CSCanada
Progress in Applied Mathematics
Vol. 5, No. 2, 2013, pp. [1–5]
DOI: 10.3968/j.pam.1925252820130502.643

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
www.cscanada.net
www.cscanada.org

The Subclasses of Characterization on Π^*-Regular Semigroups

LUO Xiaoqiang[a],*

[a]Department of Mathematics, Sichuan University of Arts and Science, China.

* Corresponding author.
Address: Department of Mathematics, Sichuan University of Arts and Science, Dazhou, 635000, China; E-Mail: lxq1128@163.com

Received: January 2, 2013/ Accepted: February 26, 2013/ Published: April 30, 2013

Abstract: In the paper, we define the equivalence relations on Π^*-regulur semigroups, to show L^*, R^*, H^*, J^*-class contains an idempotent with some characterizations.

Key words: Π^*-regular semigroup; Completely Π^*-regular semigroup; Subclass; Idempotent

Luo, X. (2013). The Subclasses of Characterization on Π^*-Regular Semigroups. *Progress in Applied Mathematics, 5*(2), 1–5. Available from http://www.cscanada.net/index.php/pam/article/view/j.pam.1925252820130502.643 DOI: 10.3968/j.pam.1925252820130502.643

1. INTRODUCTION

An element d of a semigroup S is Π^*-regular, if S_1 and S_2 are non-empty regular semigroups, and S is all non-empty subset of $S_1 \times S_2$, for any $a \in S_1$, $b \in S_2$, $(a, b) = d \in S$, there exist $m \in Z^+$, $(x, y) \in S$, such that

$$(a, b)^m = (a, b)^m (x, y)(a, b)^m.$$

A semigroup S is Π^*-regular, if every element of S is Π^*-regular. A semigroup S is completely Π^*-regular, if S is Π^*-regular, for any $(a, b) \in S$ and element (a, b) is a regular, there exist $m \in Z^+$, $(x, y) \in S$, such that $(a, b)^m(x, y) = (x, y)(a, b)^m$ [1].

In this paper, we consider some special case of Π^*-regular semigroups and completely Π^*-regular semigroups.

Remark The marks we don't illustrate in this paper please see reference [2].

Now let S be a Π^*-regular semigroup. Define the equivalence relations L^*, R^*, H^*, J^* on S by

$$(a,b)L^*(x,y) \Leftrightarrow S(a,b)^m = S(x,y)^n$$
$$(a,b)R^*(x,y) \Leftrightarrow (a,b)^m S = (x,y)^n S$$
$$H^* \Leftrightarrow L^* \cap R^*$$
$$(a,b)J^*(x,y) \Leftrightarrow S(a,b)^m S = S(x,y)^n S$$

where m,n are the smallest positive integers such that $(a,b)^m$, $(x,y)^n$ are regular, i.e., $(a,b)^m$, $(x,y)^n \in S$. In what follows we will denote by $L^*_{(a,b)}(R^*_{(a,b)}, H^*_{(a,b)}, J^*_{(a,b)})$ the L^*-, R^*-, H^*-, J^*- class containing an element (a,b) of S. According to the hypothesis, we can easily draw the following lemma.

Lemma 1. Let S be a Π^*-regular semigroup. Then every $L^*(R^*)$-class contains at least one idempotent.

Lemma 2. Let S be a Π^*-regular semigroup. Then each idempotent (e,e) of S is a right (left, two-sided) identity for regular elements from $L^*_{(e,e)}(R^*_{(e,e)}, J^*_{(e,e)})$.

Lemma 3. In a Π^*-regular semigroup S every H^*-class contains at most one idempotent.

Lemma 4. Let S be a Π^*-regular semigroup, $(a,b) \in S$ and p be the smallest positive integers such that $(a,b)^p \in S$. Then $(a,b)^p \in L^*_{(a,b)} \cap R^*_{(a,b)} = H^*_{(a,b)}$.

2. MAIN RESULTS

Let V be the set of all inverse elements of S [3], E is the set of all idempotent of S. Here we get a good result.

Theorem 1. Let (a,b) and (x,y) be element of a Π^*-regular semigroup S. Then

$$(1)(a,b)L^*(x,y) \Leftrightarrow \exists (a,b)',\ (x,y)' \in V,\ (a,b)'(a,b)^m = (x,y)'(x,y)^n;$$
$$(2)(a,b)R^*(x,y) \Leftrightarrow \exists (a,b)',\ (x,y)' \in V,\ (a,b)^m(a,b)' = (x,y)^n(x,y)';$$
$$(3)(a,b)H^*(x,y) \Leftrightarrow \exists (a,b)',\ (x,y)' \in V,\ (a,b)'(a,b)^m = (x,y)'(x,y)^n,$$
$$(a,b)^m(a,b)' = (x,y)^n(x,y)'.$$

Proof. Obviously we only need to prove (3). Let (a,b), $(x,y) \in H^*$ and $(a,b)'$, $(x,y)' \in V$. Then

$$(e,e) = (a,b)'(a,b)^m \in L^*_{(a,b)} \cap E = L^*_{(x,y)} \cap E;$$
$$(f,f) = (a,b)^m(a,b)' \in R^*_{(a,b)} \cap E = R^*_{(x,y)} \cap E;$$

So

$$(x,y)^n = (x,y)^n(e,e) = (f,f)(x,y)^n$$
$$(f,f) = (x,y)^n(u,v),\ (e,e) = (s,t)(x,y)^n\ ((u,v),(s,t) \in S).$$

Assume that $(x,y)' = (e,e)(u,v)(f,f)$ then $(x,y)' \in V$. We will find

$$(x,y)^n(x,y)'(x,y)^n = (x,y)^n(e,e)(u,v)(f,f)(x,y)^n$$
$$=(x,y)^n(u,v)(x,y)^n = (f,f)(x,y)^n = (x,y)^n;$$

$$(x,y)'(x,y)^n(x,y) = (e,e)(u,v)(f,f)(x,y)^n(e,e)(u,v)(f,f)$$
$$=(e,e)(u,v)(x,y)^n(u,v)(f,f) = (e,e)(u,v)(f,f) = (x,y)',$$

Now

$$(x,y)^n(x,y)' = (x,y)^n(e,e)(u,v)(f,f)$$
$$=(x,y)^n(u,v)(f,f) = (f,f) = (a,b)^m(a,b)';$$

$$(x,y)'(x,y)^n = (e,e)(u,v)(f,f)(x,y)^n = (s,t)(x,y)^n(u,v)(f,f)(x,y)^n$$
$$=(s,t)(f,f)(x,y)^n = (s,t)(x,y)^n = (e,e) = (a,b)'(a,b)^m.$$

Conversely, if $(a,b)'(a,b)^m = (x,y)'(x,y)^n$ and $(a,b)^m(a,b)' = (x,y)^n(x,y)'$ for some $(a,b)', (x,y)' \in V$, then

$$(a,b)^m = (a,b)^m(a,b)'(a,b)^m = (a,b)^m(x,y)'(x,y)^n = (x,y)^n(x,y)'(a,b)^m$$
$$(x,y)^n = (x,y)^n(x,y)'(x,y)^n = (x,y)^n(a,b)'(a,b)^m = (a,b)^n(a,b)'(x,y)^n$$

hence

$$S(a,b)^m = S(x,y)^n, \ (a,b)^m S = (x,y)^n S$$

whence $(a,b)H^*(x,y)$. □

Theorem 2. Let (e,e) be an idempotent of a Π^*-regular semigroup S. Then $G_{e,e} \subseteq H^*_{(e,e)}$, furthermore, if $(u,v) \in H^*_{(e,e)}$ and p is the smallest positive integer such that $(u,v)^p \in S$, then $(u,v)^q \in G_{(e,e)}$ for every $q \geq p$.

Proof. Let $(a,b) \in G_{(e,e)}$ and let (s,t) be an inverse element for (a,b) in $G_{(e,e)}$ [4]. Since $(s,t)(a,b) = (e,e) = (a,b)(s,t)$, we obtain that $(s,t) \in V$, so by theorem 1 $(a,b) \in H^*_{(e,e)}$. Hence $G_{(e,e)} \subseteq H^*_{(e,e)}$.

Assume $(u,v) \in H^*_{(e,e)}$ and let p be the smallest positive integer such that $(u,v)^p \in S$. By theorem 1 (3) there exists $(u,v)' \in V$ such that $(u,v)'(u,v)^p = (e,e) = (u,v)^p(u,v)'$. So $(u,v)^p$ is completely regular, i.e., $(u,v)^p \in G_{(f,f)}$. It is easy to show that $(e,e) = (f,f)$. Therefore $(u,v)^q \in G_{(e,e)}$, $q \geq p$. □

By theorem 1 and theorem 2, we obtain the following case.

Proposition. S is a Π^*-regular semigroup if and only if S is completely regular semigroup and H^*-class contains an idempotent.

Lemma 5. Let S be a Π^*-regular semigroup. Then

(1) every J^*-class contains at least one idempotent;

(2) $G_{(e,e)} \subseteq H_{(e,e)} \subseteq J_{(e,e)}$ for every $e \in E$.

Lemma 6. Let S be a Π^*-regular semigroup. Then (for some (e,e), $(f,f) \in E$)

$$J^*_{(e,e)} = J^*_{(f,f)}, (e,e)(f,f) = (f,f)(e,e) = (f,f) \Rightarrow (e,e) = (f,f).$$

Proof. It follows from that $S(e,e)S = S(f,f)S$ that

$$(e,e) = (u,v)(f,f)(s,t) \quad ((u,v), \ (s,t) \in S).$$

Suppose that

$$(a,b) = (e,e)(u,v)(e,e),$$

then

$$\begin{aligned}(a,b) &= (e,e)(u,v)(e,e)\\ &= (e,e)(u,v)(e,e)(f,f)(s,t)(e,e)(u,v)(e,e)\\ &= (a,b)(f,f)(s,t)(a,b).\end{aligned}$$

By the hypothesis that there exists $(a,b)' \in S$, such that

$$(a,b) = (a,b)(a,b)'(a,b),\ (a,b)'(a,b) = (a,b)(a,b)'.$$

Now we assume $(x,y) = (e,e)(s,t)(e,e)$, then

$$\begin{aligned}&(a,b)(f,f)(x,y) = (e,e)(u,v)(e,e)(f,f)(e,e)(s,t)(e,e)\\ =&(e,e)(u,v)(f,f)(s,t)(e,e) = (e,e)\end{aligned}$$

Therefore,

$$\begin{aligned}(e,e) &= (a,b)(f,f)(x,y) = (a,b)(a,b)'(a,b)(f,f)(x,y)\\ &= (a,b)(a,b)'(e,e) = (a,b)'(a,b)(e,e) = (a,b)'(a,b).\end{aligned}$$

Hence,

$$\begin{aligned}(e,e) &= (a,b)'(a,b) = (a,b)'(e,e)(a,b)\\ &= (a,b)'(a,b)(f,f)(x,y)(a,b) = (e,e)(f,f)(x,y)(a,b)(f,f)(x,y)(a,b).\end{aligned}$$

Thus

$$(e,e) = (f,f)(e,e)$$

and whence

$$(e,e) = (f,f).$$

□

Lemma 7. Let S be a Π^*-regular semigroup. Then for some $(u,v) \in S,\ (e,e) \in E$,

$$J^*_{(e,e)} = J^*_{(e,e)(u,v)(e,e)} \Rightarrow (e,e)\,(u,v)\,(e,e) \in G_{(e,e)}.$$

Proof. This is obvious. Here we don't prove it. □

Theorem 3. Let S be a Π^*-regular semigroup. Then (for some $(a,b),\ (x,y) \in S$),

$$J^*_{(a,b)(x,y)} = J^*_{(x,y)(a,b)}.$$

Proof. Let p and q be the smallest positive integers such that

$$((a,b)(x,y))^p,\ ((x,y)(a,b))^q \in S.$$

Then

$$\begin{aligned}((a,b)(x,y))^p &\in G_{(e,e)} \subseteq J^*_{(e,e)},\\ ((x,y)(a,b))^q &\in G_{(f,f)} \subseteq J^*_{(f,f)}.\end{aligned}$$

Whence by [5] we obtain

$$\begin{aligned}((a,b)(x,y))^{p+m} &\in G_{(e,e)} \subseteq J_{(e,e)},\\ ((x,y)(a,b))^{q+n} &\in G_{(f,f)} \subseteq J^*_{(f,f)}.\end{aligned}$$

For every $p,q \geq 0$, so

$$\begin{aligned}&((a,b)(x,y))^m J^*((a,b)(x,y))^{p+m},\\ &((x,y)(a,b))^n J^*((x,y)(a,b))^{q+n}.\end{aligned}$$

And for $k = \max(m, n)$, we have

$$S((a,b)(x,y))^m S = S((a,b)(x,y))^{k+1} S,$$
$$= S(a,b)((x,y)(a,b))^k (x,y) S \subseteq S((x,y)(a,b))^k S \subseteq S((x,y)(a,b))^n S.$$

Similarly,

$$S((x,y)(a,b))^n S \subseteq S((a,b)(x,y))^m S.$$

Thus,

$$S((a,b)(x,y))^m S = S((x,y)(a,b))^n S.$$

That is

$$J^*_{(a,b)(x,y)} = J^*_{(x,y)(a,b)}.$$

□

REFERENCES

[1] Luo, X. (2012). Π^*-regular semigroups. *Bulletin of Mathematical Science & Applications, 1*(1), 63-70.

[2] Petrich, M., & Reilly, N. R. (1999). *Completely regular semigroups.* New York: John Wiley & Sons, Inc. Press.

[3] Howie, J. M. (1976). *An introduction to semigroup theory.* New York: Academic Press.

[4] Luo, X. (2010). Another characterition of congruences on completely simple semigroups. *Journal of Sichuan University of Arts and Science, 20*(2), 20-22.

[5] Luo, X. (2012). Another property of congruence-free regular semigroups. *South Asian Journal of Mathematics, 2*(3), 238-241.