ and $t > 0$.

Definition 10: Two mapping A and S on fuzzy 2-metric space X are weakly commuting iff $M(ASu, SAu, a, t) \geq M(Au, Su, a, t)$ for all $u, a \in X$ and $t > 0$.

Definition 11: A binary operation $* : [0, 1] * [0, 1] * [0, 1] * [0, 1] \to [0, 1]$ is called a continuous t-norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that

$a_1 * b_1 * c_1 * d_1 \leq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ and $d_1 \leq d_2$ for $a_1, a_2, b_1, b_2, c_1, c_2$ and d_1, d_2 are in $[0, 1]$.

Definition 12: The 3- tuple $(X, M, *)$ is called a fuzzy 3-metric space if X is an arbitary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^4 \times [0, \infty)$ satisfying the following conditions, for all $x, y, z, w, u \in X$ and $t_1, t_2, t_3, t_4 > 0$.

$(FM^3 - 1)$ $M(x, y, z, w, 0) = 0$,

$(FM^3 - 2)$ $M(x, y, z, w, t) = 1$ for all $t > 0$,
[only when the three simplex (x, y, z, w) degenerate]

$(FM^3 - 3)$ $M(x, y, z, t) = M(x, w, z, y, t) = M(y, z, w, x, t) = M(z, w, x, y, t) = ...$

$(FM^3 - 4)$ $M(x, y, z, w, t_1 + t_2 + t_3 + t_4)$
$\geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2) * M(x, u, z, w, t_3) * M(u, y, z, w, t_4)$

$(FM^3 - 5)$ $M(x, y, z, w) : [0, 1) \to [0, 1]$ is left continuous.

Definition 13: Let $(X, M, *)$ be a fuzzy 3-metric space, then-

(i) A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, a, b, t) = 1, \forall a, b \in X$ and $t > 0$.

(ii) A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if $\lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1 \forall a, b \in X$ and $t > 0$, $p > 0$.

(iii) A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 14: A function M is continuous in fuzzy 3-metric space iff whenever $x_n \to x$, $y_n \to y$, then $\lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t) \forall a, b \in Xt > 0$.

Definition 15: Two mappings A and S on fuzzy 3-metric space are weakly commuting iff $M(ASu, SAu, a, b, t) \geq M(Au, Su, a, b, t) \forall u, a, b \in X$ and $t > 0$.

Fisher[12] proved the following theorem for three mappings in complete metric space.

Theorem A: Let S and T be continuous mappings of a complete metric space (X, d) into itself. Then S and T have a common fixed point in X if there exists a continuous mapping A of X into $S(X) \cap T(X)$ which commute with S and T and satisfy, $d(Ax, Ay) \leq \alpha d(Sx, Ty)$ for all $x, y \in X$ and $0 < \alpha < 1$. Indeed S, T and A have a common unique fixed point.

Sharma[21] proved the following extend theorem A to fuzzy metric space, fuzzy 2-metric space and fuzzy 3-metric space.

Theorem B: Let $(X, M, *)$ be a complete fuzzy metric space with the condition $(FM^1 - 6)$ and let S and T be continuous mappings of X in X, then S and T have a common fixed point in X if there exists a continuous mapping A of X into $S(X) \cap T(X)$ which commute with S and T and,

$$M(Ax, Ay, qt) \geq Min\{M(Ty, Ay, t), M(Sx, Ax, t), M(Sx, Ty, t)\}$$

for all $x, y \in X$, $t > 0$ and $0 < q < 1$, then S, T and A have a unique common fixed point.

2. MAIN RESULT

In this paper we generalize the result of Sharma[21], Theorem B by using $(FM^1 - 6)$ to fuzzy metric space, fuzzy 2-metric space and fuzzy 3-metric space.

First we prove the following:

Theorem 1: Let $(X, M, *)$ be a complete fuzzy metric space with the condition $(FM^1 - 6)$ and A and B be continuous mappings of X in X, then A and B have a common fixed point in X if there exists mapping T of X into $A(X) \cap B(X)$ which commute with A and B and

$$\begin{aligned} M(Tx, Ty, qt) \geq \; & \min\{M(By, Ty, t), M(Ax, Tx, t), \\ & M(Ax, By, t), [M(Tx, Bx, t/2) * M(Bx, Ty, t/2)], \\ & [M(Tx, By, t/2) * (By, Ty, t/2)]\} \end{aligned} \quad (4)$$

for all $x, y \in X$, $t > 0$ and $0 < q < 1$. Then A, B and T have a unique common fixed point.

Proof: We define a sequence $\{x_n\}$ such that

$Tx_{2n} = Ax_{2n-1}$ and $Tx_{2n-1} = Bx_{2n}, n = 1, 2, ...$

we shall prove that $\{Tx_n\}$is a Cauchy sequence.

Suppose $x = x_{2n}$ and $y = x_{2n+1}$ in (4), we write.

$$\begin{aligned} M(Tx_{2n}, Tx_{2n+1}, qt) \geq \; & \min\{M(Bx_{2n+1}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n}, t), M(Ax_{2n}, Bx_{2n+1}, t), \\ & [M(Tx_{2n}, Bx_{2n}, t/2) * M(Bx_{2n}, Tx_{2n+1}, t/2)], \\ & [M(Tx_{2n}, Bx_{2n+1}, t/2) * M(Bx_{2n+1}, Tx_{2n+1}, t/2)]\} \\ \geq \; & \min\{M(Tx_{2n}, Tx_{2n+1}, t), M(Tx_{2n+1}, Tx_{2n}, t), M(Tx_{2n+1}, Tx_{2n}, t), \\ & [M(Tx_{2n}, Tx_{2n-1}, t/2) * M(Tx_{2n-1}, Tx_{2n+1}, t/2)], \\ & [M(Tx_{2n}, Tx_{2n}, t/2) * M(Tx_{2n}, Tx_{2n+1}, t/2)]\} \\ \geq \; & \min\{M(Tx_{2n}, Tx_{2n+1}, t), M(Tx_{2n}, Tx_{2n+1}, t), \\ & M(Tx_{2n}, Tx_{2n+1}, t), M(Tx_{2n}, Tx_{2n+1}, t), M(Tx_{2n}, Tx_{2n+1}, t)\} \\ \geq \; & \min\{M(Tx_{2n-1}, Tx_{2n}, t/q), M(Tx_{2n}, Tx_{2n-1}, t/q), \\ & M(Tx_{2n-1}, Tx_{2n}, t/q), M(Tx_{2n-1}, Tx_{2n}, t/q), M(Tx_{2n-1}, Tx_{2n}, t/q)\} \end{aligned}$$

therefore

$$M(Tx_{2n}, Tx_{2n+1}, qt) \geq M(Tx_{2n-1}, Tx_{2n}, t/q)$$

By induction

$$M(Tx_{2k}, Tx_{2m+1}, qt) \geq M(Tx_{2m}, Tx_{2k-1}, t/q)$$

for every k and m in N.

Further, if $2m + 1 > 2k$, then

$$\begin{aligned} M(Tx_{2k}, Tx_{2m+1}, qt) \geq \; & M(Tx_{2k-1}, Tx_{2m}, t/q) \\ & ..... \geq M(Tx_0, Tx_{2m+1-2k}, t/q^{2k}) \end{aligned} \quad (5)$$

If $2k > 2m + 1$, then

$$M(Tx_{2k}, Tx_{2m+1}, qt) \geq M(Tx_{2k-1}, Tx_{2m}, t/q)$$

$$..... \geq M(Tx_{2k-(2m+1)}, Tx_0, t/q^{2m+1}) \tag{6}$$

By simple induction with (5) & (6) we have

$$M(Tx_n, Tx_{n+p}, qt) \geq M(Tx_0, Tx_p, t/q^n)$$

for $n = 2k$, $p = 2m + 1$ or $n = 2k + 1$, $p = 2m + 1$.

And by $(FM^1 - 4)$,

$$M(Tx_n, Tx_{n+p}, qt) \geq M(Tx_0, Tx_1, t/2q^n) * M(Tx_1, Tx_p, t/2q^n) \tag{7}$$

if $n = 2k$, $p = 2m$ or $n = 2k + 1$, $p = 2m$.

For every positive integer p and n in N, by noting that

$M(Tx_0, Tx_p, t/q^n) \to 1$ as $n \to \infty$

Thus $\{Tx_n\}$ is a Cauchy sequence. Since the space X is complete, there exists

$z = \lim_{n\to\infty} Tx_n$ and $z = \lim_{n\to\infty} Ax_{2n-1} = Tx_{2n}$

It follows that $Tz = Az = Bz$ and

$$\begin{aligned} M\left(Tz, T^2z, qt\right) \geq\ & \min\{M(BTz, TTz, t), M(Tz, Bz, t), M(Tz, BTz, t), \\ & [M(Az, Bz, t/2) * M(Bz, TTz, t/2)], [M(Tz, BTz, t/2) * (BTz, TTz, t/2)]\} \\ \geq\ & \min\{1, 1, M(Tz, BTz, t), M(Az, TTz, t), M(Tz, TTz)\} \\ \geq\ & \min\left\{M(Tz, TBz, t), M\left(Az, T^2z, t\right), M\left(Tz, T^2z, t\right)\right\} \\ \geq\ & \min\left\{M(Tz, T^2z, t), M(Tz, T^2z, t), M(Tz, T^2z, t)\right\} \\ \geq\ & M\left(Tz, T^2z, t\right) \\ & \ldots\ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots \\ \geq\ & M\left(Tz, T^2z, t/q^n\right) \end{aligned}$$

Since $\lim_{n\to\infty} M\left(Tz, T^2z, t/q^n\right) = 1$, so $Tz = T^2z$

Thus z is a common fixed point of T, A and B.

For uniqueness, let $(w \neq z)$ be another common fixed point of A, B and T. By (4), we write

$$\begin{aligned} M(Tz, Tw, qt) \geq\ & \min\{M(Bw, Tw, t), M(Az, Tz, t), M(Az, Bw, t), \\ & [M(Tz, Bz, t/2) * M(Bz, Tw, t/2)], \\ & [M(Tz, Bw, t/2) * M(Bw, Tw, t/2)]\} \end{aligned}$$

which implies that

$$M(z, w, qt) \geq M(z, w, t)$$

Therefore by lemma 3, we write $z = w$.

This completes the proof. □

We now prove Theorem 1 for fuzzy 2-metric space.

Theorem 2: Let $(X, M, *)$ be a complete fuzzy 2-metric space and let A and B be continuous mappings of X in X, then A and B have a common fixed point in X if there exists continuous mapping T of X into $A(X) \cap B(X)$ which commute with A and B and

$$M(Tx, Ty, a, qt) \geq \min\{M(By, Ty, a, t), M(Ax, Tx, a, t), M(Ax, By, a, t),$$

$$[M(Tx,Ty,Bx,t/3) * M(Tx,Bx,a,t/3) * M(Bx,Ty,a,t/3)],$$
$$[M(Tx,Ty,By,t/3) * M(Tx,By,a,t/3) * M(By,Ty,a,t/3)]\} \quad (8)$$

for all x, y, a in X, $t > 0$ and $0 < q < 1$,

$$\lim_{t\to\infty} M(x,y,z,t) = 1 \text{ for all } x,y,z \text{ in } X. \quad (9)$$

Then A, B and T have a unique fixed point.

Proof: We define a sequence $\{x_n\}$ such that

$Tx_{2n} = Ax_{2n-1}$ and $Tx_{2n-1} = Bx_{2n}$, $n = 1, 2, ...$

We shall prove that $\{Tx_n\}$ is a Cauchy sequence.

Suppose $x = x_{2n}$, $y = x_{2n+1}$, in (8), we write

$$\begin{aligned}
M(Tx_{2n},Tx_{2n+1},a,qt) \geq\ & \min\{M(Bx_{2n+1},Tx_{2n+1},a,t),M(Ax_{2n},Tx_{2n},a,t),M(Ax_{2n},Bx_{2n+1},a,t),\\
& [M(Tx_{2n},Tx_{2n+1},Bx_{2n},t/3) * M(Tx_{2n},Bx_{2n},a,t/3) *\\
& M(Bx_{2n},Tx_{2n+1},a,t/3)],\\
& [M(Tx_{2n},Tx_{2n+1},Bx_{2n+1},t/3) * M(Tx_{2n},Bx_{2n+1},a,t/3) *\\
& M(Bx_{2n+1},Tx_{2n+1},a,t/3)]\}\\
\geq\ & \min\Big\{M(Tx_{2n},Tx_{2n+1},a,t),\ M(Tx_{2n+1},Tx_{2n},a,t) * M(Tx_{2n+1},Tx_{2n},a,t),\\
& M\Big(Tx_{2n},Tx_{2n+1},a,\frac{t}{3}+\frac{t}{3}+\frac{t}{3}\Big), M\Big(Tx_{2n},Tx_{2n+1},a,\frac{t}{3}+\frac{t}{3}+\frac{t}{3}\Big)\Big\}\\
\geq\ & \min\left\{\begin{array}{l} M(Tx_{2n},Tx_{2n+1},a,t),M(Tx_{2n},Tx_{2n+1},a,t),M(Tx_{2n},Tx_{2n+1},a,t),\\ M(Tx_{2n},Tx_{2n+1},a,t),M(Tx_{2n},Tx_{2n+1},a,t)\end{array}\right\}\\
\geq\ & \min\left\{\begin{array}{l} M(Tx_{2n-1},Tx_{2n},a,t/q),M(Tx_{2n-1},Tx_{2n},a,t/q),\\ M(Tx_{2n-1},Tx_{2n},a,t/q),M(Tx_{2n-1},Tx_{2n},a,t/q),\\ M(Tx_{2n-1},Tx_{2n},a,t/q)\end{array}\right\}
\end{aligned}$$

which gives

$$M(Tx_{2n},Tx_{2n+1},a,qt) \geq M(Tx_{2n-1},Tx_{2n},a,t/q)$$

By induction

$$M(Tx_{2n},Tx_{2m+1},a,qt) \geq M(Tx_{2m},Tx_{2k-1},a,t/q)$$

for every k and m in N. Further, if $2m+1 > 2k$, then

$$\begin{aligned}
M(Tx_{2k},Tx_{2m+1},a,qt) \geq\ & M(Tx_{2k-1},T_{2m},a,t/q)\\
& .... \geq M\left(Tx_0,T_{2m+1-2k},a,t\big/q^{2k}\right)
\end{aligned} \quad (10)$$

If $2k > 2m+1$, then

$$M(Tx_{2k},Tx_{2m+1},a,qt) \geq M\left(Tx_{2k-(2m+1)},Tx_0,a,t\big/q^{2m+1}\right) \quad (11)$$

By simple induction with (10) and (11) we have

$$M\left(Tx_n,Tx_{n+p},a,qt\right) \geq M\left(Tx_0,Tx_p,a,t/q^n\right)$$

for $n = 2k$, $p = 2m+1$ or $n = 2k+1$, $p = 2m+1$

And by $\left(FM^1 - 4\right)$

$$M\left(Tx_n, Tx_{n+p}, a, qt\right) \geq M\left(Tx_0, Tx_p, Tx_1, \frac{t}{3q^n}\right) * M\left(Tx_0, Tx_1, a, \frac{t}{3q^n}\right) * M\left(Tx_1, Tx_p, a, \frac{t}{3q^n}\right) \quad (12)$$

if $n = 2k$, $p = 2m$ or $n = 2k + 1$, $p = 2m$.

For every positive integer p and n in N, by noting that

$$M\left(Tx_0, Tx_p, a, \frac{t}{q^n}\right) \to 1 \text{ as } n \to \infty.$$

Thus $\{Tx_n\}$ is a Cauchy sequence. Since the space X is complete, these exists

$$z = \lim_{n\to\infty} Tx_n \text{ and } z = \lim_{n\to\infty} Tx_{2n-1} = Tx_{2n}$$

It follows that $Tz = Az = Bz$ and

$$\begin{aligned} M\left(Tz, T^2z, a, qt\right) &\geq \min\Big\{M(BTz, TTz, a, t), M(Az, Tz, a, t), M(Az, BTz, a, t), \\ &\quad \left[M\left(Tz, TTz, Bz, \frac{t}{3}\right) * M\left(Tz, Bz, a, \frac{t}{3}\right) * M\left(Bz, TTz, a, \frac{t}{3}\right)\right], \\ &\quad \left[M\left(Tz, TTz, BTz, \frac{t}{3}\right) * M\left(Tz, BTz, a, \frac{t}{3}\right) * M\left(BTz, TTz, a, \frac{t}{3}\right)\right]\Big\} \\ &\geq M(Az, TBz, a, t) \\ &\geq M(Tz, TTz, a, t) \\ &\geq M\left(Tz, T^2z, a, t\right) \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots \\ &\geq M\left(Tz, T^2z, a, \frac{t}{q^n}\right). \end{aligned}$$

Since $\lim_{n\to\infty} M\left(Tz, T^2z, a, t/q^n\right) = 1$, so $Tz = T^2z$.

Thus z is a common fixed point of T, A and B. For uniqueness, let $w\,(w \neq z)$ be another common fixed point of A, B and T. By (8), we write –

$$\begin{aligned} M(Tz, Tw, a, qt) \geq\ & \min\{M(Bw, Tw, a, t), M(Az, Tz, a, t), M(Az, Bw, a, t), \\ & M(Tz, Tw, a, t)\ , M(Tz, Tw, a, t)\} \end{aligned}$$

which implies that

$$M(z, w, a, qt) \geq M(z, w, a, t)$$

therefore by lemma 3 we write $z = w$.

This completes the proof. □

Now we prove Theorem 1 for Fuzzy 3-metric space as follows :

Theorem 3: Let $(X, M, *)$ be a complete fuzzy 3metric space and let A and B be continuous mappings of X in X, then A and B have a common fixed point in X if there exists continuous mapping T of X into $A(X) \cap B(X)$ which commute with A and B and

$$M(Tx, Ty, a, b, qt) \geq \min\{M(By, Ty, a, b, t), M(Ax, Tx, a, b, t), M(Ax, By, a, b, t),$$

$$\left[M\left(Tx, Ty, a, Bx, \frac{t}{4}\right) * M\left(Tx, Ty, Bx, b, \frac{t}{4}\right) * M\left(Tx, Bx, a, b, \frac{t}{4}\right) * M\left(Bx, Ty, a, b, \frac{t}{4}\right)\right], \left[M\left(Tx, Ty, a, By, \frac{t}{4}\right) * M\left(Tx, Ty, By, b, \frac{t}{4}\right) * M\left(Tx, By, a, b, \frac{t}{4}\right) * M\left(By, Ty, a, b, \frac{t}{4}\right)\right]\Big\} \quad (13)$$

for all x, y, a, b in X, $t > 0$ and $0 < q < 1$,

$$\lim_{t\to\infty} M(x, y, z, w, t) = 1 \quad (14)$$

for all x, y, z, w in X. Then A, B and T have a unique common fixed point.

Proof: We define a sequence $\{x_n\}$ such that $Tx_{2n} = Ax_{2n-1}$ and $Tx_{2n-1} = Bx_{2n}$, $n = 1, 2, ...$

We shall prove that $\{Tx_n\}$ is Cauchy sequence.

Suppose $x = x_{2n}$, $y = x_{2n+1}$ in (13), we write

$$\begin{aligned} M(Tx_{2n}, Tx_{2n+1}, a, b, qt) \geq\ & \min\Big\{M(Bx_{2n+1}, Tx_{2n+1}, a, b, t), M(Ax_{2n}, Tx_{2n}, a, b, t), M(Ax_{2n}, Bx_{2n+1}, a, b, t), \\ & \left[M\left(Tx_{2n}, Tx_{2n+1}, a, Bx_{2n}, \frac{t}{4}\right) * M\left(Tx_{2n}, Tx_{2n+1}, Bx_{2n}, b, \frac{t}{4}\right) * \right. \\ & \left. M\left(Tx_{2n}, Bx_{2n}, a, b, \frac{t}{4}\right) * M\left(Bx_{2n}, Tx_{2n+1}, a, b, \frac{t}{4}\right)\right], \\ & \left[M\left(Tx_{2n}, Tx_{2n+1}, a, Bx_{2n+1}, \frac{t}{4}\right) * M\left(Tx_{2n}, Tx_{2n+1}, Bx_{2n+1}, b, \frac{t}{4}\right) * \right. \\ & \left. M\left(Tx_{2n}, Bx_{2n+1}, a, b, \frac{t}{4}\right) * M\left(Bx_{2n+1}, Tx_{2n+1}, a, b, \frac{t}{4}\right)\right]\Big\} \\ \geq\ & \min\{M(Tx_{2n}, Tx_{2n+1}, a, b, t), M(Tx_{2n+1}, Tx_{2n}, a, b, t), \\ & M(Tx_{2n+1}, Tx_{2n}, a, b, t), M(Tx_{2n}, Tx_{2n+1}, a, b, t), \\ & M(Tx_{2n}, Tx_{2n+1}, a, b, t)\} \end{aligned} \quad (15)$$

which gives

$$M(Tx_{2n}, Tx_{2n+1}, a, b, qt) \geq M(Tx_{2n-1}, Tx_{2n}, a, b, t/q).$$

By induction

$$M(Tx_{2k}, Tx_{2m+1}, a, b, qt) \geq M(Tx_{2m}, Tx_{2k-1}, a, b, t/q)$$

for every k and m in N.

Further, if $2m + 1 > 2k$, then

$$\begin{aligned} M(Tx_{2k}, Tx_{2m+1}, a, b, qt) \geq\ & M(Tx_{2k-1}, Tx_{2m}, a, b, t/q) \\ & .... \\ & .... \\ \geq\ & M\left(Tx_0, Tx_{2m+1-2k}, a, b, \frac{t}{q^{2k}}\right) \end{aligned} \quad (16)$$

If $2k > 2m + 1$, then

$$M(Tx_{2k}, Tx_{2m+1}, a, b, qt) \geq M\left(Tx_{2k-(2m+1)}, Tx_0, a, b, \frac{t}{q^{2m+1}}\right) \quad (17)$$

By simple induction with (16) and (17) we have

$$M\left(Tx_n, Tx_{n+p}, a, b, qt\right) \geq M\left(Tx_0, Tx_p, a, b, \frac{t}{q^n}\right)$$

for $n = 2k$, $p = 2m + 1$ or $n = 2k + 1$, $p = 2m + 1$.

And by $\left(FM^1 - 4\right)$

$$\begin{aligned} M\left(Tx_n, Tx_{n+p}, a, b, qt\right) \geq & M\left(Tx_0, Tx_p, a, Tx_1, \frac{t}{4q^n}\right) * M\left(Tx_0, Tx_p, Tx_1, b, \frac{t}{4q^n}\right) * \\ & M\left(Tx_0, Tx_1, a, b, \frac{t}{4q^n}\right) * M\left(Tx_1, Tx_p, a, b, \frac{t}{4q^n}\right) \end{aligned} \tag{18}$$

if $n = 2k$, $p = 2m$ or $n = 2k + 1$, $p = 2m$.

For every positive integer p and n in N, by noting that

$$M\left(Tx_0, Tx_p, a, b, \frac{t}{q^n}\right) \to 1 \text{ as } n \to \infty.$$

Thus $\{Tx_n\}$ is a Cauchy sequence. Sine the space X is complete, these exists

$$z = \lim_{n\to\infty} Tx_n \text{ and } z = \lim_{n\to\infty} Tx_{2n-1} = Tx_{2n}.$$

It follows that $Tz = Az = Bz$ and

$$\begin{aligned} M(Tz, TTz, a, b, qt) &\geq \min\{M(BTz, TTz, a, b, t), M(Az, Tz, a, b, t), M(Az, BTz, a, b, t), \\ &\quad M(Tz, TTz, a, b, t), M(Tz, TTz, a, b, t)\} \\ &\geq M(Az, TTz, a, b, t) \\ &\geq M(Tz, TTz, a, b, t/q^n) \end{aligned}$$

Since $\lim_{n\to\infty} M\left(Tz, T^2z, a, b, t/q^n\right) = 1$

So $Tz = T^2z$.

Thus z is a common fixed point of T, A and B.

For uniqueness, let $w\,(w \neq z)$ be another fixed point of A, B and T.

By (13) we write

$$M(Tz, Tw, a, b, qt) \geq \min\{M(Bw, Tw, a, b, t), M(Az, Tz, a, b, t), M(Az, Bw, a, b, t),$$

$$M(Tz, Tw, a, b, t), M(Tz, Tw, a, b, t)\}$$

which implies that

$$M(z, w, a, b, qt) \geq M(z, w, a, b, t).$$

Therefore by lemma 3, we have $z = w$.

This completes the proof. □

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