CSCanada
Progress in Applied Mathematics
Vol. 4, No. 2, 2012, pp. [44–57]
DOI: 10.3968/j.pam.1925252820120402.1785

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
www.cscanada.net
www.cscanada.org

Nonoscillation for System of Neutral Delay Dynamic Equation on Time Scales

G.H. Liu[a],* and L.CH. Liu[a]

[a] College of Science, Hunan Institute of Engineering, China.

* Corresponding author.
Address: College of Science, Hunan Institute of Engineering, 88 East Fuxing Road, Xiangtan, Hunan 411104, China; E-Mail: gh29202@163.com

Received: September 12, 2012/ Accepted: October 10, 2012/ Published: October 31, 2012

Abstract: In this paper, by fixed theorem, some sufficient conditions for nonoscillation of the system of neutral delay dynamic equations on time scales $\mathbb{T}$ are established. Our results as special case when $\mathbb{T} = R$ and $\mathbb{T} = N$, involve and improve some known results.

Key words: Nonoscillation; System; Dynamic equations; Time scales

Liu, G.H., & Liu, L.CH. (2012). Nonoscillation for System of Neutral Delay Dynamic Equation on Time Scales. *Progress in Applied Mathematics, 4*(2), 44–57. Available from http://www.cscanada.net/index.php/pam/article/view/j.pam.1925252820120402.1785 DOI: 10.3968/j.pam.1925252820120402.1785

1. INTRODUCTION

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis [1]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications [9].

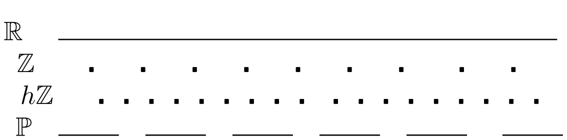

Figure 1. Some Time Scales

where $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}, h > 0\}$, $\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$.

On any time scale $\mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\}, \quad \rho(t) := \sup\{s < t : s \in \mathbb{T}\}.$$

A point $t \in \mathbb{T}, t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left- scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous function provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nondecreasing, and non-increasing on $[a, b]$, if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)$, respectively.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the delta derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient $\frac{f}{g}$ (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}.$$

$$(\frac{f}{g})^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}.$$

For $t_0, b \in \mathbb{T}$, and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_{t_0}^{b} f^{\Delta}(t)\Delta t = f(b) - f(t_0).$$

An integration by parts formula reads

$$\int_{t_0}^{b} f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]_{t_0}^{b} - \int_{t_0}^{b} f^{\Delta}(t)g^{\sigma}(t)\Delta t.$$

and infinite integral is defined as

$$\int_{t_0}^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_{t_0}^{b} f(t)\Delta t.$$

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales.We refer the reader to the papers [2–8] and the reference cited therein.

In this paper, we consider dynamic equations

$$(x(t) + px(t-\tau))^{\Delta} + Q(t)x(t-\tau) = 0, \tag{1}$$

and

$$(x(t) + Bx(t-\tau))^{\Delta} + Q(t)x(t-\tau) = 0, \tag{2}$$

Where $t \in \mathbb{T}$, $p \in R$, $0 \leqslant \tau$, $\sigma \in \mathbb{T}$, and $Q(t)$ is rd-continued $n \times n$ matrix on $[t_0, \infty) \cap \mathbb{T}$. Let $x(t)$ be the set of all rd-continuous and bounded n-dimension vector functions on $[t_0, \infty) \cap \mathbb{T}$, and B is a $n \times n$ matrix, and $\|B\| \neq 0$. Let $\|B\| = b$.

2. NONOSCILLATION THEOREMS

First we consider the case p is a constant.

Theorem 2.1 Suppose that $\int_{t_1}^{\infty} \|Q(s)\| \Delta s < \infty$, where $p \neq -1$, and $\| \bullet \|$ is any norm in $\mathbb{T}$, then Eq.(1) has a nonoscillatory solution.

Proof. The proof of this theorem will be divided into five cases depending on the five different ranges of the parameter p.

Case 1. $0 \leqslant p < 1$.

Choose a $t_1 \in \mathbb{T}$ sufficiently large such that $t_1 \geqslant t_0 + \overline{\sigma}$, where $\overline{\sigma} = \max\{\tau, \sigma\}$, and $\int_{t_1}^{\infty} \|Q(s)\| \Delta s \leqslant \dfrac{1 - p(1 + M_2) - M_1}{M_2}$ holds. Where $0 < M_1 < 1, M_2 > M_1, M_1 + M_2 < 2$, and $1 - \dfrac{M_1 + M_2}{2} \leqslant p < \dfrac{1 - M_1}{1 + M_2}$.

Let $x(t)$ be the set of all continuous and bounded vector functions on $[t_0, \infty) \cap \mathbb{T}$. Let $A = \{x \in X : M_1 \leqslant \|x(t)\| \leqslant M_2, t_0 \leqslant t\}$. Define a mapping $F : A \to X$ as follows

$$(Fx)(t) = \begin{cases} (1-p)e - px(t-\tau) + \int_t^{\infty} Q(s)x(s-\sigma)\Delta s, & t \geqslant t_1; \\ (Fx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases} \tag{3}$$

Clearly F_x is rd-continuous. For every $x \in A$ and $t \geqslant t_1$, using Eq.(3), we get

$$\begin{aligned} \|(Fx)(t)\| &= \|(1-p)e - px(t-\tau) + \int_t^{\infty} Q(s)x(s-\sigma)\Delta s\| \\ &\leqslant \|(1-p)e\| + \|px(t-\tau)\| + \|\int_t^{\infty} Q(s)x(s-\sigma)\Delta s\| \\ &\leqslant (1-p) + p\|x(t-\tau)\| + \int_t^{\infty} \|Q(s)x(s-\sigma)\|\Delta s \\ &\leqslant 1 - p + pM_2 + M_2\int_t^{\infty} \|Q(s)\|\Delta s \\ &\leqslant 1 - p + pM_2 + M_2\frac{1 - p(1+M_2) - M_1}{M_2} \\ &= 2(1-p) - M_1 \leqslant M_2 \end{aligned}$$

Further, in view of Eq.(3), we have

$$\begin{aligned} \|(Fx)(t)\| &= \|(1-p)e - [px(t-\tau) - \int_t^{\infty} Q(s)x(s-\sigma)\Delta s]\| \\ &\geqslant \|(1-p)\| - \|px(t-\tau) - \int_t^{\infty} Q(s)x(s-\sigma)\Delta s\| \end{aligned}$$

$$
\begin{aligned}
&\geqslant 1-p-p\|x(t-\tau)\|-\int_t^\infty \|Q(s)x(s-\sigma)\|\Delta s\\
&\geqslant 1-p-pM_2-M_2\int_t^\infty \|Q(s)\|\Delta s\\
&\geqslant 1-p-pM_2-M_2\frac{1-p(1+M_2)-M_1}{M_2}\\
&= M_1.
\end{aligned}
$$

Thus we proved that $FA\subset A$.

Now, for all $x_1,x_2\in A$, and $t\geqslant t_1$, we have

$$
\begin{aligned}
&\|(Fx_1)(t)-(Fx_2)(t)\|\\
=&\|-B[x_1(t-\tau)-x_2(t-\tau)]+\int_t^\infty Q(s)[x_1(s-\sigma)-x_2(s-\sigma)]\Delta s\|\\
\leqslant&\|-B[x_1(t-\tau)-x_2(t-\tau)]\|+\|\int_t^\infty Q(s)[x_1(s-\sigma)-x_2(s-\sigma)]\Delta s\|\\
\leqslant& b\|x_1-x_2\|+\|x_1-x_2\|\int_t^\infty \|Q(s)\|\Delta s\\
\leqslant& r_1\|x_1-x_2\| \quad (r_1=b+\int_t^\infty \|Q(s)\|\Delta s)
\end{aligned}
$$

Clearly,

$$q_1=p+\int_t^\infty \|Q(s)\|\Delta s\leqslant p+\frac{1-p(1+M_2)-M_1}{M_2}=\frac{1-p-M_1}{M_2}<1.$$

so

$$\|(Fx_1)(t)-(Fx_2)(t)\|\leqslant q_1\|x_1-x_2\|,$$

which proves that F is a contraction mapping. Consequently, F has the fixed point x with $Fx=x,\ \|x\|>0$, for all $t\geq t_1$, which is a nonoscillatory solution of Eq.(1).

Case 2. $1<p<\infty$.

Choose a $t_1\in\mathbb{T}$ sufficiently large such that $t_1+\tau\geqslant t_0+\sigma$, and

$$\int_{t_1+\tau}^\infty \|Q(s)\|\Delta s\leqslant\frac{p-1-pN_1-N_2}{N_2},$$

hold, where $0<N_1<1,\ N_2>N_1$ and $N_1+N_2<2,\ \dfrac{1+N_2}{1-N_1}<p\leqslant\dfrac{2}{2-N_1-N_2}$.

Let $X(t)$ be the set of all $rd-$continuous and bounded vector functions on $t_0\leqslant t\in\mathbb{T}$. Set $A=\{x\in X: N_1\leqslant\|x(t)\|\leqslant N_2,\ t_0\leqslant t\cap\mathbb{T}\}$. Define a mapping $F:A\to X$ as follows

$$(Fx)(t)=\begin{cases}(1-\frac{1}{p})e-\frac{1}{p}x(t+\tau)+\frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s, & t\geqslant t_1;\\ (Fx)(t_1), & t_0\leqslant t\leqslant t_1.\end{cases}\tag{4}$$

Clearly Fx is C_{rd} continuous. For every $x\in A$ and $t\geqslant t_1$, using (4) we get

$$\|(Fx)(t)\| = \|(1-\frac{1}{p})e-\frac{1}{p}x(t+\tau)+\frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s\|$$

$$
\begin{aligned}
&\leqslant \|(1-\frac{1}{p})e\| + \|\frac{1}{p}x(t+\tau)\| + \|\frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\
&\leqslant (1-\frac{1}{p})\|e\| + \frac{1}{p}\|x(t+\tau)\| + \frac{1}{p}\|\int_t^\infty Q(s)x(s-\sigma)\|\Delta s \\
&\leqslant 1-\frac{1}{p}+\frac{N_2}{p}+\frac{N_2}{p}\int_t^\infty \|Q(s)\|\Delta s \\
&\leqslant 1-\frac{1}{p}+\frac{N_2}{p}+\frac{N_2}{p}\frac{p-1-pN_1-N_2}{N_2} \\
&= 2(1-\frac{1}{p})-N_1 \\
&\leqslant N_2.
\end{aligned}
$$

Further

$$
\begin{aligned}
\|(Fx)(t)\| &= \|(1-\frac{1}{p})e-[\frac{1}{p}x(t+\tau)-\frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s]\| \\
&\geqslant \|(1-\frac{1}{p})\| - \|\frac{1}{p}x(t+\tau)-\frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\
&\geqslant 1-\frac{1}{p}-\frac{1}{p}\|x(t+\tau)\|-\frac{1}{p}\int_t^\infty \|Q(s)x(s-\sigma)\|\Delta s \\
&\geqslant 1-\frac{1}{p}-\frac{N_2}{p}-\frac{1}{p}\int_t^\infty \|Q(s)\|\|x(s-\sigma)\|\Delta s \\
&\geqslant 1-\frac{1}{p}-\frac{N_2}{p}-\frac{N_2}{p}\int_t^\infty \|Q(s)\|\Delta s \\
&\geqslant 1-\frac{1}{p}-\frac{N_2}{p}-\frac{N_2}{p}\frac{p-1-pN_1-N_2}{N_2} \\
&= N_2.
\end{aligned}
$$

Thus we proved that $FA \subset A$.
Now, for all $x_1, x_2 \in A$, and $t \geqslant t_1$, we have

$$
\begin{aligned}
&\|(Fx_1)(t)-(Fx_2)(t)\| \\
=&\|-B^{-1}[x_1(t-\tau)-x_2(t-\tau)]+B^{-1}\int_t^\infty Q(s)[x_1(s-\sigma)-x_2(s-\sigma)]\Delta s\| \\
\leqslant&\|-B^{-1}[x_1(t-\tau)-x_2(t-\tau)]\|+B^{-1}\|\int_t^\infty Q(s)[x_1(s-\sigma)-x_2(s-\sigma)]\Delta s\| \\
\leqslant&\frac{1}{b}\|x_1(t-\tau)-x_2(t-\tau)\|+\frac{1}{b}\int_t^\infty \|Q(s)\|\|x_1(s-\sigma)-x_2(s-\sigma)\|\Delta s \\
\leqslant&\frac{1}{b}\|x_1-x_2\|+\frac{1}{b}\|x_1-x_2\|\int_t^\infty \|Q(s)\|\Delta s \\
\leqslant&r_2\|x_1-x_2\| \quad (r_2=\frac{1}{b}+\frac{1}{b}\int_t^\infty \|Q(s)\|\Delta s).
\end{aligned}
$$

Clearly,

$$
q_2=\frac{1}{p}+\frac{1}{p}\int_t^\infty \|Q(s)\|\Delta s \leqslant \frac{1}{p}+\frac{p-1-pN_1-N_2}{N_2}=\frac{p-1-pN_1}{pN_2}<1.
$$

so

$$\|(Fx_1)(t) - (Fx_2)(t)\| \leqslant q_2\|x_1 - x_2\|,$$

which proves that F is a contraction mapping. Consequently, F has the fixed point x with $Fx = x, \ \|x\| > 0$, for all $t \geq t_1$, which is a nonoscillatory solution of Eq.(1).

Case 3. $p = 1$.

Choose a $t_1 \in \mathbb{T}$ sufficiently large such that $t_1 + \tau \geqslant t_0 + \sigma$,

$$\sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\Delta s \leqslant \frac{\|P\| - p_1}{p_2},$$

where P be a nonzero constant vector and $p_1 < p_2$ are positive constants such that, $p_1 < \|P\| \leqslant \dfrac{p_1 + p_2}{2}$.

Let $X(t)$ be the set of all $rd-$continuous and bounded vector functions on $t_0 \leqslant t \in \mathbb{T}$. Set $A = \{x \in X : p_1 \leqslant \|x(t)\| \leqslant p_2, t_0 \leqslant t\}$. Define a mapping $F : A \to X$ as follows

$$(Fx)(t) = \begin{cases} P + \sum\limits_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\sigma)\Delta s, & t \geqslant t_1; \\ (Fx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly Fx is $rd-$continuous. For every $x \in A$ and $t \geqslant t_1$,we have

$$\begin{aligned}
\|(Fx)(t)\| &= \|P + \sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\sigma)\Delta s\| \\
&\leqslant \|P\| + \|\sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\sigma)\Delta s\| \\
&\leqslant \|P\| + \sum_{i=0}^{\infty} \|\int_{t_1+(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\sigma)\Delta s\| \\
&\leqslant \|P\| + \sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\|x(s-\sigma)\|\Delta s \\
&\leqslant \|P\| + p_2 \sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\Delta s \\
&\leqslant \|P\| + p_2 \frac{\|P\| - p_1}{p_2} \\
&= 2\|P\| - p_1 \\
&\leqslant p_2.
\end{aligned}$$

Further,

$$\begin{aligned}
\|(Fx)(t)\| &= \|P + \sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\sigma)\Delta s\| \\
&\geqslant \|P\| - \|\sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\sigma)\Delta s\| \\
&\geqslant \|P\| - \sum_{i=0}^{\infty} \|\int_{t_1+(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\sigma)\Delta s\| \\
&\geqslant \|P\| - \sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\|x(s-\sigma)\|\Delta s \\
&\geqslant \|P\| - p_2 \sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\Delta s \\
&\geqslant \|P\| - p_2 \frac{\|P\| - p_1}{p_2} \\
&= p_1.
\end{aligned}$$

Thus we proved that $FA \subset A$.

Since A is a bounded, closed and convex subset of X, we prove that T is a contraction mapping on A. Now, for all $x_1, x_2 \in A$, and $t \geqslant t_1$, we have

$$\begin{aligned}
\|(Fx_1)(t) - (Fx_2)(t)\| &= \|\sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s\| \\
&\leqslant \sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} \|Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\|\Delta s \\
&\leqslant \sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\|[x_1(s-\sigma) - x_2(s-\sigma)]\|\Delta s \\
&\leqslant \|x_1 - x_2\| \sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\Delta s \\
&= q_3\|x_1 - x_2\|.
\end{aligned}$$

This immediately implies that

$$\|(Fx_1)(t) - (Fx_2)(t)\| \leqslant q_3\|x_1 - x_2\|.$$

where

$$q_3 \leqslant \sum_{i=0}^{\infty} \int_{t_1+(2i-1)\tau}^{t+2i\tau} \|Q(s)\|\Delta s \leqslant \frac{\|P\| - p_1}{p_2} < 1.$$

which proves that F is a contraction mapping. Consequently, F has the fixed point x with $Fx = x$ with $|x\| > 0$, for all $t \geq t_1$, which is a nonoscillatory solution of Eq. (1) which completes the proof of Case 3.

Case 4. $-1 < p < 0$.

Choose a $t_1 \in \mathbb{T}$ sufficiently large such that $t_1 \geqslant t_0 + \max\{\tau, \sigma\}$, and

$$\int_{t_1}^{\infty} \|Q(s)\|\Delta s \leqslant \frac{1 + p(1 + L_2) - L_1}{L_2},$$

hold, where $0 < L_1 < 1, L_1, L_2$ are positive constants such that

$$2(1+p) < L_1 + L_2 < 2, \quad \frac{L_1 - 1}{1 + L_2} < p \leqslant \frac{L_1 + L_2}{2} - 1.$$

Let $X(t)$ be the set of all $rd-$continuous and bounded vector functions on $t_0 \leqslant t \in \mathbb{T}$. Set $A = \{x \in X : L_1 \leqslant \|x(t)\| \leqslant L_2, t_0 \leqslant t\}$. Define a mapping $F : A \to X$ as follows

$$(Fx)(t) = \begin{cases} (1+p)e - px(t-\tau) + \int_t^\infty Q(s)x(s-\sigma)\Delta s, & t \geqslant t_1; \\ (Fx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly Fx is $rd-$continuous. For every $x \in A$ and $t \geqslant t_1$, we have

$$\begin{aligned} \|(Fx)(t)\| &= \|(1+p)e - px(t-\tau) + \int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\ &\leqslant \|(1+p)e\| + \|px(t-\tau)\| + \|\int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\ &\leqslant (1+p) - p\|x(t-\tau)\| + \int_t^\infty \|Q(s)x(s-\sigma)\|\Delta s \\ &\leqslant 1 + p - pL_2 + L_2 \int_t^\infty \|Q(s)\|\Delta s \\ &\leqslant 1 + p - pL_2 + L_2 \frac{1 + p(1+L_2) - L_1}{L_2} \\ &= 2(1+p) - L_1 \\ &\leqslant L_2. \end{aligned}$$

Further

$$\begin{aligned} \|(Fx)(t)\| &= \|(1+p)e - [px(t-\tau) - \int_t^\infty Q(s)x(s-\sigma)\Delta s]\| \\ &\geqslant \|(1+p)e\| - \|px(t-\tau) - \int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\ &\geqslant (1+p)\|e\| - \|-px(t-\tau)\| - \|\int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\ &\geqslant (1+p) + p\|x(t-\tau)\| - \int_t^\infty \|Q(s)x(s-\sigma)\|\Delta s \\ &\geqslant 1 + p + pL_2 - \int_t^\infty \|Q(s)\|\|x(s-\sigma)\|\Delta s \\ &\geqslant 1 + p + pL_2 - L_2 \int_t^\infty \|Q(s)\|\Delta s \\ &\geqslant 1 + p + pL_2 - L_2 \frac{1 + p(1+L_2) - L_1}{L_2} \\ &= L_1. \end{aligned}$$

Thus we proved that $FA \subset A$.

Since A is a bounded, closed and convex subset of X, we prove that F is a contraction mapping on A.

Now, for all $x_1, x_2 \in A$, and $t \geqslant t_1$, we have

$$\begin{aligned}
&\|(Fx_1)(t) - (Fx_2)(t)\| \\
=&\| - p[x_1(t-\tau) - x_2(t-\tau)] + \int_t^\infty Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s\| \\
\leqslant&\| - p[x_1(t-\tau) - x_2(t-\tau)]\| + \| \int_t^\infty Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s\| \\
\leqslant&\| - p[x_1(t-\tau) - x_2(t-\tau)]\| + \int_t^\infty \|Q(s)\|\|[x_1(s-\sigma) - x_2(s-\sigma)]\|\Delta s \\
\leqslant& - p\|x_1 - x_2\| + \|x_1 - x_2\| \int_t^\infty \|Q(s)\|\Delta s \\
\leqslant& q_4\|x_1 - x_2\| \quad (q_4 = p + \int_t^\infty \|Q(s)\|\Delta s).
\end{aligned}$$

This immediately implies that

$$\|(Fx_1)(t) - (Fx_2)(t)\| \leqslant q_4\|x_1 - x_2\|.$$

where

$$q_4 = -p + \int_t^\infty \|Q(s)\|\Delta s \leqslant -p + \frac{1 + p(1+L_2) - L_1}{L_2} = \frac{1 + p - L_1}{L_2} < 1.$$

which proves that F is a contraction mapping. Consequently, F has the fixed point x with $Fx = x$, $\|x\| > 0$, for all $t \geq t_1$, which is a nonoscillatory solution of Eq.(1) which completes the proof of Case 4.

Case 5. $-\infty < p < -1$.

Choose a $t_1 \in \mathbb{T}$ sufficiently large such that $t_1 + \tau \geqslant t_0 + \sigma$, and

$$\int_t^\infty \|Q(s)\|\Delta s \leqslant \frac{pK_1 - 1 - p - K_2}{K_2}.$$

where $0 < K_1 < K_2 < 1$, $K_1 + K_2 > 1$, and $\dfrac{2}{K_1 + K_2 - 2} < p < \dfrac{1 + K_2}{K_1 - 1}$.

Let $X(t)$ be the set of all $rd-$continuous and bounded vector functions on $t_0 \leqslant t \in \mathbb{T}$. Set $A = \{x \in X : K_1 \leqslant \|x(t)\| \leqslant K_2, t_0 \leqslant t \in \mathbb{T}\}$. Define a mapping $F : A \to X$ as follows

$$(Fx)(t) = \begin{cases} (1 + \frac{1}{p})e - \frac{1}{p}x(t+\tau) + \frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s, & t \geqslant t_1; \\ (Fx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly Fx is $rd-$continuous. For every $x \in A$ and $t \geqslant t_1 \in \mathbb{T}$, we have

$$\begin{aligned}
\|(Fx)(t)\| &= \|(1+\frac{1}{p})e - \frac{1}{p}x(t+\tau) + \frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\
&\leqslant \|(1+\frac{1}{p})e\| + \|\frac{1}{p}x(t+\tau)\| + \|\frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\
&\leqslant (1+\frac{1}{p})\|e\| - \frac{1}{p}\|x(t+\tau)\| - \frac{1}{p}\|\int_t^\infty Q(s)x(s-\sigma)\|\Delta s \\
&\leqslant 1+\frac{1}{p} - \frac{K_2}{p} - \frac{K_2}{p}\int_t^\infty \|Q(s)\|\Delta s \\
&\leqslant 1+\frac{1}{p} - \frac{K_2}{p} - \frac{K_2}{p}\frac{pK_1 - 1 - p - K_2}{K_2} \\
&= 2(1+\frac{1}{p}) - K_1 \\
&\leqslant K_2.
\end{aligned}$$

Further,

$$\begin{aligned}
\|(Fx)(t)\| &= \|(1+\frac{1}{p})e - [\frac{1}{p}x(t+\tau) - \frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s]\| \\
&\geqslant \|(1+\frac{1}{p})\| - \|\frac{1}{p}x(t+\tau) - \frac{1}{p}\int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\
&\geqslant 1+\frac{1}{p} + \frac{1}{p}\|x(t+\tau)\| + \frac{1}{p}\int_t^\infty \|Q(s)x(s-\sigma)\|\Delta s \\
&\geqslant 1+\frac{1}{p} + \frac{K_2}{p} + \frac{1}{p}\int_t^\infty \|Q(s)\|\|x(s-\sigma)\|\Delta s \\
&\geqslant 1+\frac{1}{p} + \frac{K_2}{p} + \frac{K_2}{p}\int_t^\infty \|Q(s)\|\Delta s \\
&\geqslant 1+\frac{1}{p} + \frac{K_2}{p} + \frac{K_2}{p}\frac{pK_1 - 1 - p - K_2}{K_2} \\
&= K_1.
\end{aligned}$$

Thus we proved that $FA \subset A$.

Since A is a bounded, closed and convex subset of X, we prove that F is a contraction mapping on A. Now, for all $x_1, x_2 \in A$, and $t \geqslant t_1$, we have

$$\begin{aligned}
&\|(Fx_1)(t) - (Fx_2)(t)\| \\
=&\|\frac{1}{p}[x_1(t+\tau) - x_2(t+\tau)] - \frac{1}{p}\int_t^\infty Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s\| \\
\leqslant&\|\frac{1}{p}[x_1(t+\tau) - x_2(t+\tau)]\| - \frac{1}{p}\|\int_t^\infty Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s\| \\
\leqslant& -\frac{1}{p}\|x_1(t+\tau) - x_2(t+\tau)\| - \frac{1}{p}\int_t^\infty \|Q(s)\|\|x_1(s-\sigma) - x_2(s-\sigma)\|\Delta s \\
\leqslant& -\frac{1}{p}\|x_1 - x_2\| - \frac{1}{p}\|x_1 - x_2\|\int_t^\infty \|Q(s)\|\Delta s \\
\leqslant& q_5\|x_1 - x_2\| \quad (q_5 = -\frac{1}{p} + \int_t^\infty \|Q(s)\|\Delta s).
\end{aligned}$$

This immediately implies that

$$q_5 = -\frac{1}{p} + \frac{1}{p}\int_t^{\infty} \|Q(s)\|\Delta s \leqslant \frac{1+p-pK_1}{pK_2} < 1.$$

Therefore,

$$\|(Fx_1)(t) - (Fx_2)(t)\| \leqslant q_5\|x_1 - x_2\|,$$

which proves that F is a contraction mapping. Consequently, F has the fixed point x with $Fx = x, \|x\| > 0$, for all $t \geq t_1$, which is a nonoscillatory solution of Eq. (1), which completes the proof of Case 5. □

Now we consider B is a nonsingular constant matrix. Assume that $\|B\| = b$

Theorem 2.2. Suppose that

$$\int_{t_1}^{\infty} \|Q(s)\|\Delta s < \infty,$$

Then Eq.(2) has a nonoscillatory solution.

Proof. **Case 1**. $b \in [0, 1)$

Choose a $t_1 \in \mathbb{T}$ sufficiently large such that $t_1 \geqslant t_0 + \overline{\sigma}$, $\overline{\sigma} = \max \tau, \sigma$, and $\int_{t_1}^{\infty} \|Q(s)\|\Delta s \leqslant \dfrac{1-b(1+M_2)-M_1}{M_2}$ hold. where $0 < M_1 < 1, 0 < M_2$, $1-b < M_1 + M_2 < 2$, $1 - \dfrac{M_1+M_2}{2} \leqslant b < \dfrac{1-M_1}{1+M_2}$.

Let $X(t)$be the set of all $rd-$continuous and bounded vector functions on $t_0 \leqslant t \in \mathbb{T}$. Set $A = \{x \in X : M_1 \leqslant \|x(t)\| \leqslant M_2, t_0 \leqslant t \in \mathbb{T}\}$. Define a mapping $F : A \to X$ as follows

$$(Fx)(t) = \begin{cases} q - Bx(t-\tau) + \int_t^{\infty} Q(s)x(s-\sigma)\Delta s, & t \geqslant t_1; \\ (Fx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

where q is a vector such that $\|q\| = 1 - b$.

Clearly Fx is C_{rd} continuous. For every $x \in A$ and $t \geqslant t_1$, we get

$$\begin{aligned}
\|(Fx)(t)\| &= \|q - Bx(t-\tau) + \int_t^{\infty} Q(s)x(s-\sigma)\Delta s\| \\
&\leqslant \|q\| + \|Bx(t-\tau)\| + \|\int_t^{\infty} Q(s)x(s-\sigma)\Delta s\| \\
&\leqslant (1-b) + b\|x(t-\tau)\| + \int_t^{\infty} \|Q(s)x(s-\sigma)\|\Delta s \\
&\leqslant 1 - b + bM_2 + M_2\int_t^{\infty} \|Q(s)\|\Delta s \\
&\leqslant 1 - b + bM_2 + M_2\frac{1-b(1+M_2)-M_1}{M_2} \\
&= 2(1-b) - M_1 \\
&\leqslant M_2.
\end{aligned}$$

Further,

$$\begin{aligned}
\|(Fx)(t)\| &= \|q - [Bx(t-\tau) - \int_t^\infty Q(s)x(s-\sigma)\Delta s]\| \\
&\geqslant \|q\| - \|Bx(t-\tau) - \int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\
&\geqslant 1 - b - b\|x(t-\tau)\| - \int_t^\infty \|Q(s)x(s-\sigma)\|\Delta s \\
&\geqslant 1 - b - bM_2 - M_2 \int_t^\infty \|Q(s)\|\Delta s \\
&\geqslant 1 - b - bM_2 - M_2 \frac{1-b(1+M_2)-M_1}{M_2} \\
&= M_1.
\end{aligned}$$

Thus we proved that $FA \subset A$.

Since A is a bounded, closed and convex subset of X we have to prove that F is a contraction mapping on A in order to apply the contraction principle. Now, for all $x_1, x_2 \in A, t \geqslant t_1$, we have

$$\begin{aligned}
&\|(Fx_1)(t) - (Fx_2)(t)\| \\
=&\| - B[x_1(t-\tau) - x_2(t-\tau)] + \int_t^\infty Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s\| \\
\leqslant&\| - B[x_1(t-\tau) - x_2(t-\tau)]\| + \|\int_t^\infty Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s\| \\
\leqslant& b\|x_1 - x_2\| + \|x_1 - x_2\| \int_t^\infty \|Q(s)\|\Delta s \\
\leqslant& r_1\|x_1 - x_2\| \quad (r_1 = b + \int_t^\infty \|Q(s)\|\Delta s).
\end{aligned}$$

Clearly,

$$r_1 = b + \int_t^\infty \|Q(s)\|\Delta s \leqslant b + \frac{1-b(1+M_2)-M_1}{M_2} = \frac{1-b-M_1}{M_2} < 1.$$

This immediately implies that

$$\|(Fx_1)(t) - (Fx_2)(t)\| \leqslant r_1\|x_1 - x_2\|,$$

which proves that F is a contraction mapping. Consequently, F has the fixed point x with $Fx = x, \|x\| > 0$, for all $t > t_1$, which is a nonoscillatory solution of Eq. (2) which completes the proof of Case 1.

Case 2. $1 < b < \infty$.

Choose a $t_1 \in \mathbb{T}$ sufficiently large such that $t_1 \geqslant t_0 + \overline{\sigma}$, and

$$\int_{t_1+\tau}^\infty \|Q(s)\|\Delta s \leqslant \frac{b-1-pN_1-N_2}{N_2}$$

hold, where $0 < N_1 < 1,\ N_2 > N_1$ and $N_1 + N_2 < 2,\ \dfrac{1+N_2}{1-N_1} < b \leqslant \dfrac{2}{2-N_1-N_2}$.

Let $X(t)$ be the set as in Case 1. Set $A = \{x \in X : N_1 \leqslant \|x(t)\| \leqslant N_2,\ t_0 \leqslant t \in \mathbb{T}\}$. Define a mapping $F : A \to X$ as follows

$$(Fx)(t) = \begin{cases} r - B^{-1}x(t-\tau) + B^{-1}\int_t^\infty Q(s)x(s-\sigma)\Delta s, & t \geqslant t_1; \\ (Fx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

where r is a vector such that $\|r\| = 1 - \dfrac{1}{b}$.

Clearly Fx is rd continuous. For every $x \in A$ and $t \geqslant t_1$, we get

$$\begin{aligned} \|(Fx)(t)\| &= \|r - B^{-1}x(t-\tau) + B^{-1}\int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\ &\leqslant \|r\| + \|B^{-1}x(t-\tau)\| + \|B^{-1}\int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\ &\leqslant (1-\frac{1}{b}) + \frac{1}{b}\|x(t-\tau)\| + \frac{1}{b}\|\int_t^\infty Q(s)x(s-\sigma)\|\Delta s \\ &\leqslant 1 - \frac{1}{b} + \frac{N_2}{b} + \frac{N_2}{b}\int_t^\infty \|Q(s)\|\Delta s \\ &\leqslant 1 - \frac{1}{b} + \frac{N_2}{b} + \frac{N_2}{b}\frac{b-1-bN_1-N_2}{N_2} \\ &= 2(1-\frac{1}{b}) - N_1 \\ &\leqslant N_2. \end{aligned}$$

Further

$$\begin{aligned} \|(Fx)(t)\| &= \|r - [B^{-1}x(t-\tau) - B^{-1}\int_t^\infty Q(s)x(s-\sigma)\Delta s]\| \\ &\geqslant \|(1-\frac{1}{b})e\| - \|B^{-1}x(t-\tau) - B^{-1}\int_t^\infty Q(s)x(s-\sigma)\Delta s\| \\ &\geqslant 1 - \frac{1}{b} - \frac{1}{b}\|x(t-\tau)\| - \frac{1}{b}\int_t^\infty \|Q(s)x(s-\sigma)\|\Delta s \\ &\geqslant 1 - \frac{1}{b} - \frac{N_2}{b} - \frac{1}{b}\int_t^\infty \|Q(s)\|\|x(s-\sigma)\|\Delta s \\ &\geqslant 1 - \frac{1}{b} - \frac{N_2}{b} - \frac{N_2}{b}\int_t^\infty \|Q(s)\|\Delta s \\ &\geqslant 1 - \frac{1}{b} - \frac{N_2}{b} - \frac{N_2}{b}\frac{b-1-bN_1-N_2}{N_2} \\ &= N_2. \end{aligned}$$

Thus we proved that $FA \subset A$.

Since A is a bounded, closed and convex subset of X. we have to prove that F is a contraction mapping on A in order to apply the contraction principle. Now, for all $x_1, x_2 \in A, t \geqslant t_1$, we have

$$
\begin{aligned}
&\|(Fx_1)(t) - (Fx_2)(t)\| \\
=&\| - B^{-1}[x_1(t-\tau) - x_2(t-\tau)] + B^{-1}\int_t^{\infty} Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s\| \\
\leqslant&\| - B^{-1}[x_1(t-\tau) - x_2(t-\tau)]\| + B^{-1}\|\int_t^{\infty} Q(s)[x_1(s-\sigma) - x_2(s-\sigma)]\Delta s\| \\
\leqslant&\frac{1}{b}\|x_1(t-\tau) - x_2(t-\tau)\| + \frac{1}{b}\int_t^{\infty}\|Q(s)\|\|x_1(s-\sigma) - x_2(s-\sigma)\|\Delta s \\
\leqslant&\frac{1}{b}\|x_1 - x_2\| + \frac{1}{b}\|x_1 - x_2\|\int_t^{\infty}\|Q(s)\|\Delta s \\
\leqslant&r_2\|x_1 - x_2\| \quad (r_2 = \frac{1}{b} + \frac{1}{b}\int_t^{\infty}\|Q(s)\|\Delta s).
\end{aligned}
$$

Clearly,

$$r_2 = \frac{1}{b} + \frac{1}{b}\int_t^{\infty}\|Q(s)\|\Delta s \leqslant \frac{1}{b} + \frac{b-1-bN_1-N_2}{N_2} = \frac{b-1-bN_1}{pN_2} < 1.$$

This immediately implies that

$$\|(Fx_1)(t) - (Fx_2)(t)\| \leqslant r_2\|x_1 - x_2\|,$$

which proves that F is a contraction mapping. Consequently, F has the fixed point x with $Fx = x, \|x\| > 0$, for all $t > t_1$, which is a nonoscillatory solution of Eq.(2) which completes the proof of Case 2. □

REFERENCES

[1] Hilger, S. (1990). Analysis on measure chains a unified approach to continuous and discrete calculus. *Results in Matematics, 18*, 18-56.

[2] Hilger, S. (1997). Differential and difference calculus unified. *Nonlinear Analysis, 30*(5), 2683-2694.

[3] Bohner, M., & Castillo, J.E. (2001). Mimetic methods on measure chains. *Comput. Math. Appl., 42*, 705-710.

[4] Agarwal, R.P., & Bohner, M. (1999). Basic calculus on time scales and some of its applications. *Results Math., 35*, 3-22.

[5] Dosly, O., & Hilger, S. (2002). A necessary and sufficient condition for oscillation of the Sturm-Liouville dynamic equations on time scales. *Comput. Appl. Math., 141*, 147-158.

[6] Saker (2004). Oscillation of nonlinear differential equations on time scales. *Appl. Math. Comput., 148*, 81-91.

[7] Medico, A.D., & Kong, Q.K. (2004). Kamenev-type and interval oscillation criteria for second-order linear differential equations on a measure chain. *Math. Anal. Appl., 294*, 621-643.

[8] Tanigawa, T. (2003). Oscillation and nonoscillation theorems for a class of fourth order quasilinear functional differential equations. *Hiroshima Math., 33*, 297-316.

[9] Bohner, M., & Peterson, A. (2001). *Dynamic equations on time scales: an introduction with applications*. Boston: Birkhanser.