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# Error Analysis for Numerical Solutions of Hammerstein Integral Equation With a Generalized Singular Kernel

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**Abstract:** In this work, the existence and uniqueness solution of the Hammerstein integral equation (HIE), with a generalized singular kernel, is discussed and solved numerically using Toeplitz matrix method and Product Nyström method. Moreover, the error analysis for these methods is discussed. Finally, numerical results when the kernel takes a generalized logarithmic form, Carleman function and Cauchy kernel function are investigated. Also the error, in each case, is estimated.
**Key words:** Hammerstein singular integral equation; Toeplitz matrix; Product Nyström method; Logarithmic form; Carleman function

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## 1. INTRODUCTION

Many methods are used to obtain the solution of the nonlinear integral equation. Abdou et al. [1] obtained numerically the solution of the singular Fredholm integral equation. Abdou et. al. [2] obtained a numerical solution for the nonlinear integral equation of Hammerstein type. Abdou and Hendi [3] solved the Fredholm integral equation with Hilbert kernel numerically. Abdou [4] obtained the solution of linear and nonlinear integral equation. Abdou and AL-Bugami [5] studied Fredholm integral equation with a generalized singular kernel in the linear case and solved this equation numerically. Emamzadeh and Kajani [6] solved nonlinear Fredholm integral equation by using quadrature Methods. Ahmad and Akbar [7] solved nonlinear Fredholm integral equation using Lagrange Functions.

In this paper, we consider the HIE with a generalized singular kernel of the form

$$\mu\phi(t)-\lambda\int_{-a}^{a}k(|g(t)-g(\zeta)|)\gamma(t,\zeta,\phi(\zeta))d\zeta=f(t) \tag{1}$$

where the free term $f(t)$ and the kernel $k(|g(t)-g(\zeta)|)$ are known functions, and $\phi(t)$ is the unknown function to be determined. The numerical coefficient λ is called the parameter of the integral equation, may be complex, and has many physical meaning, while the parameter μ defines the kind of the integral equation. The function $g(t)$ is a continuous with its derivatives in $t\in[-a,a]$.

We assume the following conditions:

(i) The kernel $k(|g(t)-g(\zeta)|)$ satisfies the discontinuity condition

$$\{\int_{-a}^{a}\int_{-a}^{a}|k(|g(t)-g(\zeta)|)|^2\,dtd\zeta\}^{\frac{1}{2}}=c<\infty,\ (c \text{ is a constant}).$$

(ii) The given function $f(t)$ is continuous in the space $L_2[-a,a]$, and its norm is defined as

$$\|f(t)\|_{L_2[-a,a]}=\{\int_{-a}^{a}|f(t)|^2\,dt\}^{\frac{1}{2}}=N^*,\ (N^* \text{ is a constant}).$$

(iii) The known continuous function $\gamma(t,\zeta,\phi(t))$ satisfies, for the constants $A>A_1, A>P,$ the following conditions

(a) $\{\int_{-a}^{a}|\gamma(t,\zeta,\phi(t))|^2\,dt\}^{\frac{1}{2}}\le A_1\|\phi(t)\|_{L_2[-a,a]}$,

(b) $|\gamma(t,\zeta,\phi_1(t))-\gamma(t,\zeta,\phi_2(t))|\le M(t)|\phi_1(t)-\phi_2(t)|$,

where $\|M(t)\|_{L_2[-a,a]}=P.$

## 2. EXISTENCE AND UNIQUENESS OF THE SOLUTION

In this section, we prove the existence of a unique solution of Eq. (1), under the conditions (i-iii) by using successive approximation method:

**Theorem 1:** The solution of the nonlinear two-dimensional **VIE** (1) with continuous kernel is exist and a unique under the condition

$$A|\lambda| < \frac{\mu}{c} \quad (2)$$

To proof this theorem we must state the following lemmas:

**Lemma 1:** Beside the conditions (i-iii), the infinite series $\sum_{i=0}^{\infty}\phi_i(t)$ is uniformly converge to a continuous solution $\phi(t)$.

**Proof :**

We construct the sequence of the function $\phi_n(t)$ such as:

$$\phi_n(t) = f(t) + \lambda \int_{\Omega} k(|g(t) - g(\zeta)|)\gamma(t, \zeta, \phi_{n-1}(\zeta))d\zeta \quad (3)$$

with,

$$\phi_0(t) = f(t) \quad (4)$$

Here, it is convenient to introduce

$$\theta_n(t) = \phi_n(t) - \phi_{n-1}(t) \quad (5)$$

where,

and $\theta_0(t) = f(t)$.

$$\phi_n(t) = \sum_{i=0}^{n}\theta_i(t) \quad (6)$$

Using the properties of the modulus, the relation (3) takes the form:

$$|\theta_n(t)| \le \left|\frac{\lambda}{\mu}\right| \int_{\Omega} |k(|g(t) - g(\zeta)|)||\gamma(t, \zeta, \phi_{n-1}(\zeta)) - \gamma(t, \zeta, \phi_{n-2}(\zeta))|\, d\zeta \quad (7)$$

Using the condition (iii-*b*), we obtain:

$$|\theta_n(t)| \le \left|\frac{\lambda}{\mu}\right| A \int_{\Omega} |k(|g(t) - g(\zeta)|)||\phi_{n-1}(\zeta) - \phi_{n-2}(\zeta)|\, d\zeta \quad (8)$$

with the aid of (5) and take the maximum over $t$ we get,

$$\max_{0\le t\le T}|\theta_n(t)| \le \left|\frac{\lambda}{\mu}\right| A \int_{\Omega} |k(|g(t) - g(\zeta)|)| \max_{0\le\zeta\le T}|\theta_{n-1}(\zeta)|\, d\zeta$$

Then, we have

$$\|\theta_n(t)\| \le \frac{1}{|\mu|}\|\theta_{n-1}(\zeta)\|\{|\lambda|\, A\int_{\Omega} |k(|g(t) - g(\zeta)|)|\, d\zeta\}$$

By using the condition (i), we obtain:

$$\|\theta_n(t)\| \le \frac{1}{|\mu|} c\{|\lambda|\, A\}\|\theta_{n-1}(\zeta)\|. \quad (9)$$

Inequality equation (9) takes the form:

$$\alpha_{i,2m+1} = f_i(t_{2m+1}) - \lambda\sum_{j=0}^{n} w_j k_{n,j}[\frac{h}{3}\sum_{j=0}^{2m} w_j g_{i,s}(t_{2m+1},\tau_j,\alpha_{1j},...,\alpha_{lj})$$

$$+\lambda\sum_{j=0}^{n} w_j k_{n,j}\{\frac{h}{6}[g_{i,s}(t_{2m+1},\tau_{2m},\alpha_{1,2m},...,\alpha_{l,2m})$$

$$+4g_{i,s}(t_{2m+1},\tau_{2m+1/2},(\frac{3}{8}\alpha_{1,2m}+\frac{3}{4}\alpha_{1,2m+1}$$

$$-\frac{1}{8}\alpha_{1,2m+2}),...,(\frac{3}{8}\alpha_{l,2m}+\frac{3}{4}\alpha_{l,2m+1}-\frac{1}{8}\alpha_{l,2m+2})) \tag{10}$$

where equation (10) $(\alpha_1 = \frac{1}{|\mu|}c\{|\lambda| A\} < 1)$.

If we let $n$=1 in equation (7) and using the condition (ii) we get, $\|\theta_1\| \le \alpha_1 N^*$, then, by using the mathematical induction, we obtain

$$\|\theta_n\| \le \alpha_1^n N,\ n = 0,1,2,... \tag{11}$$

This bound makes the sequence $\{\theta_n\}$ converges under the condition of equation (2), and hence the sequence $\{\phi_n(t)\}$ converges so,

$$\phi(t) = \sum_{i=0}^{\infty}\theta_i(t). \tag{12}$$

The infinite series equation (12) is uniformly convergent since the terms $\theta_i(t)$ are dominated by $C[a,b]\times C[c,d]$.

**Lemma 2:** A continuous function $\phi(t)$ represents a unique solution of equation (1)

**Proof :**

To prove that $\phi(t)$ represents a unique solution of equation (1), we prove that $\phi(t)$ defined by equation (12), satisfies Eq. (1). Set $\phi(t) = \phi_n(t) + s_n(t)$ where, as $n \to \infty$ then we get,

$$\phi(t) - s_n(t) = \frac{1}{\mu} f(t) + \frac{\lambda}{\mu}\int_{\Omega} k(|g(t)-g(\zeta)|)(\gamma(t,\zeta,\phi(\zeta)-s_{n-1}(\zeta))d\zeta$$

Therefore, using the condition (iii-$a$) we have,

$$\max_{0\le t\le T}\left|\phi(t) - \frac{1}{\mu} f(t) - \frac{\lambda}{\mu}\int_{\Omega} k(|g(t)-g(\zeta)|)\gamma(t,\zeta,\phi(\zeta))d\zeta\right|$$

$$\le \max_{0\le t\le T}|s_n(t)| - \left|\frac{\lambda}{\mu}\right| A\int_{\Omega}|k(|g(t)-g(\zeta)|)| \max_{0\le t\le T}|s_{n-1}(\zeta)|\, d\zeta \tag{13}$$

In view of the condition (i), the previous inequality takes the form:

$$\left\|\phi(t)-\frac{1}{\mu}f(t)-\frac{\lambda}{\mu}\int_{\Omega}k(|g(t)-g(\zeta)|)\gamma(t,\zeta,\phi(\zeta))d\zeta\right\|\le\|s_n(t)\|-\alpha_1\|s_{n-1}(\zeta)\| \tag{14}$$

where $\alpha_1=\frac{1}{|\mu|}c\{|\lambda|A\}$.

So that, by taking $n$ large enough, the right hand side for relation equation (14) can be made as small as desired, thus, the function $\phi(t)$ satisfies

$$\mu\phi(t)-\lambda\int_{\Omega}k(|g(t)-g(\zeta)|)\gamma(t,\zeta,\phi(\zeta))d\zeta=f(t) \tag{15}$$

and therefore it is a solution of equation (1).

Now, to show that $\phi(t)$ is the only solution, let $\bar{\phi}(t)$ is also a continuous solution of equation (1), hence:

$$\left|\phi(t)-\bar{\phi}(t)\right|\le\left|\frac{\lambda}{\mu}\right|\int_{\Omega}|k(|g(t)-g(\zeta)|)|\left|\gamma(t,\zeta,\phi(\zeta))-\gamma(t,\zeta,\bar{\phi}(\zeta))\right|d\zeta. \tag{16}$$

With the aid of conditions (i, ii-b), the equation (16) then,

$$\left\|\phi(t)-\bar{\phi}(t)\right\|\le\alpha_1\left\|\phi(\zeta)-\bar{\phi}(\zeta)\right\|,\ \alpha_1=\frac{c}{|\mu|}\{|\lambda|A\}<1. \tag{17}$$

Since $\alpha_1<1$, then the inequality equation (17) is true only if $\phi(t)=\bar{\phi}(t)$, which is the solution of equation (1).

## 3. TOEPLITZ MATRIX METHOD

To discuss the solution of equation (1) numerically, using Toeplitz matrix method, we write the integral term as:

$$\int_{-a}^{a}k(|g(t)-g(\zeta)|)\gamma(t,\zeta,\phi(\zeta))d\zeta=\sum_{n=-N}^{N}\int_{nh}^{nh+h}k(|g(t)-g(\zeta)|)\gamma(t,\zeta,\phi(\zeta))d\zeta,\ (h=\frac{a}{N}) \tag{18}$$

Then, we approximate the integral term in the right hand side by

$$\int_{nh}^{nh+h}k(|g(t)-g(\zeta)|)\gamma(t,\zeta,\phi(\zeta))d\zeta$$
$$=A_n(g(t))\gamma(t,nh,\phi(nh))+B_n(g(t))\gamma(t,nh+h,\phi(nh+h))+R \tag{19}$$

Where, $A_n(g(t)),\ B_n(g(t))$ are two arbitrary functions to be determined and $R$ is the error estimate.

As the principle idea of the Toeplitz matrix to obtain the values of the function $A_n(g(t)),\ B_n(g(t))$, we assume $\phi(\zeta)=g'(\zeta), g'(\zeta)g(\zeta)$ respectively, in equation (19), where $g'(t)$ is a monotonic increasing function. This yields a set of two equations in terms of two unknown functions where, in this case, the error is vanishing. Solving the results we have:

$$A_n(g(t)) = \frac{1}{h_1}[\gamma(t,nh+h,g'(nh+h)g(nh+h))I(t) - \gamma(t,nh+h,g'(nh+h))J(t)]$$

$$B_n(g(t)) = \frac{1}{h_1}[\gamma(t,nh,g'(nh))J(t) - \gamma(t,nh,g'(nh)g(nh))I(t)] \qquad (20)$$

Where $\begin{aligned} h_1 &= \gamma(t,nh,g'(nh))\gamma(t,nh+h,g'(nh+h)g(nh+h)) \\ &- \gamma(t,nh+h,g'(nh+h))\gamma(t,nh,g'(nh)g(nh)) \end{aligned}$,

$$\begin{aligned} I(t) &= \int_{nh}^{nh+h} k(|g(t)-g(\zeta)|).\gamma(t,\zeta,g'(\zeta))d\zeta \quad , J(t) \\ &= \int_{nh}^{nh+h} k(|g(t)-g(\zeta)|).\gamma(t,\zeta,g'(\zeta).g(\zeta))d\zeta \end{aligned} \qquad (21)$$

Using equation (20) in equation (19), the integral equation (1) becomes

$$\mu\phi(t) - \lambda \sum_{n=-N}^{N} D_n(g(t))\gamma(t,nh,\phi(nh)) = f(t) \qquad (22)$$

Using the following notation, for $t = mh$,

$$\phi(t) = \phi(mh) = \phi_m,\ D_n(g(t)) = D_n(g(mh)) = D_{n,m},\ f(t) = f(mh) = f_m \qquad (23)$$

we get the following system of nonlinear algebraic equations

$$\mu\phi_m - \lambda \sum_{n=-N}^{N} D_{n,m}\gamma(\phi_n) = f_m \qquad (24)$$

where

$$D_{n,m} = \begin{cases} A_{-N}(g(mh)), & n=-N \\ A_n(g(mh)) + B_{n-1}(g(mh)), & -N<n<N \\ B_{N-1}(g(mh)), & n=N \end{cases} \qquad (25)$$

The matrix $D_{n,m}$ may be written as $D_{n,m} = G_{n,m} - E_{n,m}$, where

$$G_{n,m} = A_n(g(mh)) + B_{n-1}(g(mh)), \quad -N \le m,n \le N \qquad (26)$$

is a Toeplitz matrix of order $2N+1$ and

$$E_{n,m} = \begin{cases} B_{-N-1}(g(mh)), & n=-N \\ 0, & -N<n<N \\ A_N(g(mh)), & n=N \end{cases}$$

Represents a matrix of order $2N+1$ whose elements are zeros except the first and the last columns (rows).
The solution of the system equation (20) can be obtained in the form

$$\phi(mh) = [\mu I - \lambda(G_{n,m} - E_{n,m})]^{-1} f(mh), \quad \left|\mu I \text{-} \lambda(\mathrm{G}_{n,m} - E_{n,m})\right| \neq 0 \tag{27}$$

### *3.1. Error analysis of Toeplitz Matrix Method*

The error term $R$ is determined from equation (19) by letting $\phi(\zeta) = g'g^2$ to get,

$$R = \left| \int_{nh}^{nh+h} \gamma(t,\zeta,g'(\zeta)g^2(\zeta))k(\left|g(t)-g(\zeta)\right|)d\zeta - A_n(g(t))\gamma(t,nh,g'(nh)g^2(nh)) \right.$$
$$\left. - B_n(g(t))\gamma(t,nh+h,g'(nh+h)g^2(nh+h)) \right| \tag{28}$$

(a) The method is said to be convergent of order $r$ in $[-a,a]$ if and only if for $N$ sufficiently large there exists a constant $D > 0$, independent of $N$, such that $\left\|\phi(t) - \phi_N(t)\right\| \leq DN^{-r}$.

(b) The nonlinear algebraic system equation (24) has a unique solution, under the convergence condition:

$$\sup_N \left\| \sum_{n=-N}^{N} D_{nm} \right\| \leq c' \quad (\ c' \text{ is a constant }) \tag{29}$$

(d) The estimate local error $R$ of (19) is determined by the following relation

$$\phi(t) - (\phi(t))_N = \sum_{n=-N}^{N} D_{n,m}[\phi(nh) - \phi_N(nh)] + R$$

where $\phi_N$ is the approximate solution of equation (1).

## 4. PRODUCT NYSTRÖM METHOD

We discuss the solution of Fredholm integral equation using the product Nyström method. Consider the integral equation:

$$\mu\phi(t) - \lambda \int_{-a}^{a} p(g(t),g(\zeta))\bar{k}(\left|g(t)-g(\zeta)\right|)\gamma(t,\zeta,\phi(\zeta))d\zeta = f(t) \tag{30}$$

when the kernel $k(\left|g(t)-g(\zeta)\right|)$ is singular within the range of integration. We can often factor out the singularity in $k(\left|g(t)-g(\zeta)\right|)$ by writing it in the form

$$k(\left|g(t)-g(\zeta)\right|) = p(\left|g(t)-g(\zeta)\right|)\bar{k}(\left|g(t)-g(\zeta)\right|) \tag{31}$$

Where $p$ and $\bar{k}$ are respectively badly behaved and well behaved functions of their arguments, respectively, $\phi(t)$ is the unknown function, while $f(t)$ is a given function. Equation (30) can be written in the form

$$\mu\phi(t_i) - \lambda \sum_{j=0}^{N} w_{ij}\bar{k}(\left|g(t_i)-g(\zeta_j)\right|)\gamma(t_i,t_j,\phi_i(\zeta_j)) = f(t_i)\ , \ i = 0,1,\ldots,N \tag{32}$$

where $t_i = \zeta_i = a+ih$, $i = 0, 1, \ldots, N$ with $h = \dfrac{2a}{N}$ and $N$ even, and $w_{ij}$ are weights. As in Delves and Mohamed [10], we write:

$$\int_{-a}^{a} p(g(t_i)\,,\, g(\zeta))\, \bar{k}(|g(t_i)-g(\zeta)|)\gamma(t,\zeta,\phi(\zeta))d\zeta$$

$$\approx \sum_{j=0}^{\frac{N-2}{2}} \int_{y_{2j}}^{y_{2j+2}} p(g(t_i),g(\zeta))\,.\,\{\frac{(g(\zeta_{2j+1})-g(\zeta))(g(\zeta_{2j+2})-g(\zeta))}{g(h)g(2h)}\,.\,\bar{k}(|g(t_i)-g(\zeta_{2j})|)\gamma(t_i,\zeta_{2j},\phi(\zeta_{2j}))$$

$$+\frac{(g(\zeta)-g(\zeta_{2j}))(g(\zeta_{2j+2})-g(\zeta))}{g(h)g(h)}\bar{k}(|g(t_i)-g(\zeta_{2j+1})|)\gamma(t_i,\zeta_{2j+1},\phi(\zeta_{2j+1}))$$

$$+\frac{(g(\zeta)-g(\zeta_{2j+1}))(g(\zeta)-g(\zeta_{2j}))}{g(2h)g(h)}\bar{k}(|g(t_i)-g(\zeta_{2j+2})|)\gamma(t_i,\zeta_{2j+2},\phi(\zeta_{2j+2}))\}d\zeta$$

$$=\sum_{j=0}^{N} w_{ij}\bar{k}(|g(t_i)-g(\zeta_j)|)\gamma(t_i,\zeta_j,\phi(\zeta_j)) \tag{33}$$

Therefore:

$$\begin{aligned} w_{i,0} &= \beta_1(\zeta_i),\ w_{i,2j+1} = 2\gamma_{j+1}(\zeta_i) \\ w_{i,2j} &= \alpha_j(\zeta_i)+\beta_{j+1}(\zeta_i),\ w_{i,N} = \alpha_{N/2}(\zeta_i) \end{aligned} \tag{34}$$

where,

$$\begin{aligned} \alpha_{ij} &= \frac{(g(h))^2}{g(2h)}\int_0^2 \eta(\eta-1)\,p(g(\zeta_{2j-2}+\eta h),g(\zeta_i))d\eta \\ \beta_{ij} &= \frac{(g(h))^2}{g(2h)}\int_0^2 (\eta-1)(\eta-2)p(g(\zeta_{2j-2}+\eta h),g(\zeta_i))d\eta \\ \gamma_{ij} &= \frac{(g(h))^2}{g(2h)}\int_0^2 \eta(2-\eta)\,p(g(\zeta_{2j-2}+\eta h),g(\zeta_i))d\eta \end{aligned} \tag{35}$$

Here, in equation (35), we introduce the variable $g(\zeta) = g(\zeta_{2j-2})+\eta\, g(h), 0 \le \eta \le 2$.

Therefore, the system equation (32) has a solution.

$$\Phi = [\mu I - \lambda W]^{-1}F \tag{36}$$

Where $I$ is the identity matrix, and $|\mu I - \lambda W| \neq 0$

## *4.1 Error Analysis of Product Nyström Method*

The product Nystrom method is said to be convergent of order $q$ in $[-a,a]$, if and only if for sufficiently large $N$ there exists $C>0$ independt of $N$ such that

$\|\phi(t)-\phi_N(t)\| \le CN^{-q}$

Also, we can easily prove that

$$\sup_N \sum_{j=0}^{n} |w_{ij}| < c'', (c'' \text{ is a constant}) \tag{37}$$

-The approximate solution $\tilde{\phi}(t)$ of equation (30) is

$$\mu\tilde{\phi}(t) = f(t) + \lambda \int_{\Omega} k(|g(x)-g(y)|)\gamma(t,y,\tilde{\phi}(\zeta))d\zeta + E(t) \tag{38}$$

Where, $E(t)$ indicates the error functional of Nyström method operating on $t$. We set $e(t) = \phi(t) - \tilde{\phi}(t),$ for the error in our result, and then we can be found by subtracting equation (38) from equation (1)

$$\mu e(t) = \lambda \int_{\Omega} k(|g(t)-g(\zeta)|)\gamma(t,\zeta,e(\zeta))d\zeta + E(t) \tag{39}$$

Taking the norm in the space $L_2(\Omega)$, and using the conditions (i-iii), we get

$$\|e(t)\| \le \frac{E(t)}{|\mu|(1-\alpha_1)}, \; (\alpha_1 = \left|\frac{\lambda}{\mu}\right|(cA)] \tag{40}$$

Provided that $\alpha_1$ is less than unity, we have therefore a rigorous bound for the error in the result. In other word, if a reasonable large number of points is used, we may replace $\phi(t)$ by $\tilde{\phi}(t)$ in computing the error term $E(t)$.

### *4.2 Convergence of the Product Nyström Method*

Rewrite equation (32) in the two equivalent forms

$$\mu\phi(t_i) = \lambda \sum_{m=0}^{N} w_{m,n}\phi(t_m) + f(t_i)$$

$$\mu\tilde{\phi}(t_i) = \lambda \sum_{m=0}^{N} w_{m,n}\tilde{\phi}(t_m) + f(t_i)$$

Subtracting the previous equations, then let $e = \phi(t_i) - \tilde{\phi}(t_i),$ we get

$$\mu e \le \lambda \sum_{m=0}^{N} w_{m,n} e + E(t) \tag{41}$$

Then, take the modulus, we get

$$|\mu||e| = |\lambda| \sup_m \sum_{m=0}^{N} |w_{m,n}||e| + |E|.$$

From the equation (37), we have

$$\|e\| \le \frac{\|E\|}{|\mu|(1-\alpha_2)}, \; (\alpha_2 = \left|\frac{\lambda}{\mu}\right| c'') \tag{42}$$

where $e$ is the finite vector modulus. Under the condition $\alpha_2 < 1,$ we have therefore the bound of the error, in other words, the Nyström method is converges.

## 5. NUMERICAL EXAMPLES

In this section, we apply the previous methods to obtain numerical results for equation (1), with logarithmic kernel, Carleman function and Cauchy kernel. By using Maple 10 the approximate solution, and the absolute error in each case, is obtained and computed, respectively.

**Example 1:** For integral equation

$$\phi(t)-\lambda\int_{-1}^{1}\ln\left|t^3-\zeta^3\right|\phi^3(\zeta)d\zeta = f(t),\ (\text{ exact solution } \phi(t)=t^6,\ |t|\le 1)$$

Toeplitz matrix method and product Nyström method are used to get the approximate solution for values of $\mu = 1$, $\lambda = 0.001$, 0.01 and $N = 10$, 20 units.

| $\lambda$ | $N$ | $t$ | Exact sol. | Appr. sol. T. | Err. T. | Appr. sol. N. | Err. N. |
|---|---|---|---|---|---|---|---|
| 0.001 | 10 | -1.00 | 1.000000 | 1.004417 | 0.004417 | 1.000093 | 0.000093 |
| | | -0.60 | 0.046656 | 0.044359 | 0.002296 | 0.046673 | 0.000166 |
| | | -0.20 | 0.000064 | 0.000057 | $0.638\times10^{-5}$ | 0.000079 | 0.000166 |
| | | 0.20 | 0.000064 | 0.000057 | $0.638\times10^{-5}$ | 0.000080 | 0.000166 |
| | | 0.60 | 0.046656 | 0.044359 | 0.002296 | 0.046677 | 0.000166 |
| | | 1.00 | 1.000000 | 1.004417 | 0.00441 | 1.000099 | 0.000345 |
| 0.01 | | -1.00 | 1.000000 | 1.038899 | 0.038899 | 1.000934 | 0.000934 |
| | | -0.60 | 0.046656 | 0.028567 | 0.018088 | 0.046834 | 0.000178 |
| | | -0.20 | 0.000064 | 0.000018 | 0.000045 | 0.000219 | 0.000155 |
| | | 0.20 | 0.000064 | 0.000018 | 0.000045 | 0.000232 | 0.000168 |
| | | 0.60 | 0.046656 | 0.028567 | 0.018088 | 0.046872 | 0.000216 |
| | | 1.00 | 1.000000 | 1.038899 | 0.038899 | 1.000993 | 0.000993 |
| 0.001 | 20 | -1.00 | 1.000000 | 1.011956 | 0.011956 | 1.000097 | 0.000097 |
| | | -0.60 | 0.046656 | 0.044938 | 0.001717 | 0.046675 | 0.000166 |
| | | -0.20 | 0.000064 | 0.000061 | $0.266\times10^{-5}$ | 0.000080 | 0.000166 |
| | | 0.20 | 0.000064 | 0.000061 | $0.266\times10^{-5}$ | 0.000081 | 0.000166 |
| | | 0.60 | 0.046656 | 0.044938 | 0.0017178 | 0.046676 | 0.000166 |
| | | 1.00 | 1.000000 | 1.011956 | 0.01195 | 1.000099 | 0.000345 |
| 0.01 | | -1.00 | 1.000000 | 1.114677 | 0.114677 | 1.000979 | 0.000979 |
| | | -0.60 | 0.046656 | 0.033380 | 0.013275 | 0.046852 | 0.000196 |
| | | -0.20 | 0.000064 | 0.000043 | 0.000020 | 0.000231 | 0.000167 |
| | | 0.20 | 0.000064 | 0.000043 | 0.000020 | 0.000235 | 0.000171 |
| | | 0.60 | 0.046656 | 0.033380 | 0.013275 | 0.046865 | 0.000209 |
| | | 1.00 | 1.000000 | 1.114677 | 0.114677 | 1.000993 | 0.000993 |

**Example 2:** For integral equation

$$\phi(t)-\lambda\int_{-1}^{1}\ln\left|e^{t}-e^{\zeta}\right|(\phi(\zeta))^{3}d\zeta = f(t),\ (\text{ exact solution }\ \phi(t)=e^{t}\ )$$

The Toeplitz matrix method and product Nystrom method are used to get the approximate solution for values of $\mu = 1$, $\lambda = 0.0001, 0.001$, and $N = 10, 20$ units.

| $\lambda$ | $N$ | $t$ | Exact sol. | Appr. sol. T. | Err. T. | Appr. sol. N. | Err. N. |
|---|---|---|---|---|---|---|---|
| 0.0001 | 10 | -1.00 | 0.367879 | 0.367899 | 0.0000201 | 0.369803 | 0.001924 |
| | | -0.60 | 0.548811 | 0.548876 | 0.0000650 | 0.550815 | 0.002003 |
| | | -0.20 | 0.818730 | 0.819072 | 0.0003414 | 0.820813 | 0.002083 |
| | | 0.20 | 1.221402 | 1.221810 | 0.0037782 | 1.224060 | 0.002664 |
| | | 0.60 | 1.822118 | 1.863059 | 0.0409411 | 1.825270 | 0.003213 |
| | | 1.00 | 2.718281 | 2.569364 | 0.1489173 | 2.721646 | 0.003965 |
| 0.001 | | -1.00 | 0.367879 | 0.368110 | 0.000231 | 0.387574 | 0.019695 |
| | | -0.60 | 0.548811 | 0.549482 | 0.000670 | 0.569246 | 0.020435 |
| | | -0.20 | 0.818730 | 0.822351 | 0.003620 | 0.840019 | 0.021289 |
| | | 0.20 | 1.221402 | 1.263828 | 0.042425 | 1.248462 | 0.027060 |
| | | 0.60 | 1.822118 | 2.277306 | 0.455188 | 1.854044 | 0.031926 |
| | | 1.00 | 2.718281 | 1.526211 | 1.192070 | 2.752605 | 0.034324 |
| 0.0001 | 20 | -1.00 | 0.367879 | 0.367909 | 0.000030 | 0.371590 | 0.003718 |
| | | -0.60 | 0.548811 | 0.548876 | 0.000065 | 0.552552 | 0.003749 |
| | | -0.20 | 0.818730 | 0.819053 | 0.000323 | 0.822991 | 0.004270 |
| | | 0.20 | 1.221402 | 1.224870 | 0.003467 | 1.226723 | 0.005337 |
| | | 0.60 | 1.822118 | 1.859004 | 0.037481 | 1.829158 | 0.007092 |
| | | 1.00 | 2.718281 | 2.432618 | 0.285662 | 2.726690 | 0.008779 |
| 0.001 | | -1.00 | 0.367879 | 0.368233 | 0.000353 | 0.406464 | 0.038585 |
| | | -0.60 | 0.548811 | 0.549479 | 0.000667 | 0.587453 | 0.038642 |
| | | -0.20 | 0.818730 | 0.822122 | 0.003391 | 0.862869 | 0.044139 |
| | | 0.20 | 1.221402 | 1.259570 | 0.038168 | 1.276723 | 0.055321 |
| | | 0.60 | 1.822118 | 2.256416 | 0.434297 | 1.895499 | 0.073381 |
| | | 1.00 | 2.718281 | 0.995810 | 1.722471 | 2.807506 | 0.089225 |

**Example 3:** Consider the integral equation

$$\phi(t)-\lambda\int_{-1}^{1}\left|t^{3}-\zeta^{3}\right|^{-\nu}\phi^{2}(\zeta)d\zeta = f\ (t),\quad (\text{ exact solution } \phi(t)=t^{5}\ )$$

Here Toeplitz matrix method and product Nystrom method are used to get the approximate solution for values of $\mu = 1$, $\lambda = 0.001$, $\nu = 0.1, 0.2$, and $N = 10, 20$ units.

| ν | $N$ | $t$ | Exact sol. | Appr. sol. T. | Err. T. | Appr. sol. N. | Err. N. |
|---|---|---|---|---|---|---|---|
| 0.1 | 10 | -1.00 | -1.000000 | -0.999862 | 0.000137 | -1.000199 | 0.000199 |
| | | -0.60 | -0.077760 | -0.078072 | 0.000312 | -0.077949 | 0.000189 |
| | | -0.20 | -0.000320 | -0.000507 | 0.000187 | -0.000507 | 0.000187 |
| | | 0.20 | 0.000320 | 0.000133 | 0.000186 | 0.000132 | 0.000187 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.60 | 0.077760 | 0.077659 | 0.000100 | 0.077571 | 0.000188 |
| | | 1.00 | 1.000000 | 0.999741 | 0.000258 | 0.999802 | 0.000197 |
| 0.2 | | -1.00 | -1.000000 | -0.999472 | 0.000527 | -1.000230 | 0.000230 |
| | | -0.60 | -0.077760 | -0.078229 | 0.000469 | -0.077958 | 0.000198 |
| | | -0.20 | -0.000320 | -0.000513 | 0.000193 | -0.000514 | 0.000194 |
| | | 0.20 | 0.000320 | 0.000128 | 0.000191 | 0.000127 | 0.000192 |
| | | 0.60 | 0.077760 | 0.077758 | $0.149\times10^{-5}$ | 0.077564 | 0.000195 |
| | | 1.00 | 1.000000 | 0.999647 | 0.000352 | 0.999779 | 0.000220 |
| 0.1 | 20 | -1.00 | -1.000000 | -0.999305 | 0.000694 | -1.000198 | 0.000198 |
| | | -0.60 | -0.077760 | -0.078021 | 0.000261 | -0.077948 | 0.000188 |
| | | -0.20 | -0.000320 | -0.000507 | 0.000187 | -0.000507 | 0.000187 |
| | | 0.20 | 0.000320 | 0.000133 | 0.000186 | 0.000132 | 0.000187 |
| | | 0.60 | 0.077760 | 0.077624 | 0.000135 | 0.077571 | 0.000188 |
| | | 1.00 | 1.000000 | 0.999457 | 0.000542 | 0.999802 | 0.000197 |
| 0.2 | | -1.00 | -1.000000 | -0.998216 | 0.001783 | -1.000223 | 0.000223 |
| | | -0.60 | -0.077760 | -0.078117 | 0.000357 | -0.077956 | 0.000196 |
| | | -0.20 | -0.000320 | -0.000512 | 0.000192 | -0.000512 | 0.000192 |
| | | 0.20 | 0.000320 | 0.000127 | 0.000192 | 0.000127 | 0.000192 |
| | | 0.60 | 0.077760 | 0.077683 | 0.000076 | 0.077564 | 0.000195 |
| | | 1.00 | 1.000000 | 0.999001 | 0.000998 | 0.999779 | 0.000220 |

**Example 4:** Consider the integral equation

$$\phi(t)-\lambda\int_{-1}^{1}\left|\sin(t)-\sin(\zeta)\right|^{-\nu}\phi^{2}(\zeta)d\zeta = f(t),\ \ \text{(exact solution } \phi(t)=\cos(t))$$

Here, the Toeplitz matrix method and product Nystrom method are used to get the approximate solution for values of $\mu = 1$, $\lambda = 0.001$, $\nu = 0.1, 0.2$ and N = 10, 20 units.

| ν | *N* | *t* | **Exact sol.** | **Appr. sol. T.** | **Err. T.** | **Appr. sol. N.** | **Err. N.** |
|---|---|---|---|---|---|---|---|
| 0.1 | 10 | -1.00 | 0.540302 | 0.533557 | 0.006745 | 0.540929 | 0.000626 |
| | | -0.60 | 0.825335 | 0.825858 | 0.000523 | 0.825907 | 0.000572 |
| | | -0.20 | 0.980066 | 0.982719 | 0.002653 | 0.980596 | 0.000529 |
| | | 0.20 | 0.980066 | 0.981365 | 0.001298 | 0.980546 | 0.000480 |
| | | 0.60 | 0.825335 | 0.824653 | 0.000682 | 0.825908 | 0.000572 |
| | | 1.00 | 0.540302 | 0.538051 | 0.002251 | 0.540993 | 0.000691 |
| 0.2 | | -1.00 | 0.540302 | 0.532973 | 0.007329 | 0.541447 | 0.001145 |
| | | -0.60 | 0.825335 | 0.825845 | 0.000509 | 0.826356 | 0.001020 |
| | | -0.20 | 0.980066 | 0.983038 | 0.002972 | 0.981011 | 0.000944 |
| | | 0.20 | 0.980066 | 0.981506 | 0.001439 | 0.980872 | 0.000806 |
| | | 0.60 | 0.825335 | 0.824548 | 0.000787 | 0.826371 | 0.001035 |
| | | 1.00 | 0.540302 | 0.537891 | 0.002410 | 0.541655 | 0.001353 |
| 0.1 | 20 | -1.00 | 0.540302 | 0.528992 | 0.011310 | 0.541061 | 0.000758 |
| | | -0.60 | 0.825335 | 0.825821 | 0.000485 | 0.826048 | 0.000712 |
| | | -0.20 | 0.980066 | 0.982674 | 0.002607 | 0.980713 | 0.000646 |
| | | 0.20 | 0.980066 | 0.981297 | 0.001230 | 0.980680 | 0.000613 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.60 | 0.825335 | 0.824603 | 0.000731 | 0.826051 | 0.000715 |
| | | 1.00 | 0.540302 | 0.537265 | 0.003036 | 0.541099 | 0.000797 |
| 0.2 | | -1.00 | 0.540302 | 0.528054 | 0.012247 | 0.541842 | 0.001540 |
| | | -0.60 | 0.825335 | 0.825800 | 0.000464 | 0.826794 | 0.001458 |
| | | -0.20 | 0.980066 | 0.983002 | 0.002936 | 0.981357 | 0.001291 |
| | | 0.20 | 0.980066 | 0.981437 | 0.001371 | 0.981255 | 0.001189 |
| | | 0.60 | 0.825335 | 0.824491 | 0.000844 | 0.826803 | 0.001467 |
| | | 1.00 | 0.540302 | 0.537045 | 0.003256 | 0.541965 | 0.001663 |

**Example 5:** Consider the integral equation

$$\phi(t)-\lambda\int_{-1}^{1}\left(\frac{1}{t^2-\zeta^2}\right)\phi^2(\zeta)d\zeta = f(t), \text{ (exact solution } \phi(t)=t^5)$$

Here, the Toeplitz matrix method and product Nystrom method are used to get the approximate solution for values of $\mu = 1$, $\lambda = 0.001, 0.01$ and $N = 10, 20$ units.

| $\lambda$ | $N$ | $t$ | Exact sol. | Appr. sol. T. | Err. T. | Appr. sol. N. | Err. N. |
|---|---|---|---|---|---|---|---|
| 0.001 | 10 | -1.00 | -1.000000 | -1.005239 | 0.005239 | -1.000245 | 0.000245 |
| | | -0.60 | -0.077760 | -0.079046 | 0.001286 | -0.077964 | 0.000204 |
| | | -0.20 | -0.000320 | -0.000566 | 0.000246 | -0.000510 | 0.000190 |
| | | 0.20 | 0.000320 | 0.000074 | 0.000245 | 0.000135 | 0.000184 |
| | | 0.60 | 0.077760 | 0.076938 | 0.000821 | 0.077563 | 0.000196 |
| | | 1.00 | 1.000000 | 0.991944 | 0.008055 | 0.999771 | 0.000228 |
| 0.01 | | -1.00 | -1.000000 | -0.957128 | 0.042871 | -1.002455 | 0.002455 |
| | | -0.60 | -0.077760 | -0.166645 | 0.088885 | -0.079805 | 0.002045 |
| | | -0.20 | -0.000320 | -0.022536 | 0.022216 | -0.002233 | 0.001913 |
| | | 0.20 | 0.000320 | -0.020154 | 0.020474 | -0.001532 | 0.001852 |
| | | 0.60 | 0.077760 | 0.035331 | 0.042428 | 0.075793 | 0.001966 |
| | | 1.00 | 1.000000 | 0.937242 | 0.062757 | 0.997715 | 0.002284 |
| 0.001 | 20 | -1.00 | -1.000000 | -1.000911 | 0.000911 | -1.005905 | 0.005905 |
| | | -0.60 | -0.077760 | -0.078092 | 0.000332 | -0.079174 | 0.001414 |
| | | -0.20 | -0.000320 | -0.000532 | 0.000212 | -0.000588 | 0.000268 |
| | | 0.20 | 0.000320 | 0.000107 | 0.000212 | 0.000046 | 0.000274 |
| | | 0.60 | 0.077760 | 0.077439 | 0.000320 | 0.076814 | 0.000946 |
| | | 1.00 | 1.000000 | 0.998864 | 0.001135 | 0.991037 | 0.008963 |
| 0.01 | | -1.00 | -1.000000 | -0.999335 | 0.000664 | -1.001360 | 0.001360 |
| | | -0.60 | -0.077760 | -0.080122 | 0.002362 | -0.011352 | 0.001112 |
| | | -0.20 | -0.000320 | -0.002216 | 0.001896 | -0.000739 | 0.001059 |
| | | 0.20 | 0.000320 | -0.001575 | 0.001895 | 0.049404 | 0.000914 |
| | | 0.60 | 0.077760 | 0.075597 | 0.002162 | 0.236309 | 0.000969 |
| | | 1.00 | 1.000000 | 0.997810 | 0.002189 | 0.998897 | 0.001114 |

**Example 6:** Consider the integral equation

$$\phi(t)-\lambda\int_{-1}^{1}\left(\frac{1}{e^t-e^\zeta}\right)\phi^2(\zeta)d\zeta = f(t), \text{ (exact solution } \phi(t)=e^t)$$

Here, the Toeplitz matrix method and product Nystrom method are used to get approximate solution for values of $\mu = 1$, $\lambda = 0.0001$, and $N = 10, 20$ units.

| $\lambda$ | $N$ | $t$ | **Exact sol.** | **Appr. sol. T.** | **Err. T.** | **Appr. sol. N.** | **Err. N.** |
|---|---|---|---|---|---|---|---|
| 0.0001 | 10 | -1.00 | 0.367879 | 0.367304 | 0.000574 | 0.368594 | 0.000715 |
| | | -0.60 | 0.548811 | 0.548408 | 0.000402 | 0.549529 | 0.000717 |
| | | -0.20 | 0.818730 | 0.818126 | 0.000603 | 0.819645 | 0.000915 |
| | | 0.20 | 1.221402 | 1.219359 | 0.002043 | 1.222251 | 0.000848 |
| | | 0.60 | 1.822118 | 1.811521 | 0.010597 | 1.823182 | 0.001063 |
| | | 1.00 | 2.718281 | 2.764157 | 0.045875 | 2.719164 | 0.000882 |
| 0.0001 | 20 | -1.00 | 0.367879 | 0.367120 | 0.000759 | 0.369964 | 0.002084 |
| | | -0.60 | 0.548811 | 0.548411 | 0.000399 | 0.550981 | 0.002169 |
| | | -0.20 | 0.818730 | 0.818149 | 0.000581 | 0.820928 | 0.002197 |
| | | 0.20 | 1.221402 | 1.219514 | 0.001887 | 1.223632 | 0.002229 |
| | | 0.60 | 1.822118 | 1.812458 | 0.009660 | 1.824337 | 0.002219 |
| | | 1.00 | 2.718281 | 2.821206 | 0.102927 | 2.720458 | 0.002176 |

## 6. CONCLUSIONS

From the previous examples, we deduce the following:
a) When the values of N are increasing, the error is increasing.
b) The error increase when the values of is increasing.
c) In the Carleman function, the error increase when the values of N and are fixed and when values of increasing.
d) As t is increasing in interval [-1,1], the errors due to Toeplitz matrix method and product Nystrom method are also increasing.

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