CSCanada
Studies in Mathematical Sciences
Vol.8, No.1 2014, pp. 27-40
DOI: 10.3968/2954

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net
www.cscanada.org

On $\Lambda_{f_p}^{2q}(\Delta_u^{mn}, A, \phi)_\mu^\eta$ Defined by Modulus

N. Subramanian[a],*

[a] Department of Mathematics, SASTRA University, Tanjore-613 401, India.
*Corresponding author.

Address: Department of Mathematics. SASTRA University, Tanjore-613 401, India; E-mail: nsmaths@yahoo.com

Received: November 12, 2013/ Accepted: January 5, 2014/ Published online: February 26, 2014

Abstract: In this paper we defined the $\Lambda_{f_p}^{2q}(\Delta_u^{mn}, A, \phi)_\mu^\eta$ defined by a modulus and exhibit some general properties of the space with an four dimensional infinite regular matrix.
Key words: Gai sequence; Analytic sequence; Modulus function; Double sequences; Difference sequence; Lacunary sequence; Statistical convergence

Subramanian, N. (2014). On $\Lambda_{f_p}^{2q}(\Delta_u^{mn}, A, \phi)_\mu^\eta$ Defined by Modulus. *Studies in Mathematical Sciences*, *8*(1), 27-40. Available from URL: http://www.cscanada.net/index.php/sms/article/view/2954
DOI: http:// dx.doi.org/10.3968/2954

1. INTRODUCTION

Throughout ω, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich (1965). Later on, they were investigated by Hardy (1917), Moricz (1991), Moricz and Rhoades (1988), Basarir and Solankan (1999), Tripathy (2003), Turkmenoglu (1999), and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : \sup_{m,n\in N}|x_{mn}|^{t_{mn}} < \infty\right\},$$
$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \; for\; some\; l \in \mathbb{C}\right\},$$
$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty}|x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \{(x_{mn}) \in w^2 : \Sigma_{m=1}^{\infty}\Sigma_{n=1}^{\infty}|x_{mn}|^{t_{mn}} < \infty\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t),$$

Where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n} \to \infty$ denote the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$, and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak (2004, 2005) have proved that $\mathcal{M}_u(t)$, and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α-, β-, γ- duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her Ph.D. thesis, Zelter (2001) has essentially studied both the theory of topological double sequence spacesw and the theory of summability of double sequences. Mursaleen and Edely (2003) have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen (2004) Mursaleen and Edely (2004) have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M - core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$ core of x. More recently, Altay and Basar (2005) have defined the spaces $\mathcal{BS}, \mathcal{BS}(t)$; $\mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the α- duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\vartheta)$- duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double seties. Quite recently Basar and Sever (2009) have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra (2010) have studied the space $\chi_M^2\ (p, q, u)$ of double sequences and gave some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner (1946), Maddox (1979), and others. The class of sequences which are strongly Cesàro summable withrespect to a modulus was introduced by Maddox (1986) as an extension of the definition of strongly Cesàro summable sequences. Connor (1989) further extended this definition to a definition of strong A-summability with respect to a modulus where

$A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A- statistical convergence. In (1900) the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \Sigma_{m=1}^{\infty}\Sigma_{n=1}^{\infty}\, a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

Let Φ_{mn} denotes the set of all subsets of $\mathbb{N}$, those do not contain more than (mn) elements. Further (Φ_{mn}) will denote a non-decreasing sequence of positive real numbers such that $mn\Phi_{m+1,n+1} \leq (m+1, n+1)\Phi_{mn}$, for all $m, n \in \mathbb{N}$.

Now, if $u = (u_{mn})$ is any sequence such that $u_{mn} \neq 0$ for each m, n and $w^2\ (X)$ denote the space of all sequences with elements in X, where (X, q) denotes a semi normed space, seminormed by q, and η, μ is any real number such that $\eta, \mu \geq 0$. This will be accomplished by presenting the following sequence space:

$$\Lambda_{f_p}^{2q}\ (\Delta_u^{mn}, \mathrm{A}, \phi)_\mu^\eta = \left\{x \in w^2 : \sup_{m,n\geq 1, \sigma\in\emptyset_{mn}} \frac{1}{\Phi_{mn}} \Sigma_{r\in\sigma}\Sigma_{s\in\sigma}(sr)^{-\eta\mu} f(q\ (|A_{rs}(\Delta_u^{mn} x)|^{1/m+n})^{p_{rs}}) < \infty\right\};$$

where f is a modulus function. Other implications, general properties and variations will also be presented.

We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0<p<1$; we have

$$(a+b)^p \leq a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$(see [1]).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\Phi = \{\text{all finite sequences}\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{|m,n|}$ of the sequence is defined by $x^{|m,n|} = \sum_{i,j=0}^{m,n} x_{ij}\mathfrak{I}_{ij}$ for all $m, n \in \mathbb{N}$; where $\mathfrak{I}_{ij}$

denotes the double sequence whose only non-zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\mathfrak{I}_{mn})$ is a Schauder basis for X. Or equivalently $x^{|m,n|} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m,n \in \mathbb{N}))$ are also continuous.

Orlicz (1936) used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri (1971) investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_P(1 \leq p \leq \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary (1994), Mursaleen et al. (1999), Bektas and Altin (2003), Tripathy et al. (2003), Rao and Subramanian (2004), and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6] an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano (1973) and further discussed by Ruckle (1986) and Maddox (1986), and many others.

An Orlicz function M is said to satisfy the $\Delta_2 -$ condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)(u \geq 0)$. The $\Delta_2 -$ condition is equivalent to $M\ (\ell u) K\ell M(u)$ for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p(1 \leq p \leq \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_P.

If X is a sequence space, we give the following definitions:

(i) X' = the continuous dual of X;

(ii) $X^{\alpha}=\{a = (a_{mn}): \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{for each } x \in X\}$;

(iii) $X^{\beta}\{a = (a_{mn}): \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convegent, for each } x \in X\}$;

(iv) $X^{\gamma} = \{a = (a_{mn}): \sup_{mn} \geq 1|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}| < \infty, \text{for each } x \in X\}$;

(v) let X bean $F\ K -$ space $\supset \emptyset$; then $X^{f} = \{f(\Im_{mn}): f \in X'\}$;

(vi) $X^{\delta} = \{a = (a_{mn}): \sup_{mn}|a_{mn}x_{mn}|^{1/m+n} < \infty, \text{for each } x \in X\}$;

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called $\alpha -$ (or Köothe $-$ Toeplitz) dual of X, β- (or generalized - Köothe $-$ Toeplitz) dual of X, γ- dual of X, δ - dual of X respectively. X^{α} is defined by Gupta and Kamptan (1981). It is clear that $x^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz (1981) as follows

$$Z(\Delta) = \{x = (x_k) \in w: (\Delta_{x_k}) \in Z\},$$

for Z = c, c_0 and ℓ_{∞}, where $\Delta_{x_k} = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_{∞} denote the classes of convergent, null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$, BaȘar and Altay in [42] and in the case $0 \leq p \leq 1$, by BaȘar and Altay in [43]. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k\geq 1}|\Delta x_k| \text{ and } \|x\|_{bv_p} = (\sum_{k=1}^{\infty}|x_k|^p)^{1/p}, (1 \leq p < \infty,).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2: (\Delta x_{mn}) \in Z\},$$

where $Z = \Lambda^2$, X^2 and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

2. DEFINITIONS AND PRELIMINATIES

$\Lambda_{f_p}^{2q}(\Delta_u^{mn}, A, \phi)_{\mu}^{\eta}$ denote the Pringscheims sense of double analytic sequence space of modulus.

2.1 Definition. A modulus function was introduced by Nakano (1953). We recall that a modulus f is a function from $[0, \infty) \to [0, \infty)$, such that

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from here that f is continuous on $[0, \infty)$.

2.2. Definition. Let $A = (a_{k,\ell}^{mn})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the k, $\ell - th$ term to Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn}\, x_{mn}$$

such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman (2011) and Toeplitz (1911). Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is P- convergent is not necessarily bounded.

2.3 Definition. For a subspace ψ of a linear space is said to be sequence algebra if $x, y \in \psi$, implies that $x \cdot y = (x_{mn}y_{mn}) \in \psi$.

2.4 Definition. A sequence E is said to be solid (Or normal) if $(\lambda_{mn}x_{mn}) \in E$, wherener $(x_{mn}) \in E)$ for all sequences of scalars $(\lambda_{mn} = k)$ with $|\lambda_{mn}| \leq 1$.

2.5 Definition. A double sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

2.6 Remark. From the above, it is clear that a sequence space E is solid implies that E is monotone.

2.7 Definition. Let X be a real or complex linear space, g be a function from X to the set $\mathbb{R}$ of real numbers. Then, the pair (X, g) is called a paranormed space and g is a paranorm for X, if the following axioms are satisfied for all elements $x, y \in E$ and for all scalars α

(PN.1) $g(x) = 0$ if $x = \theta$.
(PN.2) $g(-x) = g(x)$.
(PN.3) $g(x + y) \leq g(x) + g(y)$.
(PN.4) If (α_n) is a sequence of scalars with $\alpha_n \to \alpha$ as $n \to \infty$ and x_n , x

$\in X$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to \infty$ then $\alpha_n\, x_n \to \alpha x$ as $n \to \infty$, in the sense that $g(\alpha x_n - \alpha x) \to 0$ as $n \to \infty$.

3. MAIN RESULTS

3.1 **Theorem.** $\Lambda_{f_p}^{2q}\,(\Delta_u^{mn},\ \mathrm{A}, \phi)_\mu^\eta$ is linear space over the complex field $\mathbb{C}$.

Proof: It is easy. Therefore omit the proof.

3.2 Theorem. $\Lambda_{f_p}^{2q}\,(\Delta_u^{mn},\ \mathrm{A}, \phi)_\mu^\eta$ is a paranormed space with

$$g\,(x) = \sup{}_{m,n\geq 1, \sigma\epsilon\emptyset_{mn}} \frac{1}{\emptyset_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma} (rs)^{-\eta\mu} f(q\,(|A_{rs}(\Delta_u^{mn} x)|^{1/m+n})^{p_{rs}/M}), \quad (3.1)$$

if and only if $h = in\,f p_{rs} > 0$, where M = max (1, H) and H = supp$_{mn.}$

(ii)$\Lambda_{f_p}^{2q}\,(\Delta_u^{mn}, \mathrm{A}, \phi)_\mu^\eta$ is a complete paranormed linear metric space if the condition p in (3.1) is satisfied.

Proof(i): suffciency: Let $h > 0$. It is trivial that $g(\theta) = 0$ and $g(-x) = g(x)$.

The inequality $g(x + y) \leq g(x) + g(y)$ follows from the inequality (3.1), since $p_{rs}/M \leq 1$ for all positive integers r, s. We also may write $g(\lambda x) \leq \max(|\lambda|, |\lambda|^{h/M})\ g(x)$, since $|\lambda|^{p_{mn}} \leq \max(|\lambda|^h, |\lambda|^M)$ for all positive integers r, s and for any $\lambda \in \mathbb{C}$, the set of complex numbers. Using this inequality, it can be proved that $\lambda x \to \theta$, when x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

Necessity: $\Lambda_{f_p}^{2q}\,(\Delta_u^{mn},\ \mathrm{A}, \phi)_\mu^\eta$ Let be a paranormed space with the paranorm.

$$g\,(x) = \sup{}_{m,n\geq 1, \sigma\epsilon\emptyset_{mn}} \frac{1}{\emptyset_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma} (rs)^{-\eta\mu} f(q\,(|A_{rs}(\Delta_u^{mn} x)|^{1/m+n})^{p_{rs}/M})$$

and suppose that $h = 0$. Since $|\lambda|^{p_{rs}/M} \leq |\lambda|^{h/M} = 1$ for all positive integers r, s and $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < 1$, we have

$$g\,(x) = \sup{}_{m,n\geq 1, \sigma\epsilon\emptyset_{mn}} \frac{1}{\emptyset_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma} (rs)^{-\eta\mu} f(q\,(|\lambda|)^{p_{rs}/M}) = 1.$$

Hence it follows that

$$g\,(\lambda x) = \sup{}_{m,n\geq 1, \sigma\epsilon\emptyset_{mn}} \frac{1}{\emptyset_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma} (rs)^{-\eta\mu} f(q\,(|\lambda|)^{p_{rs}/M}) = 1,$$

for $x = (\alpha) \in \Lambda_{f_p}^{2q}\,(\Delta_u^{mn},\ \mathrm{A}, \phi)_\mu^\eta$ as $\lambda \to 0$. But this contradicts the assumption $\Lambda_{f_p}^{2q}\,(\Delta_u^{mn}, \mathrm{A}, \phi)_\mu^\eta$ is a paranormed space with $g\,(x)$.

(ii) The proof is clear.

3.3 Corollary. $\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta$ is a complete paranormed space with the natural paranorm if and only if $\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta = \Lambda_{f}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta$.

3.4 Theorem. $\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi^1)_\mu^\eta \subseteq \Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi^2)_\mu^\eta$ if and only if $\sup_{mn\geq 1}\frac{\phi_{mn}^1}{\phi_{mn}^2} < \infty$.

Proof: Let $x\in\ \Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi^1)_\mu^\eta$ and T= $\sup_{mn\geq 1}\frac{\phi_{mn}^1}{\phi_{mn}^2}$. Then

$$\sup_{m,n\geq 1,\sigma\ \in\emptyset_{mn}}\ \frac{1}{\phi_{mn}^2}\ \sum_{r\in\sigma}\sum_{s\in\sigma}(rs)^{-\eta\mu}f(q\left(|A_{rs}(\Delta_u^{mn}x)|^{1/m+n}\right)^{p_{rs}}) \leq$$

$$\sup_{mn\geq 1}\frac{\phi_{mn}^1}{\phi_{mn}^2}\sup_{m,n\geq 1,\sigma\ \in\emptyset_{mn}}\ \frac{1}{\phi_{mn}^1}\ \sum_{r\in\sigma}\sum_{s\in\sigma}(rs)^{-\eta\mu}f(q\left(|A_{rs}(\Delta_u^{mn}x)|^{1/m+n}\right)^{p_{rs}}) =$$

$$T\ \ \times\sup_{m,n\geq 1,\sigma\ \in\emptyset_{mn}}\ \frac{1}{\phi_{mn}^1}\ \sum_{r\in\sigma}\sum_{s\in\sigma}(rs)^{-\eta\mu}f(q\left(|A_{rs}(\Delta_u^{mn}x)|^{1/m+n}\right)^{p_{rs}}).$$

Therefore $x\in\ \Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\emptyset)_\mu^\eta$.

Conversely, let $\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi^1)_\mu^\eta \subseteq \Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi^2)_\mu^\eta$ and $x\in\ \Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi^1)_\mu^\eta$. We have

$$\sup_{m,n\geq 1,\sigma\ \in\emptyset_{mn}}\ \frac{1}{\phi_{mn}^1}\ \sum_{r\in\sigma}\sum_{s\in\sigma}(rs)^{-\eta\mu}f(q\left(|A_{rs}(\Delta_u^{mn}x)|^{1/m+n}\right)^{p_{rs}}) < \infty.$$

Suppose that $\sup_{mn\geq 1}\frac{\phi_{mn}^1}{\phi_{mn}^2}\ < \infty$. Then there exists a sequence of positive natural numbers $(m_i n_j)$ such that $\lim_{i,j}\to\infty\frac{\phi_{m_i n_j}^1}{\phi_{m_i n_j}^2} = \infty$. Hence we can write

$$\sup_{m,n\geq 1,\sigma\in\emptyset_{mn}}\ \frac{1}{\phi_{mn}^2}\ \sum_{r\in\sigma}\sum_{s\in\sigma}(rs)^{-\eta\mu}f(q\left(|A_{rs}(\Delta_u^{mn}x)|^{1/m+n}\right)^{p_{rs}}) \geq$$

$$\sup_{m,n\geq 1,\sigma\ \in\emptyset_{mn}}\ \frac{1}{\phi_{mn}^1}\ \sum_{r\in\sigma}\sum_{s\in\sigma}(rs)^{-\eta\mu}f(q\left(|A_{rs}(\Delta_u^{mn}x)|^{1/m+n}\right)^{p_{rs}}) =\ \infty.$$

Therefore $x\notin\ \Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi^2)_\mu^\eta$ which is a contradiction. Hence $\sup_{mn\geq 1}\frac{\phi_{mn}^1}{\phi_{mn}^2} < \infty$.

3.5 Theorem. Let f be an modulus function which satisfies the Δ_2- condition. Then $\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi^1)_\mu^\eta = \Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi^2)_\mu^\eta$ if and only if

$$\sup_{mn\geq 1}\frac{\phi_{mn}^1}{\phi_{mn}^2} < \infty \text{ and } \sup_{mn\geq 1}\frac{\phi_{mn}^2}{\phi_{mn}^1} < \infty.$$

3.6 Theorem. Let f and f_1 be modulus functions which satisfies the Δ_2- condition. Then $\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\ \subseteq\ \Lambda_{(f_0 f_1)_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta$.

Proof: Let $x \in \Lambda^{2q}_{f_p}\ (\Delta^{mn}_u, \mathrm{A}, \phi)^\eta_\mu$ and $\epsilon > 0$ be given and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \le t \le \delta$. Therefore we have writ

$$\sup_{m,n\ge 1, \sigma \in \emptyset_{mn}} \frac{1}{\phi_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma}(rs)^{-\eta\mu} f(f_1(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)^{p_{rs}})) =$$

$$\sup_{m,n\ge 1, \sigma \in \emptyset_{mn}} \frac{1}{\phi_{mn}} \sum\sum_1 (rs)^{-\eta\mu} f(f_1(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)^{p_{rs}})) +$$

$$\sup_{m,n\ge 1, \sigma \in \emptyset_{mn}} \frac{1}{\phi_{mn}} \sum\sum_2 (rs)^{-\eta\mu} f(f_1(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)^{p_{rs}})).$$

Where the summation $\sum\sum_1$ is over $f_1(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)) \le \delta$ and the summation $\sum\sum_2$ is over $f_1\left(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)\right) > \delta$. Since f is continuous, we have

$$\sup_{m,n\ge 1, \sigma \in \emptyset_{mn}} \frac{1}{\phi_{mn}} \sum\sum_1 (rs)^{-\eta\mu} f(f_1(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)^{p_{rs}})) \le$$

$$\max\{1, f(1)^H\} \sup_{m,n\ge 1, \sigma \in \emptyset_{mn}} \frac{1}{\phi_{mn}} \sum\sum_1 (rs)^{-\eta\mu} f_1(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)^{p_{rs}}) \le$$

$$\max\{1, f(1)^H\} \sup_{m,n\ge 1, \sigma \in \emptyset_{mn}} \frac{1}{\phi_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma} (rs)^{-\eta\mu} f_1(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)^{p_{rs}}).$$

For

$f_1\left(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)\right) > \delta$, we use the fact that

$$f_1\left(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)\right) < f_1\left(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)\right)\delta^{-1}$$

$$\le 1 + f_1\left(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)\right)\delta^{-1}$$

Since *f* satisfies the Δ_2- condition, then there exists $L > 1$ such that

$$f\left(f_1\left(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)\right)\right) f(1 + f_1\left(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)\right)\delta^{-1}) \le$$

$$\tfrac{1}{2} f(2) + \tfrac{1}{L} f(2\, f_1\left(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)\right)\delta^{-1}) +$$

$$\tfrac{1}{2} L f(2) f_1\left(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)\right)\delta^{-1}) =$$

$$L f(2)\, \delta^{-1} f_1\left(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)\right).$$

$$\sup_{m,n\ge 1, \sigma \in \emptyset_{mn}} \frac{1}{\phi_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma} (rs)^{-\eta\mu} f(f_1(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)^{p_{rs}}) \le$$

$$\max\{1, f(1)^H\} \sup_{m,n\ge 1, \sigma \in \emptyset_{mn}} \frac{1}{\phi_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma} (rs)^{-\eta\mu} f_1(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)^{p_{rs}})$$

$$+ \max\{1, Lf\ (2)\ \delta^{-1})^H\} \sup_{m,n\ge 1, \sigma \in \emptyset_{mn}} \frac{1}{\phi_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma} (rs)^{-\eta\mu} f_1(q\left(|A_{rs}(\Delta^{mn}_u x)|^{1/m+n}\right)^{p_{rs}}).$$

Therefore $x \in \Lambda^{2q}_{f_p}\ (\Delta^{mn}_u, \mathrm{A}, \phi)^\eta_\mu$.

3.7 Theorem. $\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta$ is not separable.

Proof: $f(q\left(|A_{rs}(\Delta_u^{mn}x)|^{1/m+n}\right)^{p_{rs}} \to 0$ as $m,n \to \infty$, so it may so happen that first row or column may not be convergent, even may not be bounded. Let *S* be the set that has double sequences such that the first row is built up of sequences of zeros and ones. Then *S* will be uncountable. Consider open balls of radius 3^{-1} units. Then these open balls will not cover $\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta$. Hence is not separable.

3.8 Remark. Let $f = (f_{mn})$ be a modulus function q_1 and q_2 be two seminorms on *X*, we have

(i) $\Lambda_{f_p}^{2q_1}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta \cap \Lambda_{f_p}^{2q_2}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta \subseteq \Lambda_{f_p}^{2(q_1+q_2)}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta$,

(ii) If q_1 is stronger than q_2 then $\Lambda_{f_p}^{2q_1}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta \cap \Lambda_{f_p}^{2q_2}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta$,

(iii) If q_1 is equivalent to q_2 then $\Lambda_{f_p}^{2q_1}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta = \Lambda_{f_p}^{2q_2}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta$.

3.9 Proposition. For every

$$p = (p_{rs}), \left\{\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}^\beta = \left\{\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}^\alpha =$$

$\left\{\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}^\gamma = \{\eta_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\}$, where $\eta_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta = \cap_{N\in N-\{1\}}$,

$\left\{x = x_{mn}: \frac{1}{\phi_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma}(rs)^{-\eta\mu} f(q\left(|A_{rs}(\Delta_u^{mn}x)|^{1/m+n}\right)^{p_{rs}}) < \infty\right\}$.

Proof(1): First we show that $\left\{\eta_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\} \subset \left\{\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}^\beta$.

Let $\in \left\{\eta_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}$ and $y \in \left\{\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}^\beta$. Then we can find a positive integer *N* such that

$\left(|y_{mn}|^{1/m+n}\right) <$

$\max\left(1, \sup_{m,n\geq 1} \frac{1}{\phi_{mn}} \sum_{r\in\sigma}\sum_{s\in\sigma}(rs)^{-\eta\mu} f(q\left(|A_{rs}(\Delta_u^{mn}x)|^{1/m+n}\right)^{p_{rs}}\right) < \infty) < N$,

for all *m*, *n*.

Hence we may write

$$\left|\sum_{m,n} x_{mn}y_{mn}\right| \leq \sum_{m,n}|x_{mn}y_{mn}| \leq \sum_{mn}\left(f(|x_{mn}y_{mn}|)\right) \leq \sum_{m,n}\left(f(|\Delta_u^{mn}\, x|N^{m+n})\right).$$

Since $x \in \left\{\eta_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}$. The series on the right side of the above inequality is convergent, whence $x \in \left\{\eta_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}$. Hence $\left\{\eta_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\} \subset \left\{\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}^\beta$.

Now we show that $\left\{\Lambda_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}^\beta \subset \left\{\eta_{f_p}^{2q}\ (\Delta_u^{mn},\mathrm{A},\phi)_\mu^\eta\right\}$.

For this, let $x \in \left\{\Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta\right\}^\beta$ and suppose that $x \notin \left\{\Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta\right\}$. Then there exists a positive integer $N > 1$ such that $\sum_{m,n}\left(f\left(|\Delta_u^{mn} x|N^{m+n}\right)\right) = \infty$.

If we define $y_{mn} = N^{m+n} Sgn \Delta_u^{mn}$ m, n =1, 2…, then $y \in \left\{\Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta\right\}$.

But, since $\left|\sum_{m,n} x_{mn} y_{mn}\right| = \sum_{mn}\left(f\left(|x_{mn} y_{mn}|\right)\right) = \sum_{m,n}\left(f\left(|\Delta_u^{mn} x|N^{m+n}\right)\right) = \infty$, we get $x \notin \left\{\Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta\right\}^\beta$, which contradicts to the assumption $x \in \left\{\Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta\right\}^\beta$. Therefore $x \in \left\{\eta_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta\right\}$.

Therefore $\left\{\Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta\right\}^\beta = \left\{\eta_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta\right\}$.

(ii) and (iii) can be shown in a similar way of (i). Therefore we omit it.

4. RESULT

4.1. Proposition. The space $\Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta$ is not monotone and such are not solid.

Proof: The space $\Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta$ is not monotone follows from following examples. Since the space $\Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta$ is not monotone, is not solid is clear from the remark 2.6.

Example: Let $X = \mathbb{C}$ and consider the sequence (x_{mn}) defined by if

$$(x_{mn}) = \begin{pmatrix} 1, 1, \ldots 1 \\ 1, 1, \ldots 1 \\ . \\ . \\ . \\ 1, 1, \ldots 1 \end{pmatrix} \text{ for all } m, n \in \mathbb{N}.$$

Consider the sequence (y_{mn}) in the preimage space defined as

$$y_{mn} = \begin{cases} i^2, \text{ if } m = n, i \in \mathbb{N} \\ 0, \quad \text{otherwise.} \end{cases}$$

Then $(x_{mn}) \notin \Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta$ but $(y_{mn}) \in \Lambda_{f_p}^{2q}\left(\Delta_u^{mn}, A, \phi\right)_\mu^\eta$ are not monotone.

4.2 Proposition. The space $\Lambda_{f_p}^{2q}$ $(\Delta_u^{mn}, \mathrm{A}, \phi)_\mu^\eta$ are not convergence free in general.

Proof: The proof follows from the following example:

Example: Consider the sequences (x_{mn}), $(y_{mn}) \in \Lambda_{f_p}^{2q}$ $(\Delta_u^{mn}, \mathrm{A}, \phi)_\mu^\eta$. Defined by $(\Delta_u^{mn}x) = (\frac{1}{m+n})^{1/m+n}$ and $(\Delta_u^{mn}y)$=$(\frac{m-n}{m+n})^{1/m+n}$. Hence $\sup_{m,n\geq 1,}(\frac{1}{m+n})^{1/m+n} < \infty$. which implies $\sup_{m,n\geq 1,}|x_{mn}|^{1/m+n} < \infty$. Also

$\sup_{m,n\geq 1,}$ $(\frac{m-n}{m+n})^{1/m+n} = 0$. Hence $\sup_{m,n\geq 1,}$ $|y_{mn}|^{1/m+n}$=0. Therefore the space $\Lambda_{f_p}^{2q}$ $(\Delta_u^{mn}, \mathrm{A}, \phi)_\mu^\eta$ are not convergence free.

4.3. Proposition. $\Lambda_{f_p}^{2q}$ $(\Delta_u^{mn}, \mathrm{A}, \phi)_\mu^\eta$ is not sequence algebra.

Proof: This result is clear the following example:

Example: Let (x_{mn}) = $(\frac{m}{m+n})^{1/m+n}$ and (y_{mn}) = $(\frac{-n}{m+n})^{1/m+n}$ for all $m, n \in \mathbb{N}$. Then we have $x, y \in \Lambda_{f_p}^{2q}$ $(\Delta_u^{mn}, \mathrm{A}, \phi)_\mu^\eta$ but

$x \cdot y \notin \Lambda_{f_p}^{2q}$ $(\Delta_u^{mn}, \mathrm{A}, \phi)_\mu^\eta$.

REFERENCES

[01] Apostol, T. (1978). *Mathematical analysis*. London: Addison-Wesley.

[02] Basarir, M., & Solancan, O. (1999). On some double sequence spaces. *J. Indian Acad. Math., 21*(2), 193-200.

[03] Bektas, C., & Altin, Y. (2003). The sequence space l_M *(p, q, s)* on seminormed spaces. *Indian J. Pure Appl. Math., 34*(4), 529-534.

[04] Bromwich, T. J. I'A. (1965). *An introduction to the theory of infinite series*. New York: Macmillan and Co. Ltd.

[05] Hardy, G. H. (1917). *On the convergence of certain multiple series* (pp.86-95). Proc. Camb. Phil. Soc.

[06] Krasnoselskii, M. A., & Rutickii, Y. B. (1961). *Convex functions and orlicz spaces*. Netherlands: Gorningen .

[07] Lindenstrauss, J., & Tzafriri, L. (1971). On orlicz sequence spaces. *Israel J. Math., 10*, 379-390.

[08] Maddox, I. J. (1986). Sequence spaces defined by a modulus. *Math. Proc. Cambridge Philos. Soc, 100*(1), 161-166.

[09] Moricz, F. (1991). Extentions of the spaces c and c0 from single to double sequences. *Acta. Math. Hung., 57*(1-2), 129-136.

[10] Moricz, F., & Rhoades, B. E. (1988). Almost convergence of double sequences and strong regularity of summability matrices. *Math. Proc. Camb. Phil. Soc., 104*, 283-294.

[11] Mursaleen, M. A. K., & Qamaruddin. (1999). Difference sequence spaces defined by Orlicz functions. *Demonstratio Math., Vol. XXXII*, 145-150.

[12] Nakano, H. (1953). Concave modular. *J. Math. Soc. Japan, 5*, 29-49.

[13] Orlicz,W. (1936). über Raume (L^M). *Bull. Int. Acad. Polon. Sci. A*, 93-107.

[14] Parashar, S. D., & Choudhary, B. (1994). Sequence spaces defined by Orlicz functions. *Indian J. Pure Appl. Math., 25*(4), 419-428.

[15] Rao, K. C., & Subramanian, N. (2004). The Orlicz space of entire sequences. *Int. J. Math. Sci., 68,* 3755-3764.

[16] Ruckle, W. H. (1973). FK spaces in which the sequence of coordinate vectors is bounded. Canad. *J. Math., 25*, 973-978.

[17] Tripathy, B. C. (2003). On statistically convergent double sequences. *Tamkang J. Math., 34*(3), 231-237.

[18] Tripathy, B. C., & Altin, Y. (2003). Generalized difference sequence spaces defferned by Orlicz function in a locally convex space. *J. Anal. Appl., 1*(3), 175-192.

[19] Turkmenoglu, A. (1999). Matrix transformation between some classes of double sequences. *J. Inst. Math. Comp. Sci. Math. Ser., 12*(1), 23-31.

[20] Kamthan, P. K., & Gupta, M. (1981). *Sequence spaces and series, lecture notes, pure and applied, mathematics.* New York: 65 Marcel Dekker, In c.

[21] Gökhan, A., & Colak, A. (2004). The double sequence spaces $c^P{}_2$ (p) and $c^{PB}{}_2$ (p). *Appl. Math. Comput., 157*(2), 491-501.

[22] Gökhan, A., & Colak, A. (2005). Double sequence spaces $l^\infty{}_2$. *ibid*., *160*(1), 147-153.

[23] Zeltser, M. (2001). *Investigation of double sequence spaces by soft and hard analitical methods.* Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu.

[24] Mursaleen, M., & Edely, O. H. H. (2003). Statistical convergence of double sequences. *J. Math. Anal. Appl., 288*(1), 223-231.

[25] Mursaleen, M. (2004). Almost strongly regular matrices and a core theorem for double sequences. *J. Math. Anal. Appl., 293*(2), 523-531.

[26] Mursaleen, M., & Edely, O. H. H. Almost convergence and a core theorem for double sequences. *J. Math. Anal. Appl., 293*(2), 532-540.

[27] Altay, B., & BaSar, F. (2005). Some new spaces of double sequences. *J. Math. Anal. Appl., 309*(1), 70-90.

[28] BaSar, F., & Sever, Y. (2009). The space *Lp* of double sequences. *Math. J. Okayama Univ, 51*, 149-157.

[29] Subramanian, N., & Misra, U. K. (2010). The semi normed space defined by a double gai sequence of modulus function. *Fasciculi Math., 46*.

[30] Kizmaz, H. (1981). On certain sequence spaces. *Cand. Math. Bull., 24*(2), 169-176.

[31] Kuttner, B. (1946). Note on strong summabilit. *J. London Math. Soc., 21*, 118-122.

[32] Maddox, I. J. (1979). On strong almost convergence. *Math. Proc. Cambridge Philos. Soc., 85*(2), 345-350.

[33] Cannor, J. (1989). On strong matrix summability with respect to a modulus and statistical convergence. *Canad. Math. Bull., 32*(2), 194-198.

[34] Pringsheim, A. (1900). Zurtheorie derzweifach unendlichen zahlenfolgen. *Math. Ann., 53*, 289-321.

[35] Hamilton, H. J. (1936). Transformations of multiple sequences. *Duke Math. J., 2,* 29-60.

[36] Hamilton, H. J. (1930). A Generalization of multiple sequences transformation. *Duke Math. J., 4*, 343-358.

[37] Hamilton, H. J. (1938). Change of Dimension in sequence transformatio , *Duke Math. J., 4,* 341-342.

[38] Hamilton, H. J. (1939). Preservation of partial Limits in Multiple sequence transformations. *Duke Math. J., 4,* 293-297.

[39] Robison, G. M. (1926). Divergent double sequences and series. *Amer. Math. Soc. Trans., 28,* 50-73.

[40] Silverman, L. L. (2011). *On the definition of the sum of a divergent series, un published thesis*. University of Missouri studies, Mathematics series. *61*(2), 245-256.

[41] Toeplitz, O. (1911). über allgenmeine linear mittel bridungen. *Prace Matemalyczno Fizyczne (warsaw), 22*.

[42] BaSar, F., & Atlay, B. (2003). On the space of sequences of *p*- bounded variation and related matrix mappings. *Ukrainian Math. J., 55*(1), 136-147.

[43] Altay, B., & BaSar, F. (2007). The fine spectrum and the matrix domain of the difference operator △ on the sequence space *lp*; $(0 < p < 1)$, Commun. *Math. Anal., 2*(2), 1-11.

[44] Colak, M. Et. R., & Malkowsky, E. (2004). *Some topics of sequence spaces, lecture notes in mathematics* (pp.1-63). Firat Univ. Elazig, Turkey, Firat Univ. Press, ISBN: 975-394-0386-6.