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Non-Central Beta Type 3 Distribution

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Abstract: Let X and Y be independent random variables, X having a gamma distribution with shape parameter a and Y having a non-central gamma distribution with shape and non-centrality parameters b and δ, respectively. Define $W = X/(X + 2Y)$. Then, the random variable W has a non-central beta type 3 distribution, $W \sim \mathrm{NCB3}(a, b; \delta)$. In this article we study several of its properties. We also give a multivariate generalization of the non-central beta type 3 distribution and derive its properties.

Key words: Beta distribution; Quotient; Gauss hypergeometric function; Multivariate distribution; Transformation

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1. INTRODUCTION

The beta type 1 distribution with parameters (a, b) is defined by the probability density function (p.d.f.)

$$\mathrm{B1}(u; a, b) = \frac{u^{a-1}(1-u)^{b-1}}{B(a, b)}, \quad 0 < u < 1, \tag{1}$$

where $a > 0$, $b > 0$, and $B(a, b)$ is the beta function defined by

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \mathrm{Re}(a) > 0, \quad \mathrm{Re}(b) > 0.$$

The beta type 1 distribution is well known in Bayesian methodology as a prior distribution on the success probability of a binomial distribution. The random variable V with the p.d.f.

$$\mathrm{B2}(v; a, b) = \frac{v^{a-1}(1+v)^{-(a+b)}}{B(a,b)}, \quad v > 0, \tag{2}$$

where $a > 0$ and $b > 0$ is said to have a beta type 2 distribution with parameters (a, b). Since (2) can be obtained from (1) by the transformation $V = U/(1-U)$ some authors call the distribution of V an *inverted beta distribution*. The beta type 1 and beta type 2 are very flexible distributions for positive random variables and have wide applications in statistical analysis, e.g., see Johnson, Kotz and Balakrishnan [8]. For an in-depth view the reader is referred to an edited volume by Gupta and Nadarajah [4] which contains a collection of essays by various authors covering many different aspects. Systematic treatment of matrix variate generalizations of the beta type 1 and the beta type 2 distributions is given in Gupta and Nagar [5]. By using the transformation $W = U/(2-U)$, the beta type 3 density is obtained as (Gupta and Nagar [6,7], Cardeño, Nagar and Sánchez [1]),

$$\mathrm{B3}(w; a, b) = \frac{2^a w^{a-1}(1-w)^{b-1}}{B(a,b)(1+w)^{a+b}}, \quad 0 < w < 1. \tag{3}$$

It is well known that if X and Y are independent random variables having a standard gamma distribution with shape parameters a and b, respectively, then $X/(X+Y) \sim \mathrm{B1}(a,b)$, $X/Y \sim \mathrm{B2}(a,b)$ and $X/(X+2Y) \sim \mathrm{B3}(a,b)$.

The random variable U is said to have a non-central beta type 1 distribution if its p.d.f. is given by

$$\mathrm{NCB1}(u; a, b; \delta) = \frac{\exp(-\delta)\, u^{a-1}(1-u)^{b-1}}{B(a,b)} {}_1F_1\left(a+b; b; \delta(1-u)\right), \tag{4}$$

where $0 < u < 1$ and the confluent hypergeometric function ${}_1F_1$ has the integral representation (Luke [9, Eq. 4.2(1)]),

$${}_1F_1(a; c; z) = \frac{1}{B(a, c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \exp(zt)\, \mathrm{d}t, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0. \tag{5}$$

Expanding $\exp(zt)$ in (5) and integrating t, the series expansion for ${}_1F_1$ is obtained as

$${}_1F_1(a; c; z) = \sum_{j=0}^{\infty} \frac{\Gamma(c)\Gamma(a+j)}{\Gamma(a)\Gamma(c+j)} \frac{z^j}{j!}. \tag{6}$$

The non-central beta type 1 distribution is used in computing power of several test statistics. Recently, Miranda De Sá [10] has shown that the sampling distribution of coherence estimate between one random and one periodic signal is type 1 non-central beta (also see Nadarajah and Kotz [12]). This distribution also appears in statistical discrimination and sequential testing of nested linear hypothesis.

Nadarajah [11] has derived distributions of sum, product, and ratios of non-central beta type 1 variables. By making the transformation $V = U/(1-U)$ in (4) the non-central beta type 2 density is derived as

$$\mathrm{NCB2}(v; a, b; \delta) = \frac{\exp(-\delta)\, v^{a-1}(1+v)^{-(a+b)}}{B(a,b)} {}_1F_1\left(a+b; b; \frac{\delta}{1+v}\right), \quad v > 0. \quad (7)$$

Further, transforming $W = U/(2-U)$ in (4), the non-central beta type 3 density is derived as

$$\mathrm{NCB3}(w; a, b; \delta) = \frac{2^a \exp(-\delta)\, w^{a-1}(1-w)^{b-1}}{B(a,b)(1+w)^{a+b}} {}_1F_1\left(a+b; b; \frac{\delta(1-w)}{1+w}\right), \quad (8)$$

where $0 < w < 1$.

In this article, we study properties of the non-central beta type 3 distribution and its multivariate generalization. In Section 2, several properties of the non-central beta type 3 distribution including mixture representation, cumulative distribution function, moment generating function and moments are derived. Finally, in Section 3, we define a multivariate generalization of the non-central beta type 3 distribution and study its properties.

2. PROPERTIES

In this section we study some properties of the non-central beta type 3 distribution.

From the non-central beta type 3 density it is straightforward to show that

$$\int_0^1 \frac{w^{a-1}(1-w)^{b-1}}{(1+w)^{a+b}} {}_1F_1\left(a+b; b; \frac{\delta(1-w)}{1+w}\right) \mathrm{d}w = 2^{-a}\exp(\delta) B(a,b). \quad (9)$$

The complementary cumulative distribution function of W is obtained as

$$\begin{aligned} P(W > w) &= \frac{2^a \exp(-\delta)}{B(a,b)} \int_w^1 \frac{v^{a-1}(1-v)^{b-1}}{(1+v)^{a+b}} {}_1F_1\left(a+b; b; \frac{\delta(1-v)}{1+v}\right) \mathrm{d}v \\ &= \frac{\exp(-\delta)}{B(a,b)} \int_0^{(1-w)/(1+w)} u^{b-1}(1-u)^{a-1} {}_1F_1(a+b; b; \delta u)\, \mathrm{d}u, \quad (10) \end{aligned}$$

where the second step has been obtained by substituting $v = (1-u)/(1+u)$. Now, using the series expansion of ${}_1F_1$ given in (6) and the definition of incomplete beta function, we obtain

$$P(W > w) = \sum_{j=0}^{\infty} \frac{\exp(-\delta)\delta^j}{j!} I_{(1-w)/(1+w)}(b+j, a),$$

where $I_x(\alpha, \beta)$ is the Pearson's incomplete beta function defined by

$$I_x(\alpha,\beta) = \frac{1}{B(\alpha,\beta)} \int_0^x v^{\alpha-1}(1-v)^{\beta-1}\, \mathrm{d}v.$$

Figure 1 shows the NCB3 density function for selected values of a, b, and δ. It can be seen that for $a < 1$ and $b < 1$, the NCB3 is U-shaped. Also, for $a < 1$

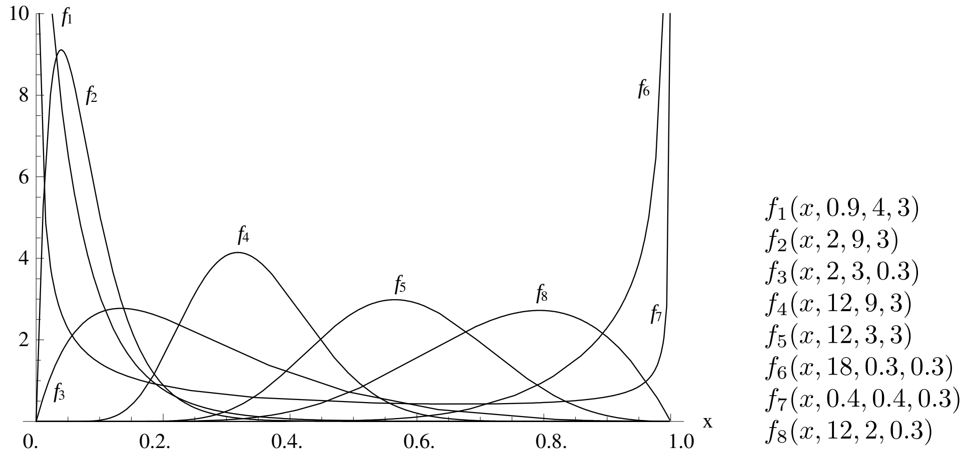

Figure 1
Graphs of the NCB3 Density Function for Selected Values of a, b, and δ.

and $b > 1$, NCB3 is strictly decreasing. The NCB3 density, for $1 < a < 10$, $b < 1$, is positively skew symmetric while for $a \geq 10$, $1 < b < 2$, it is negatively skew symmetric. For $a \geq 10$, $b > 2$, the curve of NCB3 density tends to symmetry.

Using the series expansion

$$ {}_1F_1(a+b;b;z) = \sum_{j=0}^{\infty} \frac{\Gamma(b)\Gamma(a+b+j)}{\Gamma(a+b)\Gamma(b+j)} \frac{z^j}{j!}, $$

where $z = \delta(1-w)/(1+w)$, we see that (8) can be represented as

$$ \text{NCB3}(w;a,b;\delta) = \sum_{j=0}^{\infty} \frac{\exp(-\delta)\delta^j}{j!} \text{B3}(w;a,b+j), \tag{11} $$

where $0 < w < 1$. Thus the non-central beta type 3 distribution is an infinite mixture of beta type 3 distributions. Further, by expanding

$$ \begin{aligned} (1+w)^{-(a+b+j)} &= 2^{-(a+b+j)} \left[1 - \frac{1-w}{2}\right]^{-(a+b+j)} \\ &= 2^{-(a+b+j)} \sum_{k=0}^{\infty} \frac{\Gamma(a+b+j+k)}{\Gamma(a+b+j)} \frac{(1-w)^k}{2^k k!}, \end{aligned} $$

the density $\text{B3}(w;a,b+j)$ can be written as

$$ \text{B3}(w;a,b+j) = 2^{-b-j} \sum_{k=0}^{\infty} \frac{\Gamma(b+j+k)}{\Gamma(b+j)2^k k!} \text{B1}(w;a,b+j+k). \tag{12} $$

If $W \sim \text{B3}(a, b+j)$, then the moment generating function (m.g.f.) of W is given by

$$ 2^{-b-j} \exp(t) \mathbf{\Phi}_1 \left[b+j, a+b+j; a+b+j; \frac{1}{2}, -t\right], \tag{13} $$

where the Humbert's confluent hypergeometric function $\mathbf{\Phi}_1$ is defined by

$$\mathbf{\Phi}_1[a, b_1; c; z_1, z_2] = \frac{1}{B(a, c-a)} \int_0^1 \frac{v^{a-1}(1-v)^{c-a-1} \exp(vz_2)\,\mathrm{d}v}{(1-vz_1)^{b_1}}, \tag{14}$$

with $|z_1| < 1$, $|z_2| < \infty$, $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(c-a) > 0$. Note that for $b_1 = 0$, $\mathbf{\Phi}_1$ reduces to a ${}_1F_1$ function. For properties and further results on these functions the reader is referred to Luke [9], and Srivastava and Karlsson [14]. Now, using (11) and (13), the m.g.f. of $W \sim \mathrm{NCB3}(a, b; \delta)$ is derived as

$$\exp(t) \sum_{j=0}^{\infty} \frac{\exp(-\delta)\delta^j}{2^{b+j}\, j!} \mathbf{\Phi}_1 \left[b+j, a+b+j; a+b+j; \frac{1}{2}, -t \right].$$

The relationship between non-central beta type 1, type 2 and type 3 random variables is exhibited in the following theorem. The proof is straightforward.

Theorem 2.1. *Let* $U \sim \mathrm{NCB1}(a, b; \delta)$, $V \sim \mathrm{NCB2}(a, b; \delta)$ *and* $W \sim \mathrm{NCB3}(a, b; \delta)$. *Then, (i)* $(1+U)^{-1}(1-U) \sim \mathrm{NCB3}(b, a; \delta)$, *(ii)* $2W/(1+W) \sim \mathrm{NCB1}(a, b; \delta)$ *(iii)* $(1+W)^{-1}(1-W) \sim \mathrm{NCB1}(b, a; \delta)$, *(iv)* $V/(2+V) \sim \mathrm{NCB3}(a, b; \delta)$, *(v)* $(1+2V)^{-1} \sim \mathrm{NCB3}(b, a; \delta)$, *(vi)* $2W/(1-W) \sim \mathrm{NCB2}(a, b; \delta)$, *and (vii)* $W^{-1}(1-W)/2 \sim \mathrm{NCB2}(b, a; \delta)$.

The non-central beta densities are obtained by using non-central gamma variables. The random variable Y is said to have a non-central gamma distribution with shape parameter $\kappa\,(> 0)$, and non-centrality parameter $\delta\,(\geq 0)$, denoted by $Y \sim \mathrm{Ga}(\kappa; \delta)$, if its p.d.f. is given by

$$\mathrm{Ga}(y; \kappa; \delta) = \frac{\exp(-\delta - y)\, y^{\kappa-1}}{\Gamma(\kappa)} {}_0F_1(\kappa; \delta y), \tag{15}$$

where $y > 0$ and

$${}_0F_1(a; z) = \sum_{j=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a+j)} \frac{z^j}{j!}.$$

For $\delta = 0$, the non-central gamma distribution reduces to a gamma distribution and we have $\mathrm{Ga}(y; a; 0) \equiv \mathrm{Ga}(y; a)$.

Let $Y_1 \sim \mathrm{Ga}(a)$ and $Y_2 \sim \mathrm{Ga}(b; \delta)$ be independent. Then, it is well known that (Sánchez, Nagar and Gupta [13]),

$$U \stackrel{d}{=} \frac{Y_1}{Y_1 + Y_2} \sim \mathrm{B1}(a, b; \delta), \quad V \stackrel{d}{=} \frac{Y_1}{Y_2} \sim \mathrm{B2}(a, b; \delta), \tag{16}$$

where $X \stackrel{d}{=} Z$ means that X and Z have identical distribution. Next, we state the following result from Fang, Kotz and Ng [2], and Fang and Zhang [3].

Theorem 2.2. *Let* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be n-dimensional random vectors. Further, let* $\mathbf{Y} \stackrel{d}{=} \mathbf{Z}$ *and* $f_j(\cdot), j = 1, \ldots, m$ *be Borel measurable functions. Then,*

$$\begin{pmatrix} f_1(\mathbf{Y}) \\ \vdots \\ f_m(\mathbf{Y}) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} f_1(\mathbf{Z}) \\ \vdots \\ f_m(\mathbf{Z}) \end{pmatrix}.$$

Now, using (16) and the above theorem, it is easy to see that

$$W \overset{d}{=} \frac{U}{2-U} \overset{d}{=} \frac{Y_1}{Y_1+2Y_2}. \tag{17}$$

Further, using the stochastic representations (16) and (17) and Theorem 2.2, all the results of the Theorem 2.1 can be established easily. The representation (17) suggests the obvious extension

$$W_c \overset{d}{=} \frac{Y_1}{Y_1+cY_2} \overset{d}{=} \frac{V}{V+c} \quad (c>0), \tag{18}$$

where $V \sim \mathrm{B2}(a,b;\delta)$. The p.d.f. of W_c is

$$\frac{c^a \exp(-\delta) w^{a-1}(1-w)^{b-1}}{B(a,b)[1+(c-1)w]^{a+b}} {}_1F_1\left(a+b;b;\frac{\delta(1-w)}{1+(c-1)w}\right), \quad 0<w<1. \tag{19}$$

Further, using (19), it is easy to see that for $a>0, b>0$ and $K>0$,

$$\int_0^1 \frac{w^{a-1}(1-w)^{b-1}}{(1+Kw)^{a+b}} {}_1F_1\left(a+b;b;\frac{\delta(1-w)}{1+Kw}\right) \mathrm{d}w = \frac{\exp(\delta)B(a,b)}{(1+K)^a}. \tag{20}$$

For $K=1$, the above integral reduces to (9) and for $\delta=0$ it simplifies to

$$\int_0^1 \frac{w^{a-1}(1-w)^{b-1}}{(1+Kw)^{a+b}} \mathrm{d}w = \frac{B(a,b)}{(1+K)^a}. \tag{21}$$

Next, we give the definition of the Gauss hypergeometric function ${}_2F_1$ which we need to derive moments. The integral representation of the Gauss hypergeometric function is given as (Luke [9, Eq. 3.6(1)]),

$${}_2F_1(a,b;c;z) = \frac{1}{B(a,c-a)} \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-zt)^b} \mathrm{d}t, \tag{22}$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, $|\arg(1-z)| < \pi$. Expanding $(1-zt)^{-b}$, $|zt|<1$, in (22) and integrating t, the series expansion for ${}_2F_1$ is derived as

$${}_2F_1(a,b;c;z) = \sum_{j=0}^{\infty} \frac{\Gamma(c)\Gamma(a+j)\Gamma(b+j)}{\Gamma(a)\Gamma(b)\Gamma(c+j)} \frac{z^j}{j!}. \tag{23}$$

From (22), it easily follows that

$$\int_0^1 \frac{w^{a-1}(1-w)^{b-1}}{(1+Kw)^c} \mathrm{d}w = \frac{B(a,b)}{(1+K)^c} {}_2F_1\left(b,c;a+b;\frac{K}{1+K}\right).$$

Also, by expanding ${}_1F_1$ and using the above integral, we obtain

$$\int_0^1 \frac{w^{a-1}(1-w)^{b-1}}{(1+Kw)^c} {}_1F_1\left(c;d;\frac{\delta(1-w)}{1+Kw}\right) \mathrm{d}w$$
$$= \frac{\Gamma(a)\Gamma(d)}{\Gamma(c)(1+K)^c} \sum_{r=0}^{\infty} \frac{\Gamma(b+r)\Gamma(c+r)}{\Gamma(a+b+r)\Gamma(d+r)} \frac{\delta^r}{(1+K)^r r!}$$

$$\times {}_2F_1\left(b+r, c+r; a+b+r; \frac{K}{1+K}\right).$$

For $c = a + b$, the above expression reduces to

$$\int_0^1 \frac{w^{a-1}(1-w)^{b-1}}{(1+Kw)^{a+b}} {}_1F_1\left(a+b; d; \frac{\delta(1-w)}{1+Kw}\right) \mathrm{d}w = \frac{B(a,b)}{(1+K)^a} {}_1F_1\left(b; d; \delta\right). \qquad (24)$$

For $d = b$, we obtain

$$\begin{aligned} &\int_0^1 \frac{w^{a-1}(1-w)^{b-1}}{(1+Kw)^c} {}_1F_1\left(c; b; \frac{\delta(1-w)}{1+Kw}\right) \mathrm{d}w \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)(1+K)^c} \sum_{r=0}^{\infty} \frac{\Gamma(c+r)}{\Gamma(a+b+r)} \frac{\delta^r}{(1+K)^r r!} \\ &\quad \times {}_2F_1\left(b+r, c+r; a+b+r; \frac{K}{1+K}\right). \end{aligned}$$

Theorem 2.3. *Let* $W \sim \mathrm{NCB3}(a, b; \delta)$*, then*

$$\begin{aligned} \mathrm{E}\left[\frac{W^r(1-W)^s}{(1+W)^t}\right] &= \sum_{j=0}^{\infty} \frac{\exp(-\delta)\delta^j}{j!} \frac{\Gamma(a+b+j)\Gamma(a+r)\Gamma(b+j+s)}{2^{b+j+t}\Gamma(a)\Gamma(b+j)\Gamma(a+b+j+r+s)} \\ &\quad \times {}_2F_1\left(b+j+s, a+b+j+t; a+b+j+r+s; \frac{1}{2}\right), \end{aligned} \qquad (25)$$

where $\mathrm{Re}\,(r+a) > 0$*,* $\mathrm{Re}\,(s+b) > 0$ *and* ${}_2F_1$ *is the Gauss hypergeometric function.*

Proof. Using (11), we have

$$\mathrm{E}\left[\frac{W^r(1-W)^s}{(1+W)^t}\right] = \sum_{j=0}^{\infty} \frac{\exp(-\delta)\delta^j}{j!} \mathrm{E}_C\left[\frac{W^r(1-W)^s}{(1+W)^t}\right],$$

where

$$\begin{aligned} \mathrm{E}_C\left[\frac{W^r(1-W)^s}{(1+W)^t}\right] &= \int_0^1 \frac{w^r(1-w)^s}{(1+w)^t} \mathrm{B3}(w; a, b+j)\,\mathrm{d}w \\ &= \frac{2^a}{B(a,b+j)} \int_0^1 \frac{w^{a+r-1}(1-w)^{b+j+s-1}}{(1+w)^{a+b+j+t}} \mathrm{d}w. \end{aligned}$$

Writing

$$(1+w)^{-(a+b+j+t)} = 2^{-(a+b+j+t)}\left[1 - \frac{1-w}{2}\right]^{-(a+b+j+t)}$$

and substituting $z = 1 - w$, we obtain

$$\begin{aligned} \mathrm{E}_C\left[\frac{W^r(1-W)^s}{(1+W)^t}\right] &= \frac{1}{2^{b+j+t}B(a,b+j)} \int_0^1 \frac{(1-z)^{a+r-1}z^{b+j+s-1}}{(1-z/2)^{a+b+j+t}} \mathrm{d}z \\ &= \frac{B(a+r, b+j+s)}{2^{b+j+t}B(a,b+j)} \end{aligned}$$

$$\times\, {}_2F_1\left(b+j+s, a+b+j+t; a+b+j+r+s; \frac{1}{2}\right),$$

where the last step has been obtained by using (22).

Finally, substituting appropriately, we get the desired result. □

Substituting $r = t = h$ and $s = 0$ in (25) and using the result ${}_2F_1(b, a+b+h; a+b+h; 1/2) = 2^{-b}$, we get

$$\begin{aligned} \mathrm{E}\left[\frac{W^h}{(1+W)^h}\right] &= \sum_{j=0}^{\infty} \frac{\exp(-\delta)\delta^j}{j!} \frac{\Gamma(a+b+j)\Gamma(a+h)}{2^h\Gamma(a)\Gamma(a+b+j+h)} \\ &= \frac{\Gamma(a+b)\Gamma(a+h)}{2^h\Gamma(a)\Gamma(a+b+h)} {}_1F_1(a+b; a+b+h; \delta), \end{aligned}$$

where $\mathrm{Re}\,(h+a) > 0$.

The above expression can also be obtained by observing that

$$\frac{W}{1+W} \stackrel{d}{=} \frac{Y_1}{2(Y_1+Y_2)} \stackrel{d}{=} \frac{U}{2}, \tag{26}$$

where $U \sim \mathrm{NCB1}(a, b; \delta)$.

3. NON-CENTRAL DIRICHLET TYPE 3 DISTRIBUTION

The multivariate generalizations of the non-central beta type 1 and type 2 densities are defined by

$$\frac{\exp(-\delta)\prod_{i=1}^n u_i^{a_i-1}(1-\sum_{i=1}^n u_i)^{b-1}}{B(a_1,\ldots,a_n,b)} {}_1F_1\left(\sum_{i=1}^n a_i + b; b; \delta\left(1-\sum_{i=1}^n u_i\right)\right), \tag{27}$$

where $u_i > 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n u_i < 1$, and

$$\frac{\exp(-\delta)\prod_{i=1}^n v_i^{a_i-1}(1+\sum_{i=1}^n v_i)^{-(\sum_{i=1}^n a_i+b)}}{B(a_1,\ldots,a_n,b)} {}_1F_1\left(\sum_{i=1}^n a_i+b; b; \frac{\delta}{1+\sum_{i=1}^n v_i}\right), \tag{28}$$

respectively, where $v_i > 0$, $i = 1, \ldots, n$, $a_i > 0$, $i = 1, \ldots, n$, $b > 0$ and

$$B(a_1,\ldots,a_n,b) = \frac{\Gamma(b)\prod_{i=1}^n \Gamma(a_i)}{\Gamma(\sum_{i=1}^n a_i + b)}. \tag{29}$$

These distributions, defined and derived by Troskie [15], are well known in the scientific literature as the non-central Dirichlet type 1 and type 2 distributions. We will write $(U_1, \ldots, U_n) \sim \mathrm{NCD1}(a_1, \ldots, a_n; b; \delta)$ if the joint density of $U_1, \ldots, U_n$ is given by (27) and if positive random variables $V_1, \ldots, V_n$ follow the density given by (28), then $(V_1, \ldots, V_n) \sim \mathrm{NCD2}(a_1, \ldots, a_n; b; \delta)$.

A natural multivariate generalization of the non-central beta type 3 distribution can be given as follows.

Definition 3.1. *The positive random variables $W_1,\dots,W_n$ are said to have a non-central Dirichlet type 3 distribution, denoted by $(W_1,\dots,W_n)\sim \mathrm{NCD3}(a_1,\dots,a_n;b;\delta)$, if their joint p.d.f. is given by*

$$C(a_1,\dots,a_n,b)\frac{\prod_{i=1}^n w_i^{a_i-1}(1-\sum_{i=1}^n w_i)^{b-1}}{(1+\sum_{i=1}^n w_i)^{\sum_{i=1}^n a_i+b}}\,{}_1F_1\left(\sum_{i=1}^n a_i+b;b;\frac{\delta(1-\sum_{i=1}^n w_i)}{1+\sum_{i=1}^n w_i}\right), \tag{30}$$

where $w_i>0$, $i=1,\dots,n$, $\sum_{i=1}^n w_i<1$ and $C(a_1,\dots,a_n,b)$ is the normalizing constant.

The normalizing constant in (30) is given by

$$\begin{aligned}\{C(a_1,\dots,a_n,b)\}^{-1} &= \underset{\substack{\sum_{i=1}^n w_i<1\\ w_i>0, i=1,\dots,n}}{\int\cdots\int} \frac{\prod_{i=1}^n w_i^{a_i-1}(1-\sum_{i=1}^n w_i)^{b-1}}{(1+\sum_{i=1}^n w_i)^{\sum_{i=1}^n a_i+b}}\\ &\quad\times {}_1F_1\left(\sum_{i=1}^n a_i+b;b;\frac{\delta(1-\sum_{i=1}^n w_i)}{1+\sum_{i=1}^n w_i}\right)\prod_{i=1}^n \mathrm{d}w_i\\ &= \frac{\prod_{i=1}^n\Gamma(a_i)}{\Gamma(\sum_{i=1}^n a_i)}\int_0^1 \frac{w^{\sum_{i=1}^n a_i-1}(1-w)^{b-1}}{(1+w)^{\sum_{i=1}^n a_i+b}}\\ &\quad\times {}_1F_1\left(\sum_{i=1}^n a_i+b;b;\frac{\delta(1-w)}{1+w}\right)\mathrm{d}w,\end{aligned}$$

where the last line has been obtained by using Liouville-Dirichlet integral. Now, evaluating the above integral using (9) and simplifying the result, we get

$$\{C(a_1,\dots,a_n,b)\}^{-1}=2^{-\sum_{i=1}^n a_i}\exp(\delta)B(a_1,\dots,a_n,b). \tag{31}$$

The next theorem derives the Dirichlet type 3 distribution from the Dirichlet type 1 distribution.

Theorem 3.1. *Let $(U_1,\dots,U_n)\sim \mathrm{NCD1}(a_1,\dots,a_n;b;\delta)$. Define $W_i=U_i/(2-\sum_{i=1}^n U_i)$, $i=1,\dots,n$. Then, $(W_1,\dots,W_n)\sim \mathrm{NCD3}(a_1,\dots,a_n;b;\delta)$.*

Proof. Substituting $u_i=2w_i/(1+\sum_{i=1}^n w_i)$, $i=1,\dots,n$ with the Jacobian of transformation $J(u_1,\dots,u_n\to w_1,\dots,w_n)=2^n(1+\sum_{i=1}^n w_i)^{-(n+1)}$ in (27) and simplifying, we get the desired result. □

Theorem 3.2. *Let $Y_1,\dots,Y_{n+1}$ be independent random variables, $Y_i\sim \mathrm{Ga}(a_i)$, $i=1,\dots,n$ and $Y_{n+1}\sim \mathrm{Ga}(b;\delta)$. Define $U_i=Y_i/\sum_{j=1}^{n+1}Y_j$, $i=1,\dots,n$, $V_j=Y_j/Y_{n+1}$, $j=1,\dots,n$ and $Z=\sum_{i=1}^{n+1}Y_i$. Then, Z is independent of $(U_1,\dots,U_n)$ and $(V_1,\dots,V_n)$. Further, $(U_1,\dots,U_n)\sim \mathrm{NCD1}(a_1,\dots,a_n;b;\delta)$ and $(V_1,\dots,V_n)\sim \mathrm{NCD2}(a_1,\dots,a_n;b;\delta)$.*

Let $Y_1,\dots,Y_{n+1}$ be independent random variables, $Y_i\sim \mathrm{Ga}(a_i)$, $i=1,\dots,n$ and $Y_{n+1}\sim \mathrm{Ga}(b;\delta)$. Further, let $(U_1,\dots,U_n)\sim \mathrm{NCD1}(a_1,\dots,a_n;b;\delta)$, $(V_1,\dots,V_n)\sim \mathrm{NCD2}(a_1,\dots,a_n;\,b;\delta)$ and $(W_1,\dots,W_n)\sim \mathrm{NCD3}(a_1,\dots,a_n;b;\delta)$. Then,

$$(U_1,\dots,U_n)\overset{d}{=}\left(\frac{Y_1}{\sum_{i=1}^{n+1}Y_i},\dots,\frac{Y_n}{\sum_{i=1}^{n+1}Y_i}\right)\sim \mathrm{NCD1}(a_1,\dots,a_n;b;\delta) \tag{32}$$

and

$$(V_1,\dots,V_n) \stackrel{d}{=} \left(\frac{Y_1}{Y_{n+1}},\dots,\frac{Y_n}{Y_{n+1}}\right) \sim \text{NCD2}(a_1,\dots,a_n;b;\delta). \tag{33}$$

Now, using (32), Theorem 2.2 and Theorem 3.1, it is easy to see that

$$(W_1,\dots,W_n) \stackrel{d}{=} \left(\frac{Y_1}{\sum_{i=1}^n Y_i + 2Y_{n+1}},\dots,\frac{Y_n}{\sum_{i=1}^n Y_i + 2Y_{n+1}}\right) \sim \text{NCD3}(a_1,\dots,a_n;b;\delta). \tag{34}$$

Further, from (33) and (34), it follows that

$$(W_1,\dots,W_n) \stackrel{d}{=} \left(\frac{V_1}{\sum_{i=1}^n V_i + 2},\dots,\frac{V_n}{\sum_{i=1}^n V_i + 2}\right)$$

and

$$(V_1,\dots,V_n) \stackrel{d}{=} \left(\frac{2W_1}{1-\sum_{i=1}^n W_i},\dots,\frac{2W_n}{1-\sum_{i=1}^n W_i}\right).$$

Now, using (34) and Theorem 2.2, the next theorem can easily be established.

Theorem 3.3. *Let* $(W_1,\dots,W_n) \sim \text{NCD3}(a_1,\dots,a_n;b;\delta)$. *Then for* $1 \le s \le n-1$,

$$\left(\frac{W_{s+1}}{1-\sum_{i=1}^s W_i},\dots,\frac{W_n}{1-\sum_{i=1}^s W_i}\right) \sim \text{NCD3}(a_{s+1},\dots,a_n;b;\delta).$$

If $(U_1,\dots,U_n) \sim \text{NCD1}(a_1,\dots,a_n;b;\delta)$, then it is well known that (Sánchez, Nagar and Gupta [13]) for $1 \le m \le n$, $(U_1,\dots,U_m) \sim \text{NCD1}(a_1,\dots,a_m;\sum_{i=m+1}^n a_i + b;\delta)$. In the following theorem we will derive similar result for non-central Dirichlet type 3 variables. However, the marginal distribution in this case is not a non-central Dirichlet type 3 distribution.

Theorem 3.4. *Let* $(W_1,\dots,W_n) \sim \text{NCD3}(a_1,\dots,a_n;b;\delta)$. *Then, the marginal density of* $(X_1,\dots,X_s)$ *is given by*

$$\sum_{r=0}^\infty \frac{\exp(-\delta)\delta^r}{r!} \frac{\prod_{i=1}^s w_i^{a_i-1}(1-\sum_{i=1}^s w_i)^{\sum_{j=s+1}^n a_j+b+r-1}}{2^{b+r} B(a_1,\dots,a_s,\sum_{i=s+1}^n a_i + b + r)} \times {}_2F_1\left(\sum_{i=1}^n a_i + b + r, b+r; \sum_{i=s+1}^n a_i + b + r; \frac{1-\sum_{i=1}^s w_i}{2}\right).$$

Proof. Transforming $X_i = (1-\sum_{i=1}^s W_i)^{-1} W_i$, $i = s+1,\dots,n$ with the Jacobian $J(w_{s+1},\dots,w_n \to x_{s+1},\dots,x_n) = (1-\sum_{i=1}^s w_i)^{n-s}$ in (30), the joint density of $(W_1,\dots,W_s)$ and $(X_{s+1},\dots,X_n)$ is given by

$$C(a_1,\dots,a_n,b) \prod_{i=s+1}^n x_i^{a_i-1}(1-x^{(2)})^{b-1} \frac{\prod_{i=1}^s w_i^{a_i-1}(1-w^{(1)})^{\sum_{j=s+1}^n a_j+b-1}}{[1+w^{(1)}+(1-w^{(1)})x^{(2)}]^{\sum_{i=1}^n a_i+b}} \times {}_1F_1\left(\sum_{i=1}^n a_i + b; b; \frac{\delta(1-w^{(1)})(1-x^{(2)})}{1+w^{(1)}+(1-w^{(1)})x^{(2)}}\right), \tag{35}$$

where $w^{(1)} = \sum_{i=1}^s w_i$, $x^{(2)} = \sum_{i=s+1}^n x_i$, $x_i > 0$, $i = 1, \ldots, s$, $\sum_{i=1}^s x_i < 1$, $w_i > 0$, $i = s+1, \ldots, n$, $\sum_{i=s+1}^n w_i < 1$. Now, integrating $x_{s+1}, \ldots, x_n$ in the above expression, we get the joint density of $W_1, \ldots, W_s$ as

$$\begin{aligned} & C(a_1, \ldots, a_n, b) \prod_{i=1}^s w_i^{a_i-1} (1 - w^{(1)})^{\sum_{j=s+1}^n a_j + b - 1} \\ & \times \underset{\substack{0<\sum_{i=s+1}^n x_i<1 \\ x_i>0, i=s+1,\ldots,n}}{\int \cdots \int} \frac{\prod_{i=s+1}^n x_i^{a_i-1} (1 - x^{(2)})^{b-1}}{[1 + w^{(1)} + (1 - w^{(1)}) x^{(2)}]^{\sum_{i=1}^n a_i + b}} \\ & \times {}_1F_1 \left(\sum_{i=1}^n a_i + b; b; \frac{\delta(1 - w^{(s)})(1 - x^{(2)})}{1 + w^{(1)} + (1 - w^{(1)}) x^{(2)}} \right) \prod_{i=s+1}^n \mathrm{d}x_i, \end{aligned} \tag{36}$$

where $0 < x_i < 1$, $i = 1, \ldots, s$, $\sum_{i=1}^s x_i < 1$ Now, using Liouville-Dirichlet integral and (24), the integral given in (36) is evaluated as

$$\begin{aligned} & \frac{\prod_{i=s+1}^n \Gamma(a_i)}{\Gamma(\sum_{i=s+1}^n a_i)} \int_{0<x<1} \frac{x^{\sum_{i=s+1}^n a_i - 1} (1 - x)^{b-1}}{[1 + w^{(1)} + (1 - w^{(1)}) x]^{\sum_{i=1}^n a_i + b}} \\ & \times {}_1F_1 \left(\sum_{i=1}^n a_i + b; b; \frac{\delta(1 - w^{(1)})(1 - x)}{1 + \sum_{i=1}^s w_i + (1 - w^{(1)}) x} \right) \mathrm{d}x \\ & = \frac{\Gamma(b) \prod_{i=s+1}^n \Gamma(a_i)}{2^{\sum_{i=1}^n a_i + b + r} \Gamma(\sum_{i=1}^n a_i + b)} \sum_{r=0}^\infty \frac{\Gamma(\sum_{i=1}^n a_i + b + r)}{\Gamma(\sum_{i=s+1}^n a_i + b + r)} \frac{[\delta(1 - \sum_{i=1}^s w_i)]^r}{2^r \, r!} \\ & \times {}_2F_1 \left(\sum_{i=1}^n a_i + b + r, b + r; \sum_{i=s+1}^n a_i + b + r; \frac{1 - w^{(1)}}{2} \right). \end{aligned}$$

Finally, substituting appropriately, we get the result. □

Integrating $w_1, \ldots, w_s$ in (35) using Liouville-Dirichlet integral and (24), we get the p.d.f. of $(X_{s+1}, \ldots, X_n)$ as given in Theorem 3.3. Using the result (Luke [9, Eq. 3.8.4]),

$${}_2F_1(a, b; c; x) = (1 - x)^{-b} \, {}_2F_1 \left(c - a, b; c; -\frac{x}{1 - x} \right),$$

the p.d.f. of $(W_1, \ldots, W_s)$ given in Theorem 3.4 can be re-written as

$$\begin{aligned} & \sum_{r=0}^\infty \frac{\exp(-\delta) \delta^r}{r!} \frac{2^{\sum_{i=1}^n a_i}}{B(a_1, \ldots, a_s, \sum_{i=s+1}^n a_i + b + r)} \\ & \times \frac{\prod_{i=1}^s w_i^{a_i-1} (1 - \sum_{i=1}^s w_i)^{\sum_{j=s+1}^n a_j + b + r - 1}}{(1 + \sum_{i=1}^s w_i)^{\sum_{i=1}^n a_i + b + r}} \\ & \times {}_2F_1 \left(\sum_{i=1}^n a_i + b + r, b + r; \sum_{i=s+1}^n a_i + b + r; \frac{1 - \sum_{i=1}^s w_i}{1 + \sum_{i=1}^s w_i} \right). \end{aligned}$$

It can clearly be observed that the p.d.f. of $(W_1, \ldots, W_n)$ given in Theorem 3.4 is not a non-central Dirichlet type 3 density.

The joint moments of $W_1, \ldots, W_n$ are given by

$$E[W_1^{r_1} \cdots W_n^{r_n}] = C(a_1, \ldots, a_n, b) \underset{\substack{\sum_{i=1}^n w_i < 1 \\ w_i > 0, i=1,\ldots,n}}{\int \cdots \int} \frac{\prod_{i=1}^n w_i^{a_i + r_i - 1} (1 - \sum_{i=1}^n w_i)^{b-1}}{(1 + \sum_{i=1}^n w_i)^{\sum_{i=1}^n a_i + b}}$$
$$\times {}_1F_1 \left(\sum_{i=1}^n a_i + b; b; \frac{\delta (1 - \sum_{i=1}^n w_i)}{1 + \sum_{i=1}^n w_i} \right) \prod_{i=1}^n \mathrm{d}w_i$$
$$= \frac{C(a_1, \ldots, a_n, b)}{C(\sum_{i=1}^n a_i, b)} \frac{\prod_{i=1}^n \Gamma(a_i + r_i)}{\Gamma[\sum_{i=1}^n (a_i + r_i)]} E[W^{\sum_{i=1}^n r_i}],$$

where $W \sim \mathrm{NCB3}(\sum_{i=1}^n a_i, b)$. Computing $E[W^{\sum_{i=1}^n r_i}]$ using Theorem 2.3, substituting for $C(a_1, \ldots, a_n, b)$ and $C(\sum_{i=1}^n a_i, b)$ from (31) and simplifying the resulting expression, we obtain

$$E[W_1^{r_1} \cdots W_n^{r_n}] = \frac{\prod_{i=1}^n \Gamma(a_i + r_i)}{2^b \prod_{i=1}^n \Gamma(a_i)} \sum_{j=0}^\infty \frac{\exp(-\delta) \delta^j}{j!} \frac{\Gamma(\sum_{i=1}^n a_i + b + j)}{2^j \Gamma[\sum_{i=1}^n (a_i + r_i) + b + j]}$$
$$\times {}_2F_1 \left(b + j, \sum_{i=1}^n a_i + b + j; \sum_{i=1}^n (a_i + r_i) + b + j; \frac{1}{2} \right).$$

In the next theorem we give distribution of partial sums of random variables distributed as non-central Dirichlet type 3.

Theorem 3.5. *Let* $(W_1, \ldots, W_n) \sim \mathrm{NCD3}(a_1, \ldots, a_n; b; \delta)$ *and* $n_1, \ldots, n_\ell$ *be positive integers such that* $\sum_{i=1}^\ell n_i = n$. *Further, let* $a_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} a_j$, $n_0^* = 0$, $n_i^* = \sum_{j=1}^i n_j$, $i = 1, \ldots, \ell$. *Define* $Z_j = W_j / W_{(i)}, j = n_{i-1}^* + 1, \ldots, n_i^* - 1$ *and* $W_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} W_j$, $i = 1, \ldots, \ell$. *Then,*
(i) $(Z_{n_{i-1}^*+1}, \ldots, Z_{n_i^*-1}), i = 1, \ldots, \ell$ *and* $(W_{(1)}, \ldots, W_{(\ell)})$ *are mutually independent,*
(ii) $(Z_{n_{i-1}^*+1}, \ldots, Z_{n_i^*-1}) \sim \mathrm{D1}(a_{n_{i-1}^*+1}, \ldots, a_{n_i^*-1}; a_{n_i^*})$, $i = 1, \ldots, \ell$, *and*
(iii) $(W_{(1)}, \ldots, W_{(\ell)}) \sim \mathrm{NCD3}(a_{(1)}, \ldots, a_{(\ell)}; b; \delta)$.

Proof. Let $Y_1, \ldots, Y_{n_1^*}$, $Y_{n_1^*+1}, \ldots, Y_{n_2^*}, \ldots, Y_{n_{\ell-1}^*+1}, \ldots, Y_{n_\ell^*}$ and Y_{n+1} be mutually independent random variables, $Y_j \sim \mathrm{Ga}(a_j), j = n_{i-1}^* + 1, \ldots, n_i^*$, $i = 1, \ldots, \ell$, and $Y_{n+1} \sim \mathrm{Ga}(b; \delta)$. Then

$$(W_1, \ldots, W_{n_1^*}, W_{n_1^*+1}, \ldots, W_{n_2^*}, \ldots, W_{n_{\ell-1}^*+1}, \ldots, W_{n_\ell^*})$$
$$\stackrel{d}{=} \left(\frac{Y_1}{Y}, \ldots, \frac{Y_{n_1^*}}{Y}, \frac{Y_{n_1^*+1}}{Y}, \ldots, \frac{Y_{n_2^*}}{Y}, \ldots, \frac{Y_{n_{\ell-1}^*+1}}{Y}, \ldots, \frac{Y_{n_\ell^*}}{Y} \right), \tag{37}$$

where $Y = \sum_{i=1}^\ell Y_{(i)} + 2Y_{n+1}$ with $Y_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} Y_j$. Now, using Theorem 2.2 and the above representation, we have

$$(Z_{n_{i-1}^*+1}, \ldots, Z_{n_i^*-1}) = \left(\frac{W_{n_{i-1}^*+1}}{W_{(i)}}, \ldots, \frac{W_{n_i^*-1}}{W_{(i)}} \right) \stackrel{d}{=} \left(\frac{Y_{n_{i-1}^*+1}}{Y_{(i)}}, \ldots, \frac{Y_{n_i^*-1}}{Y_{(i)}} \right).$$

Further, from Theorem 3.2, $Y_{(i)}$ and $(Z_{n^*_{i-1}+1}, \ldots, Z_{n^*_i-1})$ are independent, $Y_{(i)} \sim \text{Ga}(a_{(i)})$ and $(Z_{n^*_{i-1}+1}, \ldots, Z_{n^*_i-1}) \sim \text{NCD1}(a_{n^*_{i-1}+1}, \ldots, a_{n^*_i-1}; a_{n^*_i})$. Since, for $i \neq k$, $(Z_{n^*_{i-1}+1}, \ldots, Z_{n^*_i-1}, Y_{(i)})$ and $(Z_{n^*_{k-1}+1}, \ldots, Z_{n^*_k-1}, Y_{(k)})$ are functions of two independent sets of variables $\{Y_{n^*_{i-1}+1}, \ldots, Y_{n^*_i}\}$ and $\{Y_{n^*_{k-1}+1}, \ldots, Y_{n^*_k}\}$, respectively, mutual independence is straightforward. Using (37) and Theorem 2.2, the stochastic representation of $(W_{(1)}, \ldots, W_{(\ell)})$ is given as

$$(W_{(1)}, \ldots, W_{(\ell)}) \stackrel{d}{=} \left(\frac{Y_{(1)}}{\sum_{i=1}^{\ell} Y_{(i)} + 2Y_{n+1}}, \ldots, \frac{Y_{(\ell)}}{\sum_{i=1}^{\ell} Y_{(i)} + 2Y_{n+1}} \right),$$

where $Y_{(1)}, \ldots, Y_{(\ell)}$ and Y_{n+1} are independent, $Y_{(i)} \sim \text{Ga}(a_i)$, $i = 1, \ldots, \ell$ and $Y_{n+1} \sim \text{Ga}(b; \delta)$. Now, the desired result follows from (34). □

Corollary 3.5.1. *If* $(W_1, \ldots, W_n) \sim \text{NCD3}(a_1, \ldots, a_n; b; \delta)$, *then*

$$\sum_{i=1}^{n} W_i \sim \text{NCB3}\left(\sum_{i=1}^{n} a_i, b \right)$$

and

$$\left(\frac{W_1}{\sum_{i=1}^{n} W_i}, \ldots, \frac{W_{n-1}}{\sum_{i=1}^{n} W_i} \right) \sim \text{D1}\left(a_1, \ldots, a_{n-1}; a_n \right),$$

are independent. Furthermore, $\sum_{i=1}^{n} W_i$ *and*

$$\frac{\sum_{i=1}^{s} W_i}{\sum_{i=1}^{n} W_i} \sim \text{ B1}\left(\sum_{i=1}^{s} a_i, a_n \right), \quad 1 \leq s \leq n-1,$$

are independent.

In next six theorems we give several factorizations of the non-central Dirichlet type 3 density.

Theorem 3.6. *Let* $(W_1, \ldots, W_n) \sim \text{NCD3}(a_1, \ldots, a_n; b; \delta)$. *Define* $Y_n = \sum_{j=1}^{n} W_j$ *and* $Y_i = \sum_{j=1}^{i} W_j / \sum_{j=1}^{i+1} W_j$, $i = 1, \ldots, n-1$. *Then,* $Y_1, \ldots, Y_n$ *are independent,* $Y_i \sim \text{B1}(\sum_{j=1}^{i} a_j, a_{i+1})$, $i = 1, \ldots, n-1$, *and* $Y_n \sim \text{NCB3}(\sum_{i=1}^{n} a_i, b; \delta)$.

Proof. Substituting $w_1 = y_n \prod_{i=1}^{n-1} y_i$, $w_2 = y_n(1-y_1) \prod_{i=2}^{n-1} y_i$, ..., $w_{n-1} = y_n\,(1-y_{n-2}) y_{n-1}$ and $w_n = y_n(1-y_{n-1})$ with the Jacobian $J(w_1, \ldots, w_n \to y_1, \ldots, y_n) = \prod_{i=2}^{n} y_i^{i-1}$ in (30) we get the desired result. □

Theorem 3.7. *Let* $(W_1, \ldots, W_n) \sim \text{NCD3}(a_1, \ldots, a_n; b; \delta)$. *Define* $Z_n = \sum_{j=1}^{n} W_j$ *and* $Z_i = W_{i+1} / \sum_{j=1}^{i} W_j$, $i = 1, \ldots, n-1$. *Then* $Z_1, \ldots, Z_n$ *are independent,* $Z_i \sim \text{B2}(a_{i+1}, \sum_{j=1}^{i} a_j)$, $i = 1, \ldots, n-1$, *and* $Z_n \sim \text{NCB3}(\sum_{i=1}^{n} a_i, b; \delta)$.

Proof. The desired result follows from Theorem 3.6 by noting that $(1 - Y_i)/Y_i \sim \text{B2}(a_{i+1}, \sum_{j=1}^{i} a_j)$. □

Theorem 3.8. *Let* $(W_1, \ldots, W_n) \sim \text{NCD3}(a_1, \ldots, a_n; b; \delta)$. *Define* $Y_n = \sum_{j=1}^{n} W_j$ *and* $Y_i = \sum_{j=1}^{i} W_j / W_{i+1}$, $i = 1, \ldots, n-1$. *Then* $Y_1, \ldots, Y_n$ *are independent,* $Y_i \sim \text{B2}(\sum_{j=1}^{i} a_j, a_{i+1})$, $i = 1, \ldots, n-1$, *and* $Y_n \sim \text{NCB3}(\sum_{i=1}^{n} a_i, b; \delta)$.

Proof. The desired result follows from Theorem 3.7 by observing that $1/Z_i \sim \mathrm{B2}(\sum_{j=1}^{i} a_j, a_{i+1})$. □

Theorem 3.9. *Let* $(W_1, \ldots, W_n) \sim \mathrm{NCD3}(a_1, \ldots, a_n; b; \delta)$. *Define* $Y_n = \sum_{j=1}^{n} W_j$ *and* $Y_i = W_i / \sum_{j=i}^{n} W_j$, $i = 1, \ldots, n-1$. *Then* $Y_1, \ldots, Y_n$ *are independent,* $Y_i \sim \mathrm{B1}(a_i, \sum_{j=i+1}^{n} a_j)$, $i = 1, \ldots, n-1$, *and* $Y_n \sim \mathrm{NCB3}(\sum_{i=1}^{n} a_i, b; \delta)$.

Proof. Substituting $w_1 = y_n y_1, w_2 = y_n y_2(1 - y_1), \ldots, w_{n-1} = y_n y_{n-1}(1 - y_1) \cdots (1 - y_{n-2})$, and $w_n = y_n(1 - y_1) \cdots (1 - y_{n-1})$ with the Jacobian $J(w_1, \ldots, w_n \to y_1, \ldots, y_n) = y_n^{n-1} \prod_{i=1}^{n-2} (1 - y_i)^{n-i-1}$ in (30), we get the desired result. □

Theorem 3.10. *Let* $(W_1, \ldots, W_n) \sim \mathrm{NCD3}(a_1, \ldots, a_n; b; \delta)$. *Define* $Z_n = \sum_{j=1}^{n} W_j$ *and* $Z_i = W_i / \sum_{j=i+1}^{n} W_j$, $i = 1, \ldots, n-1$. *Then* $Z_1, \ldots, Z_n$ *are independent,* $Z_i \sim \mathrm{B2}(a_i, \sum_{j=i+1}^{n} a_j)$, $i = 1, \ldots, n-1$, *and* $Z_n \sim \mathrm{NCB3}(\sum_{i=1}^{n} a_i, b; \delta)$.

Proof. The desired result follows from Theorem 3.9 by noting that $Y_i/(1 - Y_i) \sim \mathrm{B2}(a_i, \sum_{j=i+1}^{n} a_j)$. □

Theorem 3.11. *Let* $(W_1, \ldots, W_n) \sim \mathrm{NCD3}(a_1, \ldots, a_n; b; \delta)$. *Define* $Y_n = \sum_{j=1}^{n} X_j$ *and* $Y_i = \sum_{j=i+1}^{n} X_j / X_i$, $i = 1, \ldots, n-1$. *Then* $Y_1, \ldots, Y_n$ *are independent,* $Y_i \sim \mathrm{B2}(\sum_{j=i+1}^{n} a_j, a_i)$, $i = 1, \ldots, n-1$, *and* $Y_n \sim \mathrm{NCB3}(\sum_{i=1}^{n} a_i, b; \delta)$.

Proof. The desired result follows from Theorem 3.10 by observing that $1/W_i \sim \mathrm{B2}(\sum_{j=i+1}^{n} a_j, a_i)$. □

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REFERENCES

[1] Cardeño, Liliam, Nagar, Daya K., & Sánchez, Luz Estela (2005). Beta type 3 distribution and its multivariate generalization. *Tamsui Oxford Journal of Mathematical Sciences*, *21*(2), 225–241.

[2] Fang, K.T., Kotz, S., & Ng, K.W. (1990). *Symmetric multivariate and related distributions*. London: Chapman and Hall.

[3] Fang, K.T., & Zhang, Y.T. (1990). *Generalized multivariate analysis*. Hong Kong: Springer-Verlag.

[4] Gupta, Arjun K., & Nadarajah, Saralees (Eds.) (2004). *Handbook of Beta distribution and its applications.* Statistics: Textbooks and Monographs, 174. Marcel Dekker, Inc., New York.

[5] Gupta, A.K., & Nagar, D.K. (2000). *Matrix variate distributions*. Boca Raton: Chapman & Hall/CRC.

[6] Gupta, A.K., & Nagar, D.K. (2000). Matrix variate beta distribution, *International Journal of Mathematics and Mathematical Sciences*, *23*(7), 449–459.

[7] Gupta, Arjun K., & Nagar, Daya K. (2009). Properties of matrix variate beta type 3 distribution, *International Journal of Mathematics and Mathematical Sciences*, Article ID 308518, 18 pages (doi:10.1155/2009/308518).

[8] Johnson, N.L., Kotz, S., & Balakrishnan, N. (1994). *Continuous univariate distributions-2*(2nd ed.). New York: John Wiley & Sons.

[9] Luke, Y.L. (1969). *The special functions and their approximations*(Vol. 1). New York: Academic Press.

[10] Miranda De Sá, A.M.F.L. (2004). A note on the sampling distribution of coherence estimate for the detection of periodic signals. *IEEE Signal Processing Letters*, *11*, 323–325.

[11] Nadarajah, Saralees (2005). Sums, products, and ratios of non-central beta variables. *Communications in Statistics-Theory and Methods*, *34*(1), 89–100.

[12] Nadarajah, S., & Kotz, S. (2006). Computation of the sampling distribution of coherence estimate. *Anais da Academia Brasileira de Ciências*,*78*(4), 837–840.

[13] Sánchez, L.E., Nagar, D.K., & Gupta, A.K. (2006). Properties of noncentral Dirichlet distributions. *Computers & Mathematics with Applications*, *52*(12), 1671–1682.

[14] Srivastava, H.M., & Karlsson, P.W. (1985). *Multiple Gaussian hypergeometric series*. New York: John Wiley & Sons.

[15] Troskie, C.G. (1967). Non-central multivariate Dirichlet distributions. *South African Statistical Journal*, *1*, 21–32.