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Rate of Growth of Polynomials With Zeros on the Unit Disc

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Abstract: If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n satisfying $p(z)\neq 0$ in $|z|<1$, then for $R\geq 1$. Ankeny and Rivlin [1] proved that $M(p,R)\leq \left(\frac{R^n+1}{2}\right) M(p,1)$. In this paper we obtain some results in this direction by considering polynomials of degree ≥ 2, having all its zeros on $|z|=k$, $k\leq 1$.

Key words: Polynomial; Inequality; Zeros

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1. INTRODUCTION AND STATEMENT OF RESULTS

For an arbitrary entire function (z), let $M(f,r)=\max_{|z|=r} |f(z)|$.Then for a polynomial $p(z)$ of degree n, it is a simple consequence of maximum modulus principle (for reference see [4, vol. I, p. 137, Problem III, 269]) that

$$M(p,R)\leq R^n M(p,1), \text{ for } R\geq 1 \tag{1}$$

The result is best possible and equality holds for $p(z)=\lambda z^n$, where $|\lambda|=1$. $R\geq 1$.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequality (1) can be sharpened. In fact it was shown by Ankeny and Rivlin [1] that if $p(z) \neq 0$ in $|z| < 1$, then (1) can be replaced by

$$M(p, R) \leq \left(\frac{R^n+1}{2}\right) M(p, 1), \quad R \geq 1 \tag{2}$$

The result is sharp and equality holds for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

While trying to obtain inequality analogous to inequality (2) for polynomials not vanishing in $|z| < k, k \leq 1$, K K Dewan and Arty Ahuja [2] proved the following result.

Theorem A. If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k, k \leq 1$, then for every positive integer s

$$\{M(p, R)\}^s \leq \left(\frac{k^{n-1}(1+k)+(R^{ns}-1)}{k^{n-1}+k^n}\right)\{M(p, 1)\}^s, \quad R \geq 1 \tag{3}$$

By involving the coefficients of $p(z)$, Dewan and Ahuja [2] in the same paper obtained the following refinement of Theorem A.

Theorem B. If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k, k \leq 1$, then for every positive integer s

$$\{M(p, R)\}^s$$

$$\leq \frac{1}{k^n}\left[\frac{n|a_n|\{k^n(1+k^2)+k^2(R^{ns}-r^{ns})\}+|a_{n-1}|\{2k^n+R^{ns}-r^{ns}\}}{2|a_{n-1}|+n|a_n|(1+k^2)}\right]$$

$$\times\{M(p, 1)\}^s, \quad R \geq 1 \tag{4}$$

In this paper, we restrict ourselves to the class of polynomials of degree $n \geq 2$ having all its zeros on $|z| = k, k \leq 1$ and obtain an improvement and generalization of Theorem A and Theorem B. More precisely, we prove

Theorem 1. If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k, k \leq 1$, then for every positive integer s and $R \geq 1$

$$\{M(p,R)\}^s \le \left(\frac{k^{n-1}(1+k)+(R^{ns}-1)}{k^{n-1}+k^n}\right)\{M(p,1)\}^s$$

$$-s\,|a_1|\left(\frac{R^{ns}-1}{ns}-\frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1},$$

$$if\ n>2 \qquad (5)$$

and

$$\{M(p,R)\}^s \le \left(\frac{k^{n-1}(1+k)+(R^{ns}-1)}{k^{n-1}+k^n}\right)\{M(p,1)\}^s$$

$$-s\,|a_1|\left(\frac{R^{ns}-1}{ns}-\frac{R^{ns-1}-1}{ns-1}\right)\{M(p,1)\}^{s-1},$$

$$if\ n=2 \qquad (6)$$

By choosing $s = 1$ in Theorem 1.we get the following result.

Corollary 1. If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n \ge 2$ having all its zeros on $|z| = k, k \le 1$, then for $R \ge 1$

$$\{M(p,R)\} \le \left(\frac{k^{n-1}(1+k)+(R^{n}-1)}{k^{n-1}+k^n}\right)\{M(p,1)\}$$

$$-|a_1|\left(\frac{R^{n}-1}{n}-\frac{R^{n-1}-1}{n-2}\right),$$

$$if\ n>2 \qquad (7)$$

and

$$\{M(p,R)\} \le \left(\frac{k^{n-1}(1+k)+(R^{n}-1)}{k^{n-1}+k^n}\right)\{M(p,1)\}$$

$$-|a_1|\left(\frac{R^{n}-1}{n}-\frac{R^{n-2}-1}{n-1}\right),$$

$$if\ n=2 \qquad (8)$$

Next we prove the following result which is a refinement of Theorem 1.

Theorem 2. If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros on

$|z| = k, k \leq 1$, then for every positive integer s and $R \geq 1$

$$\{M(p,R)\}^s \leq \frac{1}{k^n}\left[\frac{n|a_n|\{k^n(1+k^2)+k^2(R^{ns}-1)\}+|a_{n-1}|\{2k^n+R^{ns}-1\}}{2|a_{n-1}|+n|a_n|(1+k^2)}\right]$$

$$\times\{M(p,1)\}^s - s\,|a_1|\left(\frac{R^{ns}-1}{ns}-\frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1},$$

$$if\ n > 2 \tag{9}$$

and

$$\{M(p,R)\}^s \leq \frac{1}{k^n}\left[\frac{n|a_n|\{k^n(1+k^2)+k^2(R^{ns}-1)\}+|a_{n-1}|\{2k^n+R^{ns}-1\}}{2|a_{n-1}|+n|a_n|(1+k^2)}\right]$$

$$\times\{M(p,1)\}^s - s\,|a_1|\left(\frac{R^{ns}-1}{ns}-\frac{R^{ns-1}-1}{ns-1}\right)\{M(p,1)\}^{s-1},$$

$$if\ n = 2 \tag{10}$$

If we choose $s = 1$ in Theorem 2, we get the following result.

Corollary 2. If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n \geq 2$ having all its zeros on $|z| = k, k \leq 1$, then for every $R \geq 1$

$$\{M(p,R)\}^s \leq \frac{1}{k^n}\left[\frac{n|a_n|\{k^n(1+k^2)+k^2(R^{n}-1)\}+|a_{n-1}|\{2k^n+R^{n}-1\}}{2|a_{n-1}|+n|a_n|(1+k^2)}\right]$$

$$\times\{M(p,1)\} - |a_1|\left(\frac{R^{n}-1}{n}-\frac{R^{n-2}-1}{n-2}\right),$$

$$if\ n > 2 \tag{11}$$

and

$$\{M(p,R)\}^s \leq \frac{1}{k^n}\left[\frac{n|a_n|\{k^n(1+k^2)+k^2(R^{n}-1)\}+|a_{n-1}|\{2k^n+R^{n}-1\}}{2|a_{n-1}|+n|a_n|(1+k^2)}\right]$$

$$\times\{M(p,1)\} - |a_1|\left(\frac{R^{n}-1}{n}-\frac{R^{n-1}-1}{n-1}\right),$$

$$if\ n = 2 \tag{12}$$

2. LEMMAS

For the proof of these theorems, we need the following lemmas.

Lemma 1. If $p(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n having all its zeros on $|z| = k, k \leq 1$, then

$$\max_{|z|=1} |p^{/}(z)| \leq \frac{n}{k^{n-1} + k^n} \max_{|z|=1} |p(z)| \tag{13}$$

The above lemma is due to Govil [3].

Lemma 2. If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros on $|\mathrm{z}| = \mathrm{k}, \mathrm{k} \leq 1$, then

$$\max_{|z|=1} |p^{/}(z)| \leq \frac{n}{k^n}\left[\frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n|(1+k^2) + 2|a_{n-1}|}\right] \max_{|z|=1} |p(z)| \tag{14}$$

The above lemma is due to Dewan and Mir [5].

Lemma 3. If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree, then for all $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq R^n M(p,1) - (R^n - R^{n-2})\,|p(0)|\,,\ if\ n > 1 \tag{15}$$

And

$$\max_{|z|=R} |p(z)| \leq R\,M(p,1) - (R-1)\,|p(0)|\,,\ if\ n = 1 \tag{16}$$

The above lemma is due to Frappier, Rahman and Ruscheweyh [6].

3. PROOF OF THE THEOREMS

Proof of Theorem 1. Let $(p,1) = \max_{|z|=1} |p(z)|$. Since p(z) is a polynomial of degree n having all its zeros on $|\mathrm{z}| = \mathrm{k}, \mathrm{k} \leq 1$, therefore, by Lemma 1, we have

$$\max_{|z|=1} |p^{/}(z)| \leq \frac{n}{k^{n-1} + k^n} \quad M(p,1) \quad for\ |z| = 1 \tag{17}$$

Now applying inequality (1) to the polynomial $\mathrm{p}^{/}(\mathrm{z})$ which is of degree $\mathrm{n} - 1$ and noting (17), it follows that for all $r \geq 1$ and $0 \leq \theta < 2\pi$

$$\left|p^{/}\left(re^{i\theta}\right)\right| \leq \frac{nr^{n-1}}{k^{n-1}+k^{n}} \quad M(p,1) \tag{18}$$

Also for each θ, $0 \leq \theta < 2\pi$ and $R \geq 1$, we obtain

$$\left\{p\left(Re^{i\theta}\right)\right\}^{s} - \left\{p\left(re^{i\theta}\right)\right\}^{s} = \int_{1}^{R} \frac{d}{dt}\left\{p\left(te^{i\theta}\right)\right\}^{s} dt$$

$$= \int_{1}^{R} s\left\{p\left(te^{i\theta}\right)\right\}^{s-1} p^{/}\left(te^{i\theta}\right)e^{i\theta}\, dt$$

This implies

$$\left|\left\{p\left(Re^{i\theta}\right)\right\}^{s} - \left\{p\left(re^{i\theta}\right)\right\}^{s}\right| \leq s \int_{1}^{R}\left|p\left(te^{i\theta}\right)\right|^{s-1}\left|p^{/}\left(te^{i\theta}\right)\right| d \tag{19}$$

Since p(z) is a polynomial of degree > 2, the polynomial $p^{/}(z)$ which is of degree $n-1 \geq 2$, hence applying inequality (15) of Lemma 3 to $p^{/}(z)$, we have for $r \geq 1$ and $0 \leq \theta < 2\pi$

$$\left|p^{/}\left(re^{i\theta}\right)\right| \leq r^{n-1}M\left(p^{/},1\right) - \left(r^{n-1}-r^{n-3}\right)\left|p^{/}(0)\right| \tag{20}$$

Inequality (20) in conjunction with inequalities (19) and (1), yields for $n > 2$ and for $R \geq 1$

$$\left|\left\{p\left(Re^{i\theta}\right)\right\}^{s} - \left\{p\left(re^{i\theta}\right)\right\}^{s}\right|$$

$$\leq s \int_{1}^{R} [t^{n}M(p,1)^{s-1}][t^{n-1}M\left(p^{/},1\right) - (t^{n-1}-t^{n-3})\left|p^{/}(0)\right|]dt$$

$$= s\int_{1}^{R} t^{ns-1} \{M(p,1)\}^{s-1}M\left(p^{/},1\right)$$

$$-(t^{ns-1}-t^{ns-3})\{M(p,1)\}^{s-1}\left|p^{/}(0)\right|]dt$$

$$= s\left[\frac{R^{ns}-1}{ns}\{M(p,1)\}^{s-1}M\left(p^{/},1\right) - \left(\frac{R^{ns}-1}{ns} - \frac{R^{ns-2}}{ns-2}\right)\{M(p,1)\}^{s-1}\left|p^{/}(0)\right|\right]$$

On applying Lemma 1 to the above inequality, we get for $n > 2$

$$\left|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s\right| \leq \frac{R^{ns}-1}{k^{n-1}+k^n}\{M(p,1)\}^s$$
$$-s\left(\frac{R^{ns}-1}{ns} - \frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1}\left|p^{/}(0)\right|$$

This gives

$$\{M(p,R)\}^s \leq \frac{R^{ns}-1+k^{n-1}+k^n}{k^{n-1}+k^n}\{M(p,1)\}^s$$
$$-s\left(\frac{R^{ns}-1}{ns} - \frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1}\left|p^{/}(0)\right|$$

from which proof of inequality (5) follows.

The proof of inequality (6) follows on the same lines as that of inequality (5), but instead of using inequality (15) of Lemma 3 we use inequality (16) of Lemma 3.

Proof of Theorem 2. The proof of Theorem 2 follows on the same lines as that of Theorem 1. But for the sake of completeness we give a brief outline of the proof. We first consider the case when polynomial $p(z)$ is of degree $n > 2$, then the polynomial $p^{/}(z)$ is of degree $(n-1) \geq 2$, hence applying inequality (15) of Lemma 3 to $p^{/}(z)$, we have for $r \geq 1$ and $0 \leq \theta < 2\pi$

$$\left|p^{/}(re^{i\theta})\right| \leq r^{n-1}M(p^{/},1) - (r^{n-1}-r^{n-3})\left|p^{/}(0)\right| \tag{21}$$

Also for each $\theta, 0 \leq \theta < 2\pi$ and $R \geq 1$, we obtain

$$\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s = \int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s\,dt$$
$$= \int_1^R s\{p(te^{i\theta})\}^{s-1}\, p^{/}(te^{i\theta})e^{i\theta}\,dt$$

This implies

$$\left|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s\right| \leq s\int_1^R \left|p(te^{i\theta})\right|^{s-1}\left|p^{/}(te^{i\theta})\right|dt \tag{22}$$

Inequality (22) in conjunction with inequalities (21) and (1), yields for $n > 2$

$$\left|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s\right|$$

$$\leq s\int_1^R [t^n M(p,1)^{s-1}][t^{n-1}M(p^/,1) - (t^{n-1} - t^{n-3})|p^/(0)|]dt$$

$$= s\int_1^R t^{ns-1}\ \{M(p,1)\}^{s-1}M(p^/,1)$$

$$-(t^{ns-1} - t^{ns-3})\{M(p,1)\}^{s-1}|p^/(0)|]dt$$

$$= s\left[\frac{R^{ns}-1}{ns}\ \{M(p,1)\}^{s-1}M(p^/,1) - \left(\frac{R^{ns}-1}{ns} - \frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1}|p^/(0)|\ \right]$$

Which on combining with lemma 2, yields for $n > 2$

$$\left|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s\right|$$

$$\leq \frac{R^{ns}-1}{k^n}\left(\frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n|(1+k^2) + 2|a_{n-1}|}\right)\{M(p,1)\}^s$$

$$-s\left(\frac{R^{ns}-1}{ns} - \frac{R^{ns-2}-1}{ns-2}\right)\{M(p,1)\}^{s-1}|p^/(0)|$$

From which we get the desired result.

The proof of inequality (10) follows on the same lines as that of inequality (9), but instead of using inequality (15) of Lemma 3 we use inequality (16) of Lemma 3.

REFERENCES

[1] Ankeny, N. C., & Rivlin, T. J. (1955). On a theorem of S. Bernstein. *Pacific J. Math., 5*, 849-852.

[2] Dewan, K. K., & Ahuja, A. (2011). Growth of polynomials with prescribed zeros. *Journal of Mathematical Inequalities*, *5*(3), 355-361.

[3] Govil, N. K. (1980). On the theorem of S. Bernstein. *J. Math.and Phy.Sci., 14*, 183-187.

[4] Polya, G., & Szego, G. (1925). *Aufgabed and Lehrsatze aus der analysis*. Berlin: Springer-Verlag.

[5] Dewan, K. K., & Mir, A. (2007). Note on a theorem of S. Bernstein. *Southeast Asian Bulletin of Math., 31*, 691-695.

[6] Frapier, C., Rahman, Q. I., & Ruscheweyh, St. (1985). New inequalities for polynomials. *Trans. Amer. Math. Soc., 288*, 69-99.