CSCanada
Progress in Applied Mathematics
Vol. 6, No. 1, 2013, pp. [64–73]
DOI: 10.3968/j.pam.1925252820130601.248

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
www.cscanada.net
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Galerkin Finite Element Method by Using Bivariate Splines for Parabolic PDEs

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Supported by the National Nature Science Foundation of China (No. 61173034, No. 70971014), the Liaoning Provincial Natural Science Foundation of China (No. 201102015), the Doctor Startup Foundation of Liaoning Province (No. 20121026), the Science Research Project of the Department of Education of Liaoning Province (No. L2012177), and the Fundamental Research Funds for the Central Universities (No. 2012TD032, No. 3132013337).

Received: April 29, 2013/ Accepted: June 30, 2013/ Published: July 31, 2013

Abstract: A Galerkin finite element method by using bivariate splines (GB method) is proposed for solving parabolic partial differential equations (PPDEs). Bivariate spline proper subspace of $S_4^{2,3}(\Delta_{mn}^{(2)})$ satisfying homogeneous boundary conditions on type-2 triangulations and quadratic B-spline interpolating boundary functions are primarily constructed. PPDEs are solved by the GB method.

Key words: Finite element method; Galerkin method; Bivariate splines; Parabolic equations

Qu, K., & Jiang, B. (2013). Galerkin Finite Element Method by Using Bivariate Splines for Parabolic PDEs. *Progress in Applied Mathematics, 6*(1), 64–73. Available from http://www.cscanada.net/index.php/pam/article/view/j.pam.1925252820130601.248 DOI: 10.3968/j.pam.1925252820130601.248

1. INTRODUCTION

In this paper, a Galerkin finite element method (GB method) is applied to the following problem of parabolic type(ST problem):

Find $u = u(x, y)$ such that

$$u_y - u_{xx} = f, \quad x \in [a, b], \quad y > 0, \tag{1}$$

$$u(a, y) = u(b, y), \qquad y > 0, \tag{2}$$

$$u(x, 0) = u_0, \quad x \in [a, b], \tag{3}$$

the functions f and u_0 are given data.

Solving the ST problem is of great practical importance and has recently been considered by several research. There is a vast amount of literature devoted to it. The reader is referred to [8,14] for excellent surveys. We review some methods referred in this paper.

The Galerkin finite element method is a well established numerical method for solving PPDEs [8]. The method is based on the variational form of boundary value problem, and approximates the exact solution by using piecewise polynomials or B-spline functions. B-spline finite elements have been widely applied to solve some kinds of PPDEs [2,5]. The finite difference method is also a popular method for solving PPDEs [1,6]. Meshless methods are another powerful class of numerical methods for solving PPDEs. The method of radial basis functions is a well-known family of meshless methods [14]. For more information see [7,16].

The present work is related to our earlier work on multivariate splines [11] and continued in [12,15]. Multivariate splines have been intensively studied in the last forty years and have become a well established tool for computational geometry, numerical approximation and wavelet etc. It has been successfully used in the finite element methods and mainly used to construct various of model functions. A lot of related work on multivariate splines and its applications has been done by Wang and his research group during the recent twenty years [4,10].

In this paper, we first write an solution to the ST problem in a weak form. Then, we propose GB method for solving the weak solution. Here, we select $S_4^{2,3;0}(\Delta_{mn}^{(2)})$ as the testing function space. For the ST problem with homogeneous boundary conditions (STH problem), we propose an algorithm (H-Algorithm). For the ST problem with nonhomogeneous boundary conditions (STN problem), we use $S_2^{1;0}(\Delta_{mn}^{(2)})$ to interpolate the boundary conditions, then we obtain a particular solution of the original problem. According to the particular solution, the STN problem could be changed into a STH problem. At last, we give some examples and comparisons of the GB method and the other methods, including the RBF method mentioned in [7], the finite difference method and the finite element method, the results indicated that the GB method has some advantages:

- the GB method is adaptive, and the accuracy could be well controlled;
- the GB method is more convenient, since the form of the system which should be solved is more simple;
- the GB method has fine accuracy.

The organization of this paper is as follows. In Section 2, the bivariate spline space is introduced. The linear system is considering in Section 3. Solving the standard model problem of parabolic type is discussed in Section 4. A conclusion is drawn in Section 5.

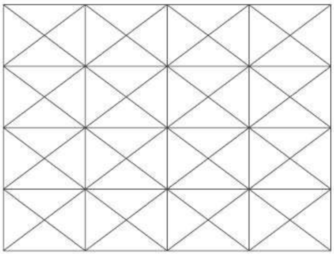

Figure 1
Uniform Type-2 Triangulation, $m = 4$**,** $n = 4$

2. Bivariate spline space on type-2 triangulations

Splines are piecewise polynomials with certain smoothness. Wang *et al.* [9,10,13] established the basic theory on multivariate splines over arbitrary partition, and presented the so-called conformality method of smoothing cofactor which is suitable for studying the multivariate spline over arbitrary partition.

Let Ω be a domain in $\mathbb{R}^2$, P_k the collection of all these bivariate polynomials with real coefficients and total degree no more than k, i.e.,

$$P_k := \{p = \sum_{i=0}^{k}\sum_{j=0}^{k-i} c_{ij}x^i y^j | c_{ij} \in R\}$$

Using a finite number of irreducible algebraic curves to carry out the partition Δ of the domain Ω, then the domain Ω is divided into M sub-domains $\delta_1, \cdots, \delta_M$, each of such sub-domains is called a cell of Δ. The space of bivariate splines with degree k and smoothness μ over Δ is defined by

$$S_k^\mu(\Delta) := \{s \in C^\mu(\Omega) | \;\; s|_{\delta_i} \in P_k, i = 1, \ldots, M\}$$

As well known, many regions including the so-called L-form regions and their combinations, can be translated to many rectangular regions. Type-2 triangulation-s are yielded by connecting two diagonals at each small rectangular cell which are based on rectangular regions. Clearly, if the original rectangular partition is unifor-m, then the induced type-2 triangulations are called uniform type-2 triangulations. All of type-2 triangulations mentioned in our paper are uniform, see Fig. 1.

Without loss the generality, let Ω be a unit square region as follows:

$$\Omega = (0, 1) \otimes (0, 1).$$

The type-2 triangulation $\Delta_{m,n}^{(2)}$ is yielded by the following partition lines:

$$mx - i = 0, ny - i = 0,$$

$$ny - mx - i = 0, ny + mx - i = 0,$$

where $i = \cdots, -1, 0, 1, \cdots$. It is from the the basic inequality [10] that the quadratic and quartic spline spaces with the highest possible uniform smoothness are $S_2^1(\Delta_{mn}^{(2)})$ and $S_4^2(\Delta_{mn}^{(2)})$, respectively.

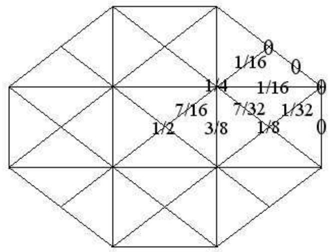

Figure 2
A Locally Supported Spline

2.1. QUADRATIC SPLINE SPACES $S_2^1(\Delta_{m,n}^{(2)})$

We first introduce a locally supported spline in $S_2^1(\Delta_{m,n}^{(2)})$ with its support octagon Q centered at (0, 0) as shown in Fig. 2. It is known that a bivariate polynomial of degree 2 on a triangle can be uniquely determined by the values of three vertices and three midpoints of the edges. In Fig. 2, the values are given on some triangles, and other values are obtained by the symmetry of lines $x = 0, y = 0, x+y = 0, x-y = 0$.

Let $\Phi(x, y)$ be a piecewise polynomial with degree 2 defined in $\mathbb{R}^2$, that is, zero outside of Q, and let its representation in every triangle of Q be determined by the values. Clearly, $\Phi(x, y) \in C^1(\mathbb{R}^2)$, and $\Phi(x, y) > 0$ inside of Q. Hence, $\Phi(x, y)$ is a bivariate B-spline over the partition. Using the conformality conditions at vertices, $\Phi(x, y)$ is uniquely determined by the symmetry of lines $x = 0, y = 0, x + y = 0, x - y = 0$, and normalized condition $\Phi(0, 0) = 1/2$. We can point out that the support of $\Phi(x, y)$ is the smallest one [10].

Denote

$$\Phi_{ij}(x, y) := \Phi(mx - i + 1/2,\ ny - j + 1/2),$$

then collection

$$A = \{\Phi_{ij}(x, y) :\ i = 0, \cdots, m + 1,\ j = 0, \cdots, n + 1\}$$

is a subspace of $S_2^1(\Delta_{m,n}^{(2)})$. From [10], we know that

$$dim S_2^1(\Delta_{m,n}^{(2)}) = (m + 2)(n + 2) - 1.$$

2.2. QUARTIC SPLINE SPACES $S_4^{2,3;0}(\Delta_{mn}^{(2)})$

In [10], they have constructed the locally supported splines in $S_4^2(\Delta_{mn}^{(2)})$ which are consist of three classes of C^2 quartic B-spline bases. Since it is not convenient in finite element method by using them immediately, Li and Wang [11] constructed a spline space $S_4^{2,3}(\Delta_{mn}^{(2)})$, where $s \in S_4^{2,3}(\Delta_{mn}^{(2)})$ is a piecewise polynomial of degree 4 with the following two continuous conditions: (i) s is C^2 continuous on the rectangle grid segments; (ii) s is C^3 continuous on the diagonal grid segments. Meanwhile, they constructed B-spline base $B(x,y)$ by the combination of three kinds of B-splines in [10]. It has been proved that all the locally supported B-splines can only span a proper subspace of $S_4^{2,3}(\Delta_{mn}^{(2)})$ [11].

Next, we discuss locally supported splines in proper subspace of $S_4^{2,3}(\Delta_{mn}^{(2)})$ with homogenous boundary conditions on type-2 triangulations. The basic idea is to use the linear combination of $B(x,y)$ in $S_4^{2,3}(\Delta_{mn}^{(2)})$ and their translations.

Let

$$B_{i,j}(x,y) = B(mx-i, ny-j).$$

Define the basis functions $\widetilde{B}_{i,j}(x,y)$ as follows:

$$\begin{cases} \widetilde{B}_{1,1}(x,y) = B_{1,1}(x,y) - B_{-1,1}(x,y) - B_{1,-1}(x,y) + B_{-1,-1}(x,y), \\ \widetilde{B}_{m-1,1}(x,y) = B_{m-1,1}(x,y) - B_{m+1,1}(x,y) - B_{m-1,-1}(x,y) + B_{m+1,-1}(x,y), \\ \widetilde{B}_{1,n-1}(x,y) = B_{1,n-1}(x,y) - B_{-1,n-1}(x,y) - B_{1,n+1}(x,y) + B_{-1,n+1}(x,y), \\ \widetilde{B}_{m-1,n-1}(x,y) = B_{m-1,n-1}(x,y) - B_{m+1,n-1}(x,y) - B_{m-1,n+1}(x,y) + B_{m+1,n+1}(x,y), \end{cases} \tag{4}$$

$$\begin{cases} \widetilde{B}_{i,1}(x,y) = B_{i,1}(x,y) - B_{i,-1}(x,y), i = 2,3,\cdots,m-2, \\ \widetilde{B}_{i,m-1}(x,y) = B_{i,m-1}(x,y) - B_{i,m+1}(x,y), i = 2,3,\cdots,m-2, \\ \widetilde{B}_{1,j}(x,y) = B_{1,j}(x,y) - B_{-1,j}(x,y), j = 2,3,\cdots,n-2, \\ \widetilde{B}_{n-1,j}(x,y) = B_{n-1,j}(x,y) - B_{n+1,j}(x,y), j = 2,3,\cdots,n-2, \end{cases} \tag{5}$$

$$\widetilde{B}_{i,j}(x,y) = B_{i,j}(x,y), i = 2,3,\cdots,m-2, j = 2,3,\cdots,n-2. \tag{6}$$

B-spline functions in Eq.(4), Eq.(5), Eq.(6) are called corner, side and interior B-spline bases, respectively. Their supports are shown in Figure 3. The B-spline functions are C^1 across the single marked mesh lines and C^0 across the double marked mesh segments.

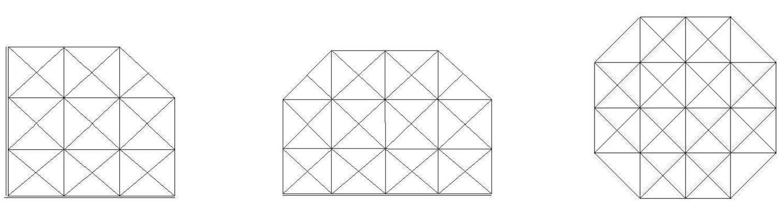

Figure 3
Corner B-spline Basis **Side B-spline Basis** **Interior B-spline Basis**

It can be proved that $\widetilde{B}_{i,j}(x,y) : 1 \le i \le m-1, 1 \le j \le n-1$ can only span the proper subspace of $S_4^{2,3}(\Delta_{mn}^{(2)})$ with homogenous boundary conditions on type-2 triangulations ($S_4^{2,3;0}(\Delta_{mn}^{(2)})$ for short).

3. SOLVING THE LINEAR PARABOLIC PROBLEMS WITH HOMOGENEOUS BOUNDARY CONDITIONS

Let u, v be scalar functions, we define the gradient by

$$\nabla v = (\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}).$$

$(\cdot,\cdot)$ denotes the usual $L_2(\Omega)$ inner product,

$$(u,v)=\int_\Omega uvdxdy.$$

$H_0^1(\Omega)$ is the usual Sobolev space consisting of all functions v vanishing on $\partial\Omega$ and having square integrable first order derivatives in Ω. In this paper, $\|\cdot\|\equiv\|\cdot\|_2$.

3.1. The GB Method

In Eq.(1), if $g(u)\equiv c$, the SP problem is called a linear SP problem. We shall now discuss the following linear SP problem: Find $u=u(x,y,t)$ such that

$$u_t-u_{xx}-u_{yy}=f,\quad in\ \Omega,\quad t>0, \tag{7}$$

$$u=0,\quad on\ \partial\Omega,\quad t>0, \tag{8}$$

$$u(\cdot,0)=u_0,\quad in\ \Omega, \tag{9}$$

Eq.(7)–Eq.(9) can be given the following equivalent weak formulation: Find u: $R_+\to H_0^1(\Omega)$ such that

$$\begin{array}{c}(u_t,v)+(\nabla u,\nabla v)=(f,v),\forall v\in H_0^1(\Omega),t>0,\\ u(\cdot,0)=u_0.\end{array} \tag{10}$$

The Discontinuous Galerkin method is defined as follows: Let $0=t_0<t_1<\cdots<t_l<\cdots$ be a (not necessarily) partition of the positive t-axis R_+ into subintervals $I_l=(t_{l-1},t_l]$, and define with q a nonnegative integer the corresponding set of piecewise polynomials of degree at most q in t with values in $H_0^1(\Omega)$ by

$$W=\{v:v\mid_{I_l}=\sum_{i=0}^{q}a_{i,l}t^i,a_{i,l}\in H_0^1(\Omega),i=0,1,\cdots,q,l=1,2,\cdots\}.$$

In this note we shall consider only the case $q=0$, it means that the solution of problem Eq.(7)–Eq.(9) in each subintervals I_l (for some suitable) is not changed. So the Discontinuous Galerkin method for Eq.(7)–Eq.(9) takes the form[3]:

$$(U_l-U_{l-1},v)+k_l(\nabla U_l,\nabla v)=\int_{I_l}(f,v)dt,\quad \forall v\in H_0^1(\Omega),l=1,2,\cdots, \tag{11}$$

$$U_0=u_0, \tag{12}$$

where $k_l=t_l-t_{l-1}$ is the length of the subinterval I_l. U_l is the numerical solution of the linear SP problem (Eq.(7)–Eq.(9)) when $t\in I_l$.

Since $S_4^{2,3;0}(\Delta_{mn}^{(2)})$ can be embedded into $H_0^1(\Omega)$, we can select it as the testing function space.

The finite element method is to find a solution $U_l\in S_4^{2,3;0}(\Delta_{mn}^{(2)})$ such that

$$(U_l-U_{l-1},v)+k_l(\nabla U_l,\nabla v)=\int_{I_l}(f,v)dt,\quad \forall v\in S_4^{2,3;0}(\Delta_{mn}^{(2)}), \tag{13}$$

It is obvious that Eq.(13) is equivalent to the following formula:

$$(U_l - U_{l-1}, \widetilde{B}_{s,t}) + k_l(\nabla U_l, \nabla \widetilde{B}_{s,t}) = \int_{I_l} (f, \widetilde{B}_{s,t})dt, \quad \forall \widetilde{B}_{s,t} \in S_4^{2,3;0}(\Delta_{mn}^{(2)}), \quad (14)$$

By using the B-spline bases on $S_4^{2,3;0}(\Delta_{mn}^{(2)})$, we can write

$$U_l = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \lambda_{i,j,l} \widetilde{B}_{i,j}(x, y),$$

and insert to Eq.(14), we have the following linear system

$$\sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \lambda_{i,j,l}(\widetilde{B}_{i,j}, \widetilde{B}_{s,t}) + k_l \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \lambda_{i,j,n}(\nabla \widetilde{B}_{i,j}, \nabla \widetilde{B}_{s,t})$$

$$= \int_{I_l} (f, \widetilde{B}_{s,t})dt + (U_{l-1}, \widetilde{B}_{s,t}), \quad \forall 1 \le s \le m-1, 1 \le t \le n-1. \quad (15)$$

Therefore, the coefficients $\lambda_{i,j,l}$ can be determined by the system of linear equations Eq.(15).

4. SOLVNG TWO-DIMENSIONAL PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

In this section, we discuss the ST problem (Eq.(1)–Eq.(3)). We should construct a particular solution satisfying Eq.(2) at first. Using the GB method, we could obtain the numerical solutions of the ST problem.

4.1. Interpolation the Boundary Conditions by Using Bivariate Spline

Considering the problem Eq.(2) in few time, i.e. T, we want to construct the quadratic spline bases on $\partial\Omega$. See Section 2.1, the quadratic spline bases on $\partial\Omega_1 = \{(x, 0) \mid x \in [0, 1]\}$ are constructed as follows:

$$s^{(1)}(x, t) = h \mid_{\partial\Omega_1} = \sum_{i=0}^{m+1} \sum_{j=0}^{n+1} c_{ij}^{(1)} \Phi_{ij}(x, t),$$

where $s^{(1)}(x, t)$ satisfies the interpolation conditions

$$s^{(1)}(w, v\tau) = h(w, 0, v\tau), w = 0, 1, \cdots, m+1, v = 0, 1, \cdots, n+1,$$

where $\tau = T/n$.

Similar to the above construction, the quadratic B-spline bases $s^{(2)}, s^{(3)}$ and $s^{(4)}$ defined on $\partial\Omega_2 = \{(x, 1) \mid x \in [0, 1]\}$, $\partial\Omega_3 = \{(0, y) \mid y \in [0, 1]\}$ and $\partial\Omega_4 = \{(1, y) \mid y \in [0, 1]\}$ can be easily constructed, respectively.

Given $4(m+2)(n+2)$ function values $h(\cdot)$ at the points

$$P = \Big\{(x, y, t) | (w, 0, v\tau), (w, 1, v\tau), (0, w, v\tau), (1, w, v\tau), w = 0, 1, \cdots, m+1, v = 0, 1, \cdots, n+1\Big\},$$

we construct the following particular solution

$$u_b(x,y,t) = \sum_{i=0}^{m+1} \left(c_{ij}^{(1)} \Phi_{ij}(x,t) + c_{ij}^{(2)} \Phi_{ij}(x,t) + c_{ij}^{(3)} \Phi_{ij}(y,t) + c_{ij}^{(4)} \Phi_{ij}(y,t) \right)$$

satisfying the interpolation conditions

$$u_b(p_i) = h(p_i), \ \forall \ p_i \in P.$$

4.2. Solving the Nonlinear Parabolic Problems with Nonhomogeneous Boundary Conditions

From Section 4.1, we can compute the spline $u_b(x,\ y)$ such that it interpolates the boundary function $h(x,\ y,\ t)$. If set $\tilde{u} = u - u_b$, then $\tilde{u}$ satisfies the corresponding nonlinear parabolic problems with homogeneous boundary condition, i.e.,

$$\tilde{u}_t - div(\tilde{g}(\tilde{u})\nabla\tilde{u}) = \tilde{f}, \quad in\ \Omega, \quad t > 0, \tag{16}$$

$$\tilde{u} = 0, \quad on\ \partial\Omega, \quad t > 0, \tag{17}$$

$$\tilde{u}(\cdot, 0) = \tilde{u}_0, \quad in\ \Omega, \tag{18}$$

Eq.(16)–Eq.(18) can be given the following equivalent weak formulation: Find $\tilde{u}$: $R_+ \to H_0^1(\Omega)$ such that

$$\begin{gathered} (\tilde{u}_t, v) + (\tilde{g}(\tilde{u})\nabla\tilde{u}, \nabla v) = (\tilde{f}, v), \forall v \in H_0^1(\Omega), t > 0, \\ \tilde{u}(\cdot, 0) = \tilde{u}_0. \end{gathered} \tag{19}$$

Using the Galerkin finite element method by using bivariate splines, we want to find a solution $\tilde{U}_l \in S_4^{2,3;0}(\Delta_{mn}^{(2)})$ such that

$$(\tilde{U}_l - \tilde{U}_{l-1}, v) + k_l(\tilde{g}(\tilde{U}_{l-1})\nabla\tilde{U}_l, \nabla v) = \int_{I_l} (\tilde{f}, v)dt, \forall v \in S_4^{2,3;0}(\Delta_{mn}^{(2)}), \tag{20}$$

$$\tilde{U}_0 = \tilde{u}_0, \tag{21}$$

where $k_l = t_l - t_{l-1}$ is the length of the subinterval I_l.

By using the B-spline bases on $S_4^{2,3;0}(\Delta_{mn}^{(2)})$, we can write

$$\tilde{U}_l = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} \lambda_{i,j,l}\widetilde{B}_{i,j}(x,y),$$

and insert to Eq.(20), we have the following linear system

$$\begin{aligned} &\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} \lambda_{i,j,l}(\widetilde{B}_{i,j}, \widetilde{B}_{s,t}) + k_l \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} \lambda_{i,j,l}(\tilde{g}(\tilde{U}_{l-1})\nabla\widetilde{B}_{i,j}, \nabla\widetilde{B}_{s,t}) \\ &= \int_{I_l} (f, \widetilde{B}_{s,t})dt + (U_{l-1}, \widetilde{B}_{s,t}), \forall 1 \le s \le m-1, 1 \le t \le n-1. \end{aligned} \tag{22}$$

Therefore, the coefficients $\lambda_{i,j,l}$ can be determined by the system of linear equations Eq.(22).

Remark 4.1. *The initial condition $\tilde{U}_0$ is always interpolated by $S_4^{2,3;0}(\Delta_{mn}^{(2)})$.*

5. CONCLUSION AND FUTURE WORK

A Galerkin finite element method by using bivariate splines (GB method) is proposed for solving parabolic partial differential equations. B-spline finite element method have been widely applied to solve parabolic equations. The present work is related the Galerkin method. The numerical results that the GB method provides more accurate solutions.

The difference from the theory when we use the whole space $S_4^{2,3}(\Delta_{mn}^{(2)})$ and the proper space of $S_4^{2,3}(\Delta_{mn}^{(2)})$ for solving parabolic equations remains to be our future work.

The accuracy of the interpolation on boundary conditions affects the accuracy of the numerical solutions of the parabolic equations. In future, we will use $S_4^{2,3}(\Delta_{mn}^{(2)})$ to interpolation the boundary conditions.

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