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Pricing Foreign Exchange Option Under Fractional Jump-Diffusions

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Abstract: Foreign exchange option, as a financial derivative, plays an important role in the financial market. It is of great theoretical and practical significance to study the foreign exchange options, especially its pricing model. In order to more accurately portray the authenticity of foreign exchange market, this paper applies fractional Brown motion in the fractal market hypothesis and combines with jump diffusion process so as to establish the pricing model of foreign exchange option. Moreover, this paper put forward the pricing formulas of European foreign exchange call and put option, as well as their relationships by using the method of insurance actuary pricing. No matter whether the financial market has arbitrage or not, no matter it is complete or not, this conclusion is valid.

Key words: Fractional Brownian motion; Jump-diffusion; Insurance actuary pricing; Foreign exchange option

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1. INTRODUCTION

Since Philadelphia Stock Exchange of the United States introduced the foreign exchange option transaction in 1982, foreign exchange options have gotten rapid development. It is a mature and widely-used foreign exchange derivatives instrument, which can effectively avoid and control foreign exchange risk. It is of great theoretical and practical significance to study the foreign exchange options, especially its pricing model. There is no doubt that BSGK model, which was obtained in 1983 when Garman and Kohlhagen extended Black-Scholes option pricing model to the field of foreign exchange options, is the most influential foreign exchange option pricing model. BSGK model is built on the assumptions of normal distribution in an efficient market. This assumption believes that the fluctuations in the exchange rate follow the Brownian motion and its rate of return takes on a normal distribution. However, the long-term market validation and a large number of empirical studies have found that the empirical distribution of most of the changes of market variable (especially changes in exchange rates) has a thick tail, and shows a positive high Kurtosis, and has a long-term correlation between the exchange rate at the different times. It shows that the foreign exchange market is not an efficient market. BSGK model itself has some weaknesses, and the subsequent studies are mostly based on the amendment, improvement and expansion of the random process which the price of the underlying asset is subject to.

In 1989, Peters proposed a fractal market hypothesis, and proved that there was fractal structure with non-periodic cycle in different capital market. Fractal market hypothesis does not depend on such assumptions of exchange rate fluctuations as the independent and normal distribution assumptions. What's more, fractional Brownian motion can better explain many phenomena in foreign exchange market, such as the "thick tail" and long-term correlation, which the efficient market hypothesis can not account for. Therefore, creating a foreign exchange option pricing model by fractional Brownian motion in the fractal market hypothesis, we can more accurately portray the authenticity of the market. In 2000, Hu [1] introduced Wick integral in fractional Brownian motion, and in 2003 [2], developed Wick integral in fractional Brownian motion when Hurst exponent $H > 0.5$ through fractional white noise. Bender [3] and Elliott [4] promote Wick integral of fractional Brownian motion to the case of Hurst exponent $H \in (0, 1)$, and set up fractional Black-Scholes model of European contingent claim. What is inadequate is that fractional Brownian motion describes only the continuous change of the price of the underlying asset, that is, the normal changes of asset prices under normal market conditions, but it cannot interpret the abnormal changes of assets prices caused by the unusual circumstances in the market (non-economic factors), namely the discontinuous and wide "jump". In order to make up for this deficiency, this paper, adopting the fractional Brownian motion in the fractal market hypothesis, combining with the jump-diffusion process [5,6] and using the method of insurance actuary pricing, discusses the pricing issue of European foreign exchange options.

2. THE CONSTRUCTION OF MATHEMATICAL EVALUATION MODEL

2.1. Fractional Brownian Motion

The fractional jump-diffusion model made in this article is based on fractional Brownian motion. Fractional Brownian motion was firstly put forward by Mandelbrot. Compared with the standard Brownian motion, it is a notable feature that the time-related function is not zero, namely "long-range correlation". Next we will introduce the definition and some basic properties of fractional Brownian motion.

Definition 2.1 Stochastic process $B_H(t)$ is called fractional Brown motion, if it is continuous and meet $P(B_H(0) = 0) = 1$, $B_H(t) - B_H(s) \sim N(0, |t-s|^{2H})$, where H is Hurst exponent and $H \in (0,1)$.

Fractional Brownian motion is a continuous zero-mean Gaussian process, and its covariance function meet $E(B_H(t)B_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$.

If $0 < H < 0.5$, the correlation coefficient is negative, $B_H(t)$ is anti-persistent; if $0.5 < H < 1$, the correlation coefficient is positive, $B_H(t)$ is persistent; if $H = 0.5$, the correlation coefficient is zero, $B_H(t)$ is Brownian motion. Fractional Brown motion has self-similarity, long-term dependence and other characteristics, which makes it a more appropriate tool for mathematical finance research.

2.2. Fractional Jump-Diffusion Model of Foreign Exchange Option Pricing

Given the financial market in continuous time, taking 0 as now and T as the due date; Given a complete probability space (Ω, F, P), assume that foreign Bond price $P^f(t)$ and domestic Bond price $P^d(t)$ respectively meet

$$\mathrm{d}P^f(t) = P^f(t)r^f(t)\mathrm{d}t,\ P^f(T) = 1; \tag{1}$$

$$\mathrm{d}P^d(t) = P^d(t)r^d(t)\mathrm{d}t,\ P^d(T) = 1. \tag{2}$$

where $r^d(t)$, $r^f(t)$ represent the domestic currency risk-free interest rate and foreign currency risk-free interest rate. Easy to know, $P^f(t) = \exp\{-\int_t^T r^f(t)\mathrm{d}t\}$, $P^d(t) = \exp\{-\int_t^T r^d(t)\mathrm{d}t\}$.

Assume that the price process of foreign exchange rate $S(t)$ meets stochastic differential equation as follows:

$$\mathrm{d}S(t) = S(t)\left[\mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}B_H(t) + (e^{J(t)} - 1)\mathrm{d}Q_t\right] \tag{3}$$

where $B_H(t)$ is fractional Brownian motion, Hurst exponent $H \in (0,1)$; $\mu(t)$, $\sigma(t)$ are continuous functions of the time t; Q_t represents the random jump number of foreign exchange price in the period of time $[0,t]$, assuming that it obeys the Poisson process and its parameter is λ; suppose the random variable $J(t)$ subject to the special normal distribution $N(-\sigma_J^2/2, \sigma_J^2)$, $e^{J(t)} - 1$ represents the relative height of the jump; further assume that $J(t)$, $B_H(t)$ and Q_t are independent of each other.

When Hurst exponent $H \neq 0.5$, fractional Brown motion is neither a Markov process, nor a semi-martingale. Therefore, we cannot use normal stochastic integrals to analysis. Wick integral of fractional Brownian motion, a special integral, is used in this paper [2,3]:

$$\int_a^b f(t,\omega)\mathrm{d}B_H(t) = \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} f(t_k,\omega)\diamondsuit(B_H(t_{k+1}) - B_H(t_k)),\quad 0.5 < H < 1 \tag{4}$$

where $\diamondsuit$ represents Wick integral, suppose $0.5 < H < 1$. According to the stochastic integral (4), the definition and properties of the function $\exp^{\diamondsuit}(X)$ [3], we can get:

$$\begin{aligned} S(T) &= S(0)\exp^{\diamondsuit}\left(\int_0^T \mu(t)\mathrm{d}t + \int_0^T \sigma(t)\mathrm{d}B_H(t) + \sum_{i=1}^{Q_T} J(i)\right) \\ &= S(0)\exp\left\{\int_0^T \left(\mu(t) - H\sigma^2(t)t^{2H-1}\right)\mathrm{d}t + \int_0^T \sigma(t)\mathrm{d}B_H(t) + \sum_{i=1}^{Q_T} J(i)\right\} \end{aligned} \tag{5}$$

$$\mathrm{E}\left(S(T)\right) = S(0)\exp\left(\int_0^T \mu(t)\mathrm{d}t\right) \tag{6}$$

3. THE METHOD OF INSURANCE ACTUARIAL PRICING

Bladt and Rydberg firstly put forward the method of insurance actuarial pricing in 1998 [7]. Compared with the traditional method of martingale pricing, this method's greatest merit is that it doesn't make any economic assumptions to the financial market, that is to say it has nothing to do with the basic assumption of market without arbitrage. No matter there is arbitrage or whether it is complete, it is effective. This text is to discuss the pricing of foreign exchange option by insurance actuarial pricing method. Next we will introduce the basic method of insurance actuarial pricing.

Definition 3.1. Suppose $S(t)$ is the pricing process of time $[0, T]$, $\int_0^T \beta(t)\mathrm{d}t$ is the expected rate of return, define: $\int_0^T \beta(t)\mathrm{d}t = \ln\dfrac{\mathrm{E}[S(T)]}{S(0)}$.

Definition 3.2. Let $C(K, T)$ be the value of European call option at time now, and let $P(K, T)$ be the value of European put option, then we have

$$\begin{aligned} &C(K,T) \\ = \ &\mathrm{E}\left[\left(\exp(-\int_0^T \beta(t)\mathrm{d}t)S(T)P^f(0) - KP^d(0)\right)\mathrm{I}_{\{\exp(-\int_0^T\beta(t)\mathrm{d}t)S(T)P^f(0)>KP^d(0)\}}\right] \end{aligned} \tag{7}$$

$$\begin{aligned} &P(K,T) \\ = \ &\mathrm{E}\left[\left(KP^d(0) - \exp(-\int_0^T \beta(t)\mathrm{d}t)S(T)P^f(0)\right)\mathrm{I}_{\{KP^d(0)>\exp(-\int_0^T\beta(t)\mathrm{d}t)S(T)P^f(0)\}}\right] \end{aligned} \tag{8}$$

where K is the exercise price, T is the expiration date, $S(t)$ is the price process of exchange rate, $\mathrm{E}(\cdot)$ represents mathematical expectation, $\mathrm{I}_A(\omega) = \begin{cases} 1, & if \omega \in A \\ 0, & if \omega \notin A \end{cases}$.

4. PRICING FOREIGN EXCHANGE OPTION UNDER FRACTIONAL JUMP-DIFFUSIONS

Theorem 4.1. Assume that foreign Bond price $P^f(t)$ and domestic Bond price $P^d(t)$ respectively meet Equation (1) and (2), the price process of foreign exchange rate $S(t)$ meet Equation (3), K is the exercise price, T is the expiration date, then we obtain the insurance actuarial pricing formulas of foreign exchange option as follows:

$$C(K,T)=\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}\left(P^f(0)S(0)\Phi(d_2^{(n)})-KP^d(0)\Phi(d_1^{(n)})\right) \tag{9}$$

$$P(K,T)=\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}\left(KP^d(0)\Phi(-d_1^{(n)})-P^f(0)S(0)\Phi(-d_2^{(n)})\right) \tag{10}$$

$$C(K,T)+KP^d(0)=P(K,T)+P^f(0)S(0) \tag{11}$$

where $d_1^{(n)}=\dfrac{\ln(\frac{S(0)P^f(0)}{KP^d(0)})-\dfrac{1}{2}\sigma_H^2}{\sigma_H}$, $d_2^{(n)}=d_1^{(n)}+\sigma_H$, $\Phi(x)=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-(1/2)\cdot s^2}\mathrm{d}s$ represents normal distribution function, $\sigma_H=\sqrt{2\int_0^T H\sigma^2(t)t^{2H-1}\mathrm{d}t+n\sigma_J^2}$.

Proof. For convenience, assume that $A\triangleq\left\{\exp\left(-\int_0^T\beta(t)\mathrm{d}t\right)S(T)P^f(0)>KP^d(0)\right\}$. According to Definition 3.2, the value of foreign exchange call option $C(K,T)$ satisfies:

$$C(K,T)=\mathrm{E}\left[\exp\left(-\int_0^T\beta(t)\mathrm{d}t\right)S(T)P^f(0)\mathrm{I}_A\right]-\mathrm{E}\left[KP^d(0)\mathrm{I}_A\right] \tag{12}$$

The expected rate of return $\int_0^T\beta(t)\mathrm{d}t$ meets $\exp\left(\int_0^T\beta(t)\mathrm{d}t\right)=\mathrm{E}\left[S(T)\right]/S(0)=\exp\left(\int_0^T\mu(t)\mathrm{d}t\right)$.

$$\begin{aligned}
A&\triangleq\{\exp(-\int_0^T\beta(t)\mathrm{d}t)S(T)P^f(0)>KP^d(0)\}\\
&=\left\{\exp\left(-\int_0^T\mu(t)\mathrm{d}t\right)\exp\{\int_0^T\left(\mu(t)-H\sigma^2(t)t^{2H-1}\right)\mathrm{d}t+\int_0^T\sigma(t)\mathrm{d}B_H(t)+\sum_{i=1}^{Q_T}J(i)\}\right.\\
&\qquad\qquad\left.>\frac{KP^d(0)}{S(0)P^f(0)}\right\}\\
&=\left\{\int_0^T\sigma(t)\mathrm{d}B_H(t)+\sum_{i=1}^{Q_T}J(i)>\mathrm{In}\frac{KP^d(0)}{S(0)P^f(0)}+\int_0^T H\sigma^2(t)t^{2H-1}\mathrm{d}t\right\}
\end{aligned}$$

Note that $\int_0^T\sigma(t)\mathrm{d}B_H(t)\sim N\left(0,2\int_0^T H\sigma^2(t)t^{2H-1}\mathrm{d}t\right)$, $\sum\limits_{i=1}^{Q_T}J(i)\sim N\left(-Q_T\sigma_J^2/2,Q_T\sigma_J^2\right)$, $J(t)$, $B_H(t)$ and Q_t are independent of each other, therefore

$$\xi\triangleq\int_0^T\sigma(t)\mathrm{d}B_H(t)+\sum_{i=1}^{Q_T}J(i)\sim N\left(-Q_T\sigma_J^2/2,Q_T\sigma_J^2+2\int_0^T H\sigma^2(t)t^{2H-1}\mathrm{d}t\right)$$

Let $\eta \triangleq (\xi + Q_T\sigma_J^2/2)/\sigma_H$, $\sigma_H = \sqrt{2\int_0^T H\sigma^2(t)t^{2H-1}\mathrm{d}t + n\sigma_J^2}$, then $A = \bigcup\limits_{n=0}^{\infty}\{\eta > -d_1^{(n)}\}$, $\eta \sim N(0,1)$.

First we compute $\mathrm{E}\left[\exp\left(-\int_0^T \beta(t)\mathrm{d}t\right)S(T)P^f(0)\mathrm{I}_A\right]$:

$$\begin{aligned}
&\mathrm{E}\left[\exp(-\int_0^T \beta(t)\mathrm{d}t)S(T)P^f(0)\mathrm{I}_A\right]\\
=&\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}\mathrm{E}\left[\exp(-\int_0^T \mu(t)\mathrm{d}t)S(T)P^f(0)\mathrm{I}_A|\, Q_T = n\right]\\
=&\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}P^f(0)S(0)\mathrm{E}\left[\exp(-\int_0^T H\sigma^2(t)t^{2H-1}\mathrm{d}t + \int_0^T \sigma(t)\mathrm{d}B_H(t)\right.\\
&\left.+\sum_{i=1}^{n}J(i))\mathrm{I}_{\{\eta>-d_1^{(n)}\}}\right]\\
=&\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}P^f(0)S(0)\mathrm{E}\left[\exp(-\int_0^T H\sigma^2(t)t^{2H-1}\mathrm{d}t + \sigma_H\eta - \frac{n\sigma_J^2}{2})\mathrm{I}_{\{\eta>d_1^{(n)}\}}\right]\\
=&\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}P^f(0)S(0)\int_{-d_1^{(n)}}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-\frac{\sigma_H^2}{2} + \sigma_H x - \frac{x^2}{2})\mathrm{d}x\\
=&\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}P^f(0)S(0)\int_{-(d_1^{(n)}+\sigma_H)}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})\mathrm{d}x\\
=&\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}P^f(0)S(0)\Phi(d_1^{(n)} + \sigma_H)
\end{aligned} \tag{13}$$

Next we compute $\mathrm{E}\left[KP^d(0)\mathrm{I}_A\right]$:

$$\begin{aligned}
&\mathrm{E}\left[KP^d(0)\mathrm{I}_A\right]\\
=&\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}\mathrm{E}\left[KP^d(0)\mathrm{I}_A|\, Q_T = n\right]\\
=&\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}\mathrm{E}\left[KP^d(0)\mathrm{I}_{\{\eta>-d_1^{(n)}\}}\right]\\
=&\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}KP^d(0)\Phi(d_1^{(n)})
\end{aligned} \tag{14}$$

Integrated (12), (13), (14), the insurance actuarial pricing formulas of foreign exchange call option are obtained. Similarly, the pricing formula (10) can also be proved. Formula (9) minus formula (10),

$$C(K,T) - P(K,T) = \sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda T}(\lambda T)^n}{n!}(P^f(0)S(0) - KP^d(0)) = P^f(0)S(0) - KP^d(0)$$

So the formula (11) is proved. □

5. CONCLUSION

Foreign exchange option, as a financial derivative, plays an important role in the financial market. Thus, how to accurately price both in theory and practice is extremely important. In order to more accurately portray the authenticity of foreign exchange market, this paper applies fractional Brown motion in the fractal market hypothesis and combines with jump diffusion process so as to establish the pricing model of foreign exchange option under fractional jump-diffusion. Moreover, this paper discusses the pricing issue of European foreign exchange according to the insurance actuary pricing method and put forward the pricing formula of foreign exchange call and put option, as well as their relationships. No matter whether the financial market has arbitrage or not, no matter it is complete or not, this conclusion is valid.

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