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Exact Solutions for the Incompressible Viscous Fluid of a Rotating Disk Flow

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Abstract: The present paper is devoted to a class of exact solutions to the steady Navier-Stokes equations for the incompressible Newtonian viscous fluid flow motion due to a disk rotating with a constant angular speed. In place of the traditional von Karman's axisymmetric evolution of the flow, the rotational non-axisymmetric stationary flow is taken into account here. As a consequence, the governing equations allow an exact solution to develop over the disk, which is influenced by a fixed point on the disk and also bounded everywhere in the normal direction to the wall. The three-dimensional equations of motion are treated analytically yielding derivation of exact solutions, which differ from those of corresponding to the classical von Karman's flow. Making use of this solution, analytical formulas for the wall shear stresses are extracted. The critical peripheral locations at which extrema of the local skin friction occur are also determined. Analytic calculations show that for the specific flow the thicknesses take the same value. Interaction of the resolved flow field with the surrounding temperature is further analyzed via the energy equation. According to the Fourier's heat law, a constant heat transfer from the disk to the fluid occurs, which is proportional to product of the thermal conduction, Reynolds number and temperature difference.

Key Words: Exact Solution; Rotating-Disk Flow; Shear Stresses; Heat Transfer

1. INTRODUCTION

The cases when an exact solution for the Navier-Stokes equation can be obtained, are of particular interest in investigations to describe fluid motion of the viscous fluid flows. However, since the Navier-Stokes equations are nonlinear in character, there is no known general method to solve the equations in full, nor the superposition principle for nonlinear partial differential equations does work. Exact solutions, on the other hand, are very important for many reasons. They provide a standard for checking the accuracies of many approximate methods such as numerical or empirical. Although, nowadays, computer techniques make the complete integration of the Navier-Stokes equations feasible, the accuracy of the results can be established only by a comparison with an exact solution.

The Navier-Stokes equations were intensively studied in the literature commencing with the paper [1], see also the review by Constantin in [2]. There are well-known exact unidirectional or parallel shear flows, a few sample cases contain the steady Poiseuille and Couette flow. In addition to this, there are exact cylindrically-symmetric solutions with closed plane streamlines, see for instance [3, 4]. Further exact solutions that we already know possess certain feature of the fluid motion, such as rectilinear motion, motions

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of the duct flows, axisymmetric flows and stagnation flows on plate with slip, etc., see for instance, the works of [5, 6].

Von Karman's swirling viscous flow[7] is a well-documented classical problem in fluid mechanics, which has several technical and industrial applications. The original problem raised by von Karman, which is the most studied by researchers in the literature, is the viscous flow motion induced by an infinite rotating disk where the fluid far from the disk is at rest. Then the problem is generalized to include the case where the fluid itself is rotating as a solid body far above the disk. Another generalization is to consider the viscous flow between two infinite coaxial rotating disks. All these problems and also stability issues are attacked, theoretically, numerically and experimentally, by many researchers amongst many others, such as [8–19]. However, all of these results are either numerical or analytical-numerical. Moreover, [20] established the existence of an infinite set of solutions to the flow of two infinite parallel plates rotating with the same angular velocity about an axis. Extension to porous plates was given in [21, 22], refer also to [23]. Additionally, heat transfer problem over a rotating disk was also studied, see for instance [24–26].

The purpose of this work is to present details of some new three-dimensional solutions of the Navier-Stokes equations governing the steady state stationary viscous flow of an incompressible Newtonian fluid associated with a single rotating disk. Physical interest is due to the applications in the areas of geophysics, rotating machinery and rheology. Dissimilar to the classical von Karman's axisymmetric similarity solution, the current interest lies in searching for the velocity and pressure field representing the non-axisymmetric flow about the axis of rotation. We show in this study that under certain well-defined conditions the Navier-Stokes equations reduce to a system of ordinary differential equations whose solution can be obtained in a closed form. The analytical solution is in fact found by imposing on the infinite rotating disk no-slip condition for the velocity field (after setting no normal velocity at all) together with an assumption that the field is bounded with respect to the normal axis. The resulting equations are then used to obtain exact expressions for the existing wall shear stresses. We further analyze the subject of heat transfer from the disk to the fluid by solving the energy equation, which, for the flow under consideration, is shown to be in balance with the dissipation function.

The following strategy is pursued in the rest of the paper. In §2. the full governing equations representing the flow motion are outlined in cylindrical coordinates. §3. contains the analytical results, both for the steady state flow and wall shear stresses, obtained under self-consistent assumptions. Heat conducting case is analyzed in §4., in which, an expression for the heat transfer is derived using the Fourier's heat law. Finally our conclusions follow in §5..

2. FORMULATION OF THE PROBLEM

We consider the three dimensional viscous flow of an incompressible fluid on an infinite disk which rotates about its axis with a constant angular velocity Ω. The Navier-Stokes equations are non-dimensionalized with respect to a length scale $L = r_e^*$, velocity scale $U_c = L\Omega$, time scale L/U_c and pressure scale ρU_c^2, where ρ is the fluid density. This leads to a Reynolds number $Re = \frac{U_c L}{\nu}$, ν being the kinematic viscosity of the fluid. Thus, relative to non-dimensional cylindrical polar coordinates (r, θ, z) which rotate with the disk, the full time-dependent, unsteady Navier-Stokes equations governing the viscous fluid flow are the usual momentum, continuity and energy equations given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial r} + \frac{1}{R_e}\left(\nabla^2 u - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2}\right), \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{R_e}\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right), \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{R_e}\nabla^2 w, \tag{3}$$

$$\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0, \tag{4}$$

$$c_p\rho(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta} + w\frac{\partial T}{\partial z}) = k\nabla^2 T + \mu\Phi. \tag{5}$$

The Laplacian operator in cylindrical coordinates is defined as

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\right). \tag{6}$$

The parameters appearing in equations (5) are, respectively, c_p is the specific heat, k is the thermal conductivity (assumed constant here) and μ is the viscosity coefficient of the fluid. Moreover, function Φ on the right-hand side of equation (5) represents the dissipation effects.

In this analysis the fluid is assumed to lie in the $z \geq 0$ semi-infinite space. Boundary conditions accompanying the equations (1-5) are such that the fluid adheres to the wall at $z = 0$ with a prescribed wall temperature, and the quantities are bounded at far distances from the wall, with the temperature being uniformly distributed there.

In the classical von Karman flow, the rotational symmetry assumption is used which removes the θ-dependence of the variables in equations (1-5). However, in the present case we allow the θ-dependence, enabling motion of the non-axisymmetric flow to develop, but instead assume $w = 0$ throughout. It should be remarked that the latter no-axial flow assumption is automatically satisfied by the third of the momentum equations (1), provided that the pressure p is not a function of z. The above constraints render the equations (1-5) to a simplified version

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{R_e}\left(\nabla^2 u - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2}\right), \tag{7}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{R_e}\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right), \tag{8}$$

$$\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} = 0, \tag{9}$$

$$c_p\rho\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta}\right) = k\nabla^2 T + \mu\Phi. \tag{10}$$

The boundary conditions accompanying steady state case of governing equations (7-10) are given by

$$u = 0, \qquad v = r, \qquad T = T_w \quad \text{at} \quad z = 0, \tag{11}$$

$$u = U(\theta), \quad v = r + V(\theta), \qquad T = T_\infty \quad \text{as} \quad z \to \infty, \tag{12}$$

for which T_w is the prescribed wall boundary, T_∞ is the constant temperature far above the disk and, $U(\theta)$ and $V(\theta)$ are to be determined as a result of the subsequent analysis.

3. STEADY-STATE MEAN FLOW

We here restrict ourselves to the stationary mean flow relative to the rotating disk. Since the energy equation in (5) is decoupled from the equations (1), this section focuses only on the solution of velocities and pressure. Within this framework, its is particularly clear that

$$u = 0, \quad v = r, \quad p = \frac{r^2}{2} + p_0, \tag{13}$$

where p_0 is a constant, constitute a solution for (7).

Next, via a coordinate transformation $\eta = \sqrt{\frac{R_e}{2}}z$, the solution (13) is perturbed and we assume a new solution of the form

$$u = r_0F(\theta,\eta), \quad v = r + r_0G(\theta,\eta), \quad p = \frac{r^2}{2} - rr_0\cos(\theta-\alpha) + p_1, \tag{14}$$

such that, as opposed to the well-known Karman solution, periodic solutions with respect to θ of F and G are sought, subjected to a pressure field given by equation (14). Moreover, parameters r_0 and α correspond to polar representation of a fixed point on the disk surface and p_1 is a constant.

Substituting equations (14) into (7), we see that the continuity equation is automatically satisfied, while the momentum equations, with the help of (14), give rise to

$$\begin{aligned} F_{\eta\eta} + 2G &= -2\cos(\theta-\alpha), \\ G_{\eta\eta} - 2F &= 2\sin(\theta-\alpha), \end{aligned} \tag{15}$$

whose solutions bounded with respect to η can be immediately expressed as

$$\begin{aligned} G &= C_1e^{-(1+i)\eta} + C_2e^{-(1-i)\eta} - \cos(\theta-\alpha), \\ F &= i(C_1e^{-(1+i)\eta} - C_2e^{-(1-i)\eta}) - \sin(\theta-\alpha), \end{aligned} \tag{16}$$

where C_1 and C_2 are complex integration constants depending on θ. The latter are determined by the no-slip velocity condition on the wall, as a result of which, F and G in (16) are found to be

$$\begin{aligned} F &= f(\eta)\cos(\theta-\alpha) + g(\eta)\sin(\theta-\alpha), \\ G &= -f(\eta)\sin(\theta-\alpha) + g(\eta)\cos(\theta-\alpha), \end{aligned} \tag{17}$$

where the form of the functions f and g in (17) are given as $f = e^{-\eta}\sin(\eta)$ and $g = -1 + e^{-\eta}\cos(\eta)$. We should mention here that the present analytical solutions are much physically related to the eccentric rotations of a disk and a fluid at infinity. The possibility of an exact solution of the Navier-Stokes equations for this type of flow was implied by Berker[27]. Although Berker's solution without an extra condition is not unique, the single rotating disk case considered here leads to the unique solution (17). Moreover, the structure of the solution obtained here points to a close similarity with the well-known Ekman boundary layer flow. Furthermore, we should also note that the present work is different from the studies of [28, 29] in that in the latter the flow is superimposed onto the classical Karman's solutions and, the self-similarity together with a boundary layer approximation allowed the non-axisymmetric solutions with axisymmetric boundary conditions.

Accounting for the asymptotic behaviors of f and g in the large η limit in equations (17), it can be immediately deduced that the velocities far away from the disk turn out to be $u = r_0\sin(\theta-\alpha)$ and $v = r + r_0\cos(\theta-\alpha)$, which differ from the no-slip velocities. The path along which the velocities vanish exactly through the space can be easily determined by setting zero the velocities in equation (14) with the consideration of (17). In the particular case when $\alpha = \theta$, the implication is that $F = f$ and $G = g$, whose graphs are displayed in figures 1 and 2.

These graphs clearly indicate the development of a boundary layer like structure near the surface of the disk. The thicknesses defined by

$$\delta_\theta = \int_0^\infty (1+g)d\eta, \tag{18}$$

as the displacement thickness in the tangential direction, by

$$\delta_r = \int_0^\infty f d\eta, \tag{19}$$

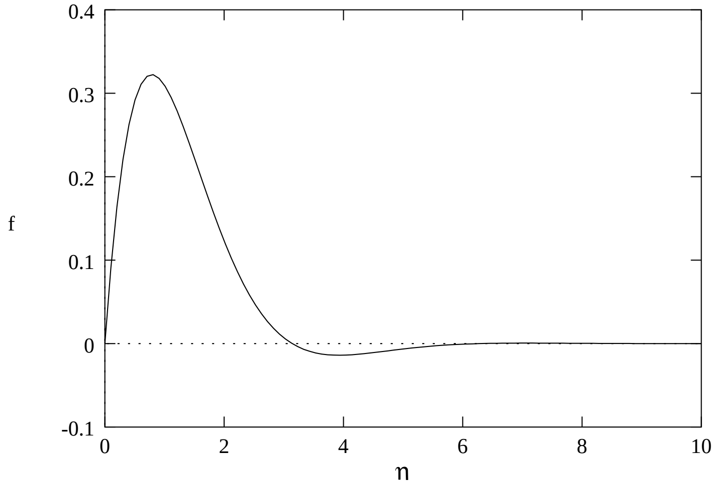

Figure 1: The graphs of f given in equation (17)

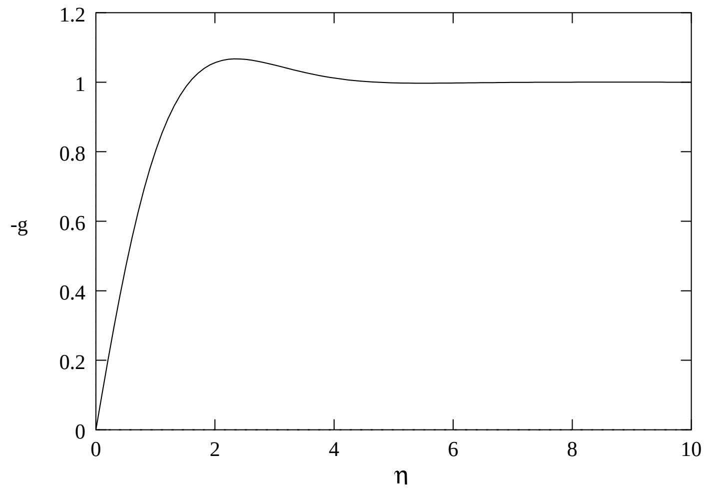

Figure 2: The graphs of g given in equation (17)

as the displacement thickness in the radial direction and also by

$$\delta_t = \int_0^\infty \Theta d\eta, \tag{20}$$

as the thermal layer thickness are all coincide, whose values can be directly evaluated as 1/2.

Having found the form of the velocities, the components of the stress tensor, which are non-zero, are only the radial τ_r and tangential τ_θ shear stresses on the wall, which can be calculated as

$$\tau_r = \left.\frac{\partial u}{\partial z}\right|_{z=0} = r_0\sqrt{\frac{R_e}{2}}[\cos(\theta-\alpha) - \sin(\theta-\alpha)], \tag{21}$$

and

$$\tau_\theta = \left.\frac{\partial v}{\partial z}\right|_{z=0} = -r_0\sqrt{\frac{R_e}{2}}[\cos(\theta-\alpha) + \sin(\theta-\alpha)]. \tag{22}$$

Note that the shear stresses obtained here are non-sxisymmetric and hence deviate from the classical Karman's case. When α is set to zero, the findings point out the fact that the minimum and maximum skin frictions offered against the flow take place respectively at the locations $\theta = 3\pi/4$ and $\theta = 7\pi/4$ for the radial stress and, $\theta = \pi/4$ and $\theta = 5\pi/4$ for the azimuthal stress, over one period of rotation. Moreover, the equal amount of resistance takes place at the locations $\theta = \pi/2$ and $\theta = 3\pi/2$. Likewise, for non-zero α, the locations corresponding to the greatest and smallest stresses can be obtained.

It should be emphasized that, analogous solutions for the Newtonian fluid flow induced by two parallel infinite rotating disks can also be obtained in a similar manner, though not implemented here. Moreover, the solutions obtained in this work are not similarity solutions as given in [7].

4. HEAT CONDUCTING CASE

Analysis of the previous section enables us to look for temperature solution of equation (10) depending only on the normal direction. As a consequence, for the flow in consideration the temperature will balance with the dissipation function which reduces only to

$$\Phi = \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2, \tag{23}$$

Additionally, by virtue of the form of velocities in (14), the energy equation (10) reduces further to

$$\begin{aligned} T_{\eta\eta} &= -r_0^2 \tfrac{\mu}{k} \left(F_\eta^2 + G_\eta^2\right), \\ T(0) = T_w, &\quad T(\infty) = T_\infty, \end{aligned} \tag{24}$$

Via the replacement $\Theta = \frac{T - T_\infty}{T_w - T_\infty}$ and taking into consideration the form of the solutions F and G in (17), it is straightforwardly found that the solution satisfying the equation (24) is $\Theta = e^{-2\eta}$. At this stage the heat transfer q from the disk to the fluid can be computed in accordance with the Fourier's heat law

$$q = -k \frac{\partial T}{\partial z}\bigg|_{z=0}, \tag{25}$$

which results in $q = k\sqrt{2R_e}(T_w - T_\infty)$. In conclusion, a constant heat transfer takes place on the wall without any dependence upon the azimuthal direction as in the case of von Karman flow [7], but in proportional to the product of thermal conductivity, Reynolds number and the temperature difference.

5. CONCLUDING REMARKS

An exact solution of the motion of incompressible viscous flow over a rotating disk has been obtained which differs from that of corresponding to the classical von Karman's flow, in that, the physical quantities are allowed to develop non-axisymmetrically, within the no normal flow confinement. The solution is influenced by a fixed point on the disk, whose polar representation is (r_0, α). The particular case $r_0 = 0$ is associated with the rigid-body rotation. The non-zero choice of r_0 has enabled us to achieve solutions bounded far away from the disk.

Solutions point out that a boundary layer structure develops near the surface of the disk, whose far-field behavior is distinct from the near-wall solution. Moreover, closed form expressions for the non-vanishing shear stresses on the disk have been worked out, which clearly demonstrate that stresses at the surface of the disk applied against the flow varies with the peripheral angle. In addition to this, the critical locations corresponding to the extremums of the skin frictions have also been determined. Furthermore, the properly defined displacement thicknesses in the radial and tangential directions as well as the thermal layer thickness have been shown to coincide for the flow under consideration.

The assumptions made on the velocity and pressure dictate a balance on the temperature and the dissipation function in the energy equation. Hence, the temperature field pertinent to the obtained form of solutions has been analytically derived. In addition to this, a heat transfer expression has been obtained using the Fourier's law. This relation reveals that a constant heat transfer, in proportional to the product of different dynamic parameter, takes place from the disk to the fluid.

The present solutions are, in no doubt closely relevant to the fully laminar flow of an incompressible viscous flow motion of the Newtonian fluid contained between two parallel plates rotating about the same axis of rotation. The particular problem may also cover the flow cases, such as, Couette flow, radial flow and spiral flow.

REFERENCES

[1] Leray, J. (1934). Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Mech., 63*, 193-248.

[2] Engquist, B., & Schmid, W. (2001). *Mathematics unlimited–2001 and beyond*. Berlin: Springer-Verlag.

[3] Batchelor, G. K. (1967). *An introduction to fluid dynamics*. Cambridge: Cambridge University Press.

[4] Polyanin, A. D. (2001). Exact solutions to the Navier-Stokes equations with generalized separation of variables. *Dokl. Phys., 46*(10), 726-731.

[5] Wang, C. Y. (2002). Flow due to a stretching boundary with partial slip–an exact solution of the Navier-Stokes equations. *Chem. Eng. Sci., 57*(17), 3745-3747.

[6] Tsangaris, S., Kondaxakis, D., & Vlachakis, N. W. (2006). Exact solution of the Navier-Stokes equations for the pulsating dean flow in a channel with porous walls. *Int. J. Eng. Sci., 44*(20), 1498-1509.

[7] Kármán, T. V. (1921). Uber laminare und turbulente Reibung. *Z. Angew. Math. Mech., 1*(4), 233-252.

[8] Cochran, W. G. (1934). The flow due to a rotating disk. *Proc. Camb. Phil. Soc., 30*(3), 365-375.

[9] Boedewadt, U. T. (1940). Die Drehstroemung uber festem Grund. *Z. Angew. Math. Mech., 20*, 241-253.

[10] Gregory, N., Stuart, J. T., & Walker, W. S. (1955). On the stability of three dimensional boundary layers with applications to the flow due to a rotating-disk. *Phil. Trans. Roy. Soc. London Ser. A, 248*, 155-199.

[11] Rogers, M. H., & Lance, G. N. (1960). The rotationaly symmetric flow of a viscous fulid in the presence of an infinite rotating disk. *J. Fluid Mech., 7*(4), 617-631.

[12] Benton Edward R. (1966). On the flow due to a rotating disk. *J. Fluid Mech., 24*(4), 781-800.

[13] Federov, B. I., Plavnik, G. Z., Prokhorov, I. V., & Zhukhovitskii, L. G.(1976). Transitional flow conditions on a rotating-disk. *J. Eng. Phys., 31*, 1448-1453.

[14] Kohama, Y. (1984). Study on boundary layer transition of a rotating-disk. *Acta Mech., 50*(3-4), 193-199.

[15] Hall, P. (1986). An asymptotic investigation of the stationary modes of instability of the boundary layer on a rotating-disk. *Proc. Roy. Soc. London Ser. A, 406*, 93-106.

[16] Zandbergen, P. J., & Dijkstra, D. (1987). Von kármán swirling flows. *Ann. Rev. Fluid Mech., 19*, 465-491.

[17] Jarre, S., Le Gal, P., & Chauve, M. P. (1996). Experimental study of rotating-disk instability. I. Natural flow. *Phys. Fluids., 8*(2), 496-508.

[18] Jarre, S., Le Gal, P., & Chauve, M. P. (1996). Experimental study of rotating-disk instability. II. Forced flow. *Phys. Fluids., 8*(11), 2985-2994.

[19] Turkyilmazoglu, M. (1998). Linear absolute and convective instabilities of some two- and three-dimensional flows (Doctoral dissertation, University of Manchester).

[20] Berker, R. (1982). An exact solution of the Navier-Stokes equation the vortex with curvilinear axis. *Int. J. Eng. Sci., 20*(2), 217-230.

[21] Rajagopal, K. R. (1984). A class of exact solutions to the Navier-Stokes equations. *Int. J. Eng. Sci., 22*(4), 451-455.

[22] Rajagopal, K. R. (1988). Swirling flows of viscoelastic fluids. *Int. J. Eng. Sci., 30*(1), 143-149.

[23] Sherman, R. S. (1990). *Viscous flow*. New York, NY: McGraw-Hill.

[24] Millsaps, K., & Pohlhausen, K. (1952). Heat transfer by Laminar flow from a rotating-plate. *J. Aero. Sci., 19*(2), 120-126.

[25] Sparrow, E. M., & Gregg, J. L. (1959). Heat transfer from a rotating disk to fluids of any Prandtl number. *J. Heat Transfer, 81*(13), 249-251.

[26] Riley, N. (1964). The heat transfer from a rotating-disk. *Q. J. Mech. Appl. Math., 17*(3), 331-349.

[27] Berker, R. (1963). *Handbook of fluid dynamics*. Berlin: Springer.

[28] Lai, C. Y., Rajagopa, K. R., & Szeri, A. Z. L. (1984). Asymmetric flow above a rotating disk. *J. Fluid Mech., 157*, 471-492.

[29] Hewitt, R. E., & Duck, P. W. (2000). Non-axisymmetric rotating-disk flows: nonlinear travelling wave states. *J. Fluid Mech., 413*, 287-316.