CSCanada
Progress in Applied Mathematics
Vol. 4, No. 2, 2012, pp. [58–64]
**DOI:** 10.3968/j.pam.1925252820120402.1981

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
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# An Introduction to the Generator of the Function Space

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**Supported by** 973-Project of China (No. 2006cb303102).

Received: September 10, 2012/ Accepted: October 15, 2012/ Published: October 31, 2012

**Abstract:** The paper would introduce a concept of the generator of the function space. The generator is a more fundamental function than the basis, that the function space can be generated by the shifts and the linear combination of the generator. Various related properties of the generator are presented.

**Key words:** Function; Generator; Space

Wu, Z.M. (2012). An Introduction to the Generator of the Function Space. *Progress in Applied Mathematics, 4*(2), 58–64. Available from http://www.cscanada.net/index.php/pam/article/view/j.pam.1925252820120402.1981 DOI: 10.3968/j.pam.1925252820120402.1981

## 1. INTRODUCTION

We will begin with a trivial question—what is applied mathematics? From the references, there is no standard definition. Someone said: almost all research topics are applied mathematics. If a research topic doesn't use mathematics, then the topic has not achieved a scientific level. Someone said: almost all of the mathematics are applied mathematics. Most of the basic problems of mathematics came from applications. On the other side, the results of the mathematics can be or will (in future) be used in applications. We will not give a standard definition of applied mathematics here. However the phrase "applied mathematics" contains two parts—the application and the mathematics. At least the interface of application and mathematics belongs to applied mathematics. Therefor compared to the people who doing application and those who doing pure mathematics, applied mathematicians will pay more attention to the interface of application and mathematics, applied mathematicians will focus on the interaction between application and mathematics.

The media for the interaction via the interface of application and mathematics are data. Data are usually huge amount, however must be finite. Applied mathematicians work at first on the data to find out or to represent the underlying functions, this is called numerical approximation. Then applied mathematicians take mathematical operator to the functions to get mathematical equations, usually to get partial differential equations, this is called mathematical modeling or in a pure mathematical language "differential operator approximation". The result equations are often interpreted as physical laws. The Kepler's law was established in this way. Finally properties of solutions of mathematical equations will be investigated and used to interpret the physical phenomena.

More in details, if we have an unknown differential operator which is applied on a series of function $\{f_j(x)\}$ and get a series of function $\{g_j(x)\}$, we can discrete the functions to be vectors (or points) — $\{f_j(x) \to \{f_j(x_k)\} = f_j\}$ and $\{g_j(x) \to \{g_j(y_k)\} = g_j\}$ (because one can only sample the finite data $\{f_j\}$ and $\{g_j\}$ in the application), so that the differential operator can be approximated by an algebraic operator, which maps the points $\{f_j\}$ to the points $\{g_j\}$. Therefor the approach of differential operator approximation can be stated as a numerical approximation problem too in a high dimensional space.

Since the differential operator approximation can be stated as a numerical approximation problem, in this paper we will focus on the numerical approximation, which approximates the underlying function based on the sampling data $\{f(x_k)\}$. A more complicated problem is that we can only sample the data of the right hand sides of a partial differential equation and the boundary conditions. For example, we can sample the data $\{\Delta f(x_k)\}_{k=1}^n$ in a domain and $\{f(x_k)\}_{k=n+1}^{n+m}$ on the boundary. More generally, we can sample the data $\{L_k f\}$, where $\{L_k\}$ are (linear or non linear) functionals, and want to find or approximate the underlying function $f(x)$.

To approximate the function, we require a nested function space. e.g. $\cdots \subset V_n \subset V_{n+1} \subset \cdots$. For example $V_n = P_n$, $V_n =$ the spline on the knots $\{j/2^n\}$, $V_n =$ the wavelets on the knots $\{j/2^n\}$, etc. The numerical approximation can be formulated as that, to find the function $f_n(x) \in V_n$ based on the data $\{f(x_j)\}_{j=0}^N$ or $\{L_j f\}_{j=0}^N$, such that the norm of $\|f_n(x) - f(x)\|$ will be minimized. If the distance or the norm is Euclidean, Hilbertian or Sobolev's, then we require only to find the projection of the function $f(x)$ to $V_n$.

We begin with an example $V_n = P_n$, the polynomial of degree less than or equal to $n$. The most common method is to find the function $f_n(x) \in V_n$ by least square, that $\sum_{k=0}^N (f_n(x_k) - f(x_k))^2$ is minimized, which is an approximation of the $L_2$ norm of $\|f_n(x) - f(x)\|$. If $n < N$, it is the classical least square, however the result can not represent the underlying function very well usually. If $n \geq N$, then the least square will turn to the interpolation. If $n = N$ the interpolation is successful and unique solvable. If $n > N$, the interpolation is successful too, but possesses no uniqueness. In this case one can find a solution with a punisher of the $L_0$ norm of the coefficients of the function $f_n(x)$ to get a shortest representation, which is very popular recently. Because to find the simplest shortest representation of the underlying function is the basic problem in applied mathematics. The least squares are convergent to the underlying function $f(x)$ as $n$ and $N$ tend to infinity, if all the derivatives of the function $f(x)$ are uniformly bounded.

Usually $n$ is very large (too large!), especially for multi-variate problems. To run this algorithm we require to solve a large scaled linear system of equations, or even with a searching process of non linear programming. The knots or data points $\{x_j\}_{j=0}^N$ are usually scattered. A lot of the cases in the application, the data are even linear functionals. (e.g. by remote sensing, seismic, etc.). We would like to mention the reader again that for engineer, both $n$ and $N$, the large of the data interacted through the interface of application and mathematics, are finite.

The key feature of the function approximation is that we want to use less basic functions (basis) to represent or approximate the underlying target complicate function. To solve such problems we require at first a basis in $V_n$. It is better to have an orthonomal basis, otherwise we should find the dual basis to make the projection. Our function $f_n(x)$ is a (or can be constructed by) linear combination of basis. Again we go back to a trivial question — what is the basis? The basis is the smallest set in $V_n$, which can be linear combined to represent all the functions in $V_n$. A trivial question again — how many basis for 100-variate parabolic function space? The answer is $3^{100}$, these lead to the the catastrophe of the space-dimension. The parabolic function space is a very weak space, we can even not use it to represent or approximate a function with more than two local maximums.

To solve the problem, we would adopt the result from Physics. The fundamental problem in Physics is to answer: What is the substance? Substance is constructed (linear combined) by Atoms or Elements.

e.g. Water = 2H + O, Salt = Na + Cl.

Basis in mathematics is somewhat like atoms or elements in Physics. However the physicist know that the atoms or elements are constructed by more fundamental particles! These lead to a question in mathematics that, whether in function space there exists such more basic function than basis? The answer is YES for most of known function spaces, especially for the function spaces, which can be used in application. For example, the function $x^n$ in the space of polynomial of the degree less than or equal to $n$. Any $n+1$ pairwise distinct shifts $\{(x-x_j)^n\}$ is a basis. This means $x^n$ is a more fundamental function than basis, any basis and function in polynomial space can be generated by $x^n$ via shifts and its linear combinations.

**Definition 1.1.** For a given function space, if there exists a function $G(x)$, such that the shifts and its linear combinations can generate the whole function space, then $G(x)$ is called the **generator** of the function space.

*Remark* 1.2. Generator is a more basic function than the basis of the function space, which contains all the DNA of the function space. One requires only to take the shifts (copy it self) and make liner combinations to generate whole function space. This phenomenon appears often in Biology too. The Proteins, the Cells and the Life copy it self and make combination to get the Cells, the Life and the Society respectively.

In the following we will give some examples of generator in some common function spaces.

- $x^n$, Polynomial.

- $\sin(2x) - 2\sin(x)$, Trigonometric Polynomial.
- $2e^x - e^{2x}$, Exponential Polynomial.
- $|x|^{2k+1}$, Polynomial Splines.
- $|x|^{2k}\ln(|x|)$, Thin-plate, Poly-harmonic Splines.
- $|x|^{2k+1}e^x\sin^2 x$, Tschebycheffian Splines.
- $e^{-x^2}$, Gaussian.
- $\sqrt{c^2+x^2}$, Multi Quadratics.

In the application, Engineers haven't learned a lot of functions. Above are almost all kinds of the functions, which have been learned by Engineers. This is perhaps the reason, why the spline wavelets are mostly used in wavelets application.

## 2. FINITE DIMENSIONAL FUNCTION SPACE

More than $n+1$ shifts of $x^n$ are linearly dependent.

**Theorem 2.1.** *(Necessary and Sufficient Condition) Function which can only generate a finite dimensional space must be a solution of linear ordinary (partial) differential equation with constant coefficients.*

*For multivariate cases, the condition is necessary but not sufficient.*

The following are some of the ordinary differential equation operators related to such function spaces.

- $D^n f(x) = 0$, Polynomial of degree $n-1$
- $(D^2 + c^2 I)f(x) = 0$, Trigonometry polynomial
- $(D^2 - c^2 I)f(x) = 0$, Exponential polynomial
- $D^n(D^4 - c^4 I)f(x) = 0$, Algebra of above functions
- $P(D)f(x) = 0$, If $P(\lambda) = 0$ possesses roots $\lambda_j$, then $\{\exp(\lambda_j x)\}$ are basis, $\sum \exp(\lambda_j x)$ is a generator.

*Remark* 2.2. These contain almost all the function, which can be supplied to the Engineers for application. Using rational form we can get more other function spaces, e.g. $\tan(x) = \{\sin(x), \cos(x)\}$ the functions pair of above functions.

**Theorem 2.3.** *For the solution space of $P(D)G(x) = 0$, if $G(x)$ satisfies*

$$G^{(k)}(0) = \delta_{k,n-1},$$

*we can prove: $G(x)$ is a generator of the solution space and is called the* ***standard generator*** *for the solutions space.*

**Theorem 2.4.** *The function $G(x)$ is the first coefficient of interpolatory polynomial that the value at $\lambda_j$ is $e^{\lambda_j x}$, which can be represented in the form of divided difference, therefore we get an explicit representation of the standard generator:*

$$G(x) = [\lambda_1, \cdots, \lambda_n]e^{\lambda x}, \tag{2.1}$$

*if $\lambda_j$ are the roots of the character function of $P(\lambda) = 0$.*

The estimates of approximations order of polynomial approximation are based on the Taylor's expanses and the Roll's Lemma, here we have the parallel results[7].

**Theorem 2.5.** *(generalized least square) If we have a series of the ordinary differential operators $P_n(D) = P_{n-1}(D)Q_n(D)$, then we have a nested solutions spaces $V_n = \mathrm{Ker}(P_n(D))$, parallel least square and the interpolation can be made for these function spaces too, if*

$$\max\{|x_j - x_k|\} \min |Im(\lambda_j)| < 2\pi.$$

**Theorem 2.6.** *(generalized Taylor's expanses) For the solution space of $P(D)G(x) = 0$, if $G_j(x)$ satisfies*

$$G^{(k)}(0) = \delta_{k,j},$$

*we have a generalized Taylor's expanses that*

$$f(x) \sim \sum f^{(j)}(\bar{x})G_j(x - \bar{x}) + \mathcal{O}^{(n+1)}(x - \bar{x}).$$

**Theorem 2.7.** *(generalized Roll's Lemma) If the character function $P(\lambda)$ is a polynomial of degree $n$, for given function $f(x)$, there is a function $f^*(x)$ in the solution space that $f(x_j) - f^*(x_j) = 0$ possesses $n + 1$ zero points $\{x_j\}$, then there exist a point $\xi \in (\min\{x_j\}, \max\{x_j\})$ that*

$$P(D)(f(\xi) - f^*(\xi)) = 0.$$

**Theorem 2.8.** *The most parallel result for the polynomial space such as the Bernstein's polynomial, Bernstein's approximation can be generalized to such solutions space too[7].*

## 3. INFINITE DIMENSIONAL FUNCTION SPACE

The approximation capacity is limited, if the generator can only generate a finite dimensional function space.

How can we construct a new generator based on the standard generator of the solution space of the ordinary differential equations, such that it can generate an infinite dimensional space and can approximate (represent) almost all the functions?

This time we would ask the Biologist!

In Biology an important concept is Bio-multiformity, which tells us the reason, why a new species appear. This event will happen via gene mutation.

Based on the generator $x$ we can construct a new function $|x|$ with gene mutation at zero, then Euclidian Hat

$$\Lambda_j(x) = \frac{(|x - x_{j+1}| - |x - x_j|)}{2(x_{j+1} - x_j)} - \frac{(|x - x_j| - |x - x_{j-1}|)}{2(x_j - x_{j-1})}$$

and piecewise linear interpolation

$$f^*(x) = \sum f(x_j)\Lambda_j(x)$$

can be constructed, if we take $|x|$ as a new generator. It is well known that any continuous function can be approximated by piecewise linear interpolation!

**Theorem 3.1.** *Any standard generator with gene mutation:*

$$G(x) := \frac{1}{2}\text{sign}(x)G(x) \tag{3.1}$$

*will generate an infinite dimensional space, which generates a Tschebycheffian spline function space: the piecewise function of algebra of polynomial, trigonometric polynomial and exponential polynomial and can approximate (represent) almost any function[6].*

**Theorem 3.2.** *For Tschebycheffian spline generated by such generator, parallel works such as the dual basis, the interpolation and the approximation, B-spline form, Energy minimization, subdivisions algorithm, wavelets decomposition have been done in [6].*

For multivariate problems, the simplest generator is the radial function: Let $\phi : R_+ \to R, \Phi(x) = \phi(\|x\|)$, then we get the radial basis space $\{\Phi(x - x_j)\}$ [3,4]. [5] first constructed compact supported radial generator with only one piece of the polynomial. [2] answers the open problem in [5] to get such compact radial basis with minimal degree. We can construct the dual basis for given Hilbert, Sobolev norm. We can design a new norm (Kriging norm [4], or native space) to form a orthogonal basis to get the result that the interpolation minimizes the Kriging norm or is a projection in the native space by using the concept of reproducing kernel Hilbert space. [1] shows that, if an image space, whose elements can be represented as $\{f(x) = \int h(x,t)F(t)dt\}$, then we have a generator (reproducing kernel) $\phi(x-y) = \int h(x,t)h^*(y,t)dt$ that the image space can be generated by the generator $\phi(x)$ via the shifts and the linear combinations in respect to the native space norm or the Kriging norm.

Any shift invariant Hilbert space possesses a reproducing generator

$$\phi(x-y) = \sum b_j(x)\overline{b_j(y)},$$

where $\{b_j(x)\}$ is any orthonormal basis, that $f(x) =< f(\cdot), \phi(x - \cdot) >$ respect to the native space norm or Kriging norm.

The key problem is to find a simple mathematical representation of the generator for using in application. The most commonly used generators in application are Gaussian $e^{-x^2/\sigma^2}$, the spline, the poly-harmonic spline and the Tschebycheffian spline (2.1, 3.1), the compactly supported radial generator [2,5] and the multiquadric function $\sqrt{c^2+x^2}$, which was used first by Hardy for aircraft design in Boing Co..

## 4. CONCLUSIONS

In this paper, we have shown a brief introduction of the generator of the function space. Some more related works and the details in respect to the concept can be found in the references and the references therein. The open problem is that, in the application, how can we find a proper generator, which can represent or approximate the underlying function simply.

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