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Equivalence of Linear Systems of Two Second-Order Ordinary Differential Equations

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Abstract: An equivalence problem is solved completely for a linear system of two second-order ordinary differential equations. Using Lie's infinitesimal method we construct the basis of differential invariants for this class of equations and provide the operators of invariant differentiation. Certain types of linear systems are described in terms of their invariants. Some examples are given to illustrate our results.

Key Words: Linear Equation; System of Second-Order Ordinary Differential Equations; Equivalence Problem; Differential Invariant

1. INTRODUCTION

Linear equations are considered to be the simplest class of easily solvable equations. Indeed, any scalar ordinary differential equation (ODE) of the second order $x'' = p(t)x' + l(t)x$ is reducible to the form $\bar{x}'' = 0$ by a suitable change of variables $\bar{t} = \theta(t)$, $\bar{x} = x\phi(t)$[1]. Therefore, given nonlinear second-order ODE is treated as an integrable one when it turns out to be linearizable. Linearization problem for this class of equations has been solved by S. Lie[2].

The situation is different for a system of two linear equations

$$\begin{aligned} x'' &= p_1(t)x' + q_1(t)y' + l_1(t)x + m_1(t)y, \\ y'' &= p_2(t)y' + q_2(t)x' + l_2(t)y + m_2(t)x. \end{aligned} \tag{1}$$

Class of equations (1) is closed with respect to point transformations

$$\bar{t} = \theta(t), \quad \bar{x} = \phi_{11}(t)x + \phi_{12}(t)y, \quad \bar{y} = \phi_{21}(t)x + \phi_{22}(t)y, \quad \det\|\phi_{ij}(t)\| \neq 0, \tag{2}$$

which form the group E of equivalence transformations of system (1). Transformation (2) relates system (1) to a linear system of the same form

$$\begin{aligned} \bar{x}'' &= \bar{p}_1(\bar{t})\bar{x}' + \bar{q}_1(\bar{t})\bar{y}' + \bar{l}_1(\bar{t})\bar{x} + \bar{m}_1(\bar{t})\bar{y}, \\ \bar{y}'' &= \bar{p}_2(\bar{t})\bar{y}' + \bar{q}_2(\bar{t})\bar{x}' + \bar{l}_2(\bar{t})\bar{y} + \bar{m}_2(\bar{t})\bar{x}. \end{aligned} \tag{3}$$

However, unlike a scalar equation, arbitrary linear system may be irreducible by a point transformation to the simplest form

$$\bar{x}'' = 0, \qquad \bar{y}'' = 0. \tag{4}$$

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It may be illustrated by the number of admitted symmetries, which is invariant under invertible change of variables. Namely, system (1) can possess 5 to 8 or 15 Lie point symmetries[3], the maximum number being achieved by equations (4)[4]. It has been established in [4] that system (1) is equivalent to (4) if and only if its relative invariants

$$\alpha_0 = \frac{1}{4}(p_1' - p_2') + \frac{1}{2}(l_2 - l_1) + \frac{1}{8}(p_2^2 - p_1^2), \quad \alpha_j = \frac{1}{2}q_j' - m_j - \frac{1}{4}(p_1 + p_2)q_j \quad (j = 1, 2), \tag{5}$$

equal to zero.

Let for a given nonlinear system of two second-order ODEs we ascertain its linearizability using, for example, the criteria of [5] or [6]. Linearizing this system one obtains equations (1), which need to be reduced to the most simple form that is possible. The problems of reducibility of a linear system to an autonomous system, system with a separating equation or a decoupled system are particular cases of equivalence problem. It can be solved with the use of invariants of the class of equations (1). Systems (1) and (3) are equivalent with respect to a point transformation (2) iff all their invariants are equal. As invariant of system (1) we name the invariant of its group E of equivalence transformations. Invariant of some subgroup of E we name relative invariant of system (1).

In Section 2 of the present paper equivalence problem is solved completely for the class of equations (1). We construct all invariants of system (1) in nondegenerate case, when $i_0 = \alpha_0^2 + \alpha_1\alpha_2$ is nonzero, and in degenerate cases as well. Note that some invariants depending on t only have been found in [7] for the case $i_0 \neq 0$. To find differential invariants we use Lie's infinitesimal method[8] (see also [9]). It has been developed in detail in [1]. This approach is used in [10] to solving equivalence problem for scalar third-order ODEs. Examples of its application to constructing invariants of some classes of linear partial differential equations (PDEs) one can find in recent papers [11, 12] (see also references therein). Group of transformations can possess infinitely many differential invariants[1], but there exists the finite basis of invariants such that all other higher order invariants of the group are obtained from the basis ones by algebraic operations and invariant differentiation[13]. The order of an invariant I is defined by the highest order of derivatives of functions p_j, q_j, l_j, m_j, $j = 1, 2$, involved in I. Operator $\mathcal{D}$ of invariant differentiation satisfies the condition: if I is an invariant of system (1), then $\mathcal{D}I$ is invariant too.

In Section 3 we describe some classes of linear systems by means of their invariants (decoupled systems, autonomous systems and so on). In Section 4 a few examples show how the invariants can be applied in solving the equivalence problem.

2. INVARIANTS OF LINEAR SYSTEM

The theorem below shows that linear system can possess invariants of two kinds. Namely, invariants $I_j(t)$, $j \geq 1$, which depend on variable t only, and invariant $I_0(t, x, y)$ having the form of ratio of two homogeneous polynomials in x, y. Equivalent systems (1) and (3) should have equal invariants. It means the consistency of algebraic equalities

$$I_0(t, x, y) = \bar{I}_0(\bar{t}, \bar{x}, \bar{y}), \quad I_1(t) = \bar{I}_1(\bar{t}), \quad I_2(t) = \bar{I}_2(\bar{t}), \quad I_3(t) = \bar{I}_3(\bar{t}), \ \ldots \tag{6}$$

where I_0, I_j, $j \geq 1$ are invariants of system (1) and $\bar{I}_0$, $\bar{I}_j$, $j \geq 1$ are invariants of system (3). If these invariants are not identically constant, then solving system (6) for $\bar{t}$ and $\bar{x}/\bar{y}$ one can find possible form of transformation (2), which relates equivalent systems (1), (3).

Invariants may be effective when we need to prove nonequivalence of two given systems. For example, for a system (1) with constant coefficients all invariants $I_j(t)$ are identically constant. Therefore, if system (3) has invariants $\bar{I}_j(\bar{t})$, which depend on $\bar{t}$ explicitly, then no transformation (2) can reduce it to an autonomous form, since in this case system (6) is inconsistent.

For the most compact representation of invariants of system (1) we use its relative invariants of the first order (5) and

$$\delta = \frac{1}{2}(p_1' + p_2' - q_1 q_2) - l_1 - l_2 - \frac{1}{4}(p_1^2 + p_2^2), \tag{7}$$

relative invariants of the second order

$$\beta_0 = \alpha_0' + \frac{1}{2}(q_2\alpha_1 - q_1\alpha_2), \quad \beta_1 = \alpha_1' + q_1\alpha_0 + \frac{1}{2}(p_2 - p_1)\alpha_1, \quad \beta_2 = \alpha_2' - q_2\alpha_0 + \frac{1}{2}(p_1 - p_2)\alpha_2, \tag{8}$$

third order

$$\begin{array}{ll} \gamma_0 = \beta_0' + \frac{1}{2}(q_2\beta_1 - q_1\beta_2) - 2\delta\alpha_0, & \gamma_1 = \beta_1' + q_1\beta_0 + \frac{1}{2}(p_2 - p_1)\beta_1 - 2\delta\alpha_1, \\ \gamma_2 = \beta_2' - q_2\beta_0 + \frac{1}{2}(p_1 - p_2)\beta_2 - 2\delta\alpha_2, & \end{array} \tag{9}$$

fourth order

$$\begin{array}{ll} \varepsilon_0 = \gamma_0' + \frac{1}{2}(q_2\gamma_1 - q_1\gamma_2) - 5\delta\beta_0, & \varepsilon_1 = \gamma_1' + q_1\gamma_0 + \frac{1}{2}(p_2 - p_1)\gamma_1 - 5\delta\beta_1, \\ \varepsilon_2 = \gamma_2' - q_2\gamma_0 + \frac{1}{2}(p_1 - p_2)\gamma_2 - 5\delta\beta_2, & \end{array} \tag{10}$$

and their combinations

$$\begin{array}{lll} B_0 = \alpha_1\beta_2 - \alpha_2\beta_1, & B_1 = \alpha_2\beta_0 - \alpha_0\beta_2, & B_2 = \alpha_0\beta_1 - \alpha_1\beta_0, \\ G_0 = \alpha_1\gamma_2 + \alpha_2\gamma_1 - \frac{5}{2}\beta_1\beta_2, & \Gamma_k = \alpha_k\gamma_k - \frac{5}{4}\beta_k^2, \quad k = 0, 1, 2, & \\ G_1 = \alpha_2\gamma_0 + \alpha_0\gamma_2 - \frac{5}{2}\beta_0\beta_2, & e_1 = \alpha_2\varepsilon_0 - \alpha_0\varepsilon_2 + \frac{9}{2}(\beta_0\gamma_2 - \beta_2\gamma_0), & \end{array} \tag{11}$$

$$\begin{array}{lll} i_0 = \alpha_0^2 + \alpha_1\alpha_2, & j_0 = y^2\alpha_1 + 2xy\alpha_0 - x^2\alpha_2, & j_1 = y^2 B_2 + xyB_0 - x^2 B_1, \\ i_1 = B_0^2 + 4B_1B_2, & i_2 = 2\Gamma_0 + G_0, & i_3 = B_0\gamma_0 + B_1\gamma_1 + B_2\gamma_2, \end{array} \tag{12}$$

$$\begin{array}{lll} E_2 = \alpha_2^2\varepsilon_2 - \frac{9}{2}\alpha_2\beta_2\gamma_2 + \frac{15}{4}\beta_2^3, & K = \alpha_2 G_1 - 2\alpha_0\Gamma_2, & \\ k_0 = K + 3yB_1^2(x\alpha_2 - y\alpha_0)^{-1}, & k_1 = \frac{1}{3}K^2 + B_1^2\Gamma_2, & k_2 = \alpha_2^2 e_1 - 3B_1\Gamma_2. \end{array} \tag{13}$$

Theorem 2.1 *Every linear system of two second-order ODEs (1) belongs to one of five types of linear systems. For each type the basis of invariants and invariant differentiation operator are given by following formulas:*

(1) Systems of the first type ($i_0 \neq 0$) have two second-order and two third-order basis invariants

$$I_0 = j_1 j_0^{-1} i_0^{-3/4}, \quad I_1 = i_1 i_0^{-5/2}, \quad I_2 = i_2 i_0^{-3/2}, \quad I_3 = i_3 i_0^{-9/4}, \quad \mathcal{D} = i_0^{-1/4} D_t; \tag{14}$$

(2) Systems of the second type ($i_0 = 0$, $\alpha_2 \neq 0$, $B_1 \neq 0$) have two third-order and one fourth-order basis invariants

$$I_0 = k_0\alpha_2^{-2/3} B_1^{-4/3}, \quad I_1 = k_1\alpha_2^{-4/3} B_1^{-8/3}, \quad I_2 = k_2\alpha_2^{-4/3} B_1^{-5/3}, \quad \mathcal{D} = \alpha_2^{1/3} B_1^{-1/3} D_t; \tag{15}$$

(3) Systems of the third type ($i_0 = 0$, $\alpha_2 \neq 0$, $B_1 = 0$, $\Gamma_2 \neq 0$) have one basis invariant of the fourth order

$$I_1 = E_2\Gamma_2^{-3/2}, \qquad \mathcal{D} = \alpha_2\Gamma_2^{-1/2} D_t; \tag{16}$$

(4) Systems of the fourth type ($i_0 = 0$, $\alpha_2 \neq 0$, $B_1 = 0$, $\Gamma_2 = 0$) have no invariants, but they have an invariant manifold

$$(x\alpha_2 - y\alpha_0)\alpha_2^{-1/2} = 0; \tag{17}$$

(5) Systems of the fifth type ($\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$) are equivalent to the simplest system $x'' = 0$, $y'' = 0$.

The proof of Theorem 2.1 is given in Appendix. In (14)–(16) D_t is the operator of total differentiation with respect to t.

Remarks: If for a given system (1) relations $i_0 = 0$, $\alpha_2 = 0$, $\alpha_1 \neq 0$ hold, then interchanging x and y one obtains the linear system with $\alpha_2 \neq 0$. So we need not consider the case $i_0 = 0$, $\alpha_1 \neq 0$.

3. INVARIANTS OF SOME CLASSES OF LINEAR SYSTEMS

Here we describe invariants for some classes of easily integrable linear systems. Since all systems with vanishing relative invariants (5) are equivalent to $\bar{x}'' = 0$, $\bar{y}'' = 0$, we will not consider the systems of the fifth type.

3.1 System with Constant Coefficients

For an autonomous system (1) all invariants I_j, $j \geq 1$ are identically constant while invariant I_0 depend on x/y only. To avoid the cumbersome formulas, below we consider some subcases of coupled autonomous systems.

Case I: For a linear system

$$x'' = q_1 y' + l_1 x, \qquad y'' = q_2 x' + l_2 y, \tag{18}$$

relative invariants (5) are equal to

$$\alpha_0 = (l_2 - l_1)/2, \qquad \alpha_1 = 0, \qquad \alpha_2 = 0.$$

When $l_1 \neq l_2$ system (18) is of the first type having the invariants

$$I_0 = \frac{q_1 y^2 - q_2 x^2}{\sqrt{2(l_2 - l_1)}xy}, \qquad I_1 = \frac{8q_1 q_2}{l_2 - l_1}, \qquad I_2 = \frac{13 q_1 q_2 + 8(l_1 + l_2)}{l_2 - l_1}, \qquad I_3 = 0. \tag{19}$$

Case II: The relative invariants of system

$$x'' = q_1 y' + m_1 y, \qquad y'' = q_2 x' + m_2 x, \tag{20}$$

are equal to

$$\alpha_0 = 0, \qquad \alpha_1 = -m_1, \qquad \alpha_2 = -m_2, \qquad i_0 = m_1 m_2, \qquad B_1 = m_2(q_2 m_1 - q_1 m_2)/2.$$

If $m_1 \neq 0$, $m_2 \neq 0$, it is a system of the first type with invariants

$$\begin{aligned} I_0 &= \frac{(q_1 m_2 - q_2 m_1)(m_2 x^2 + m_1 y^2)}{2(m_1 m_2)^{3/4}(m_2 x^2 - m_1 y^2)}, \qquad I_1 = -\frac{(q_1 m_2 - q_2 m_1)^2}{(m_1 m_2)^{3/2}}, \\ I_2 &= \frac{9}{8} I_1 + \frac{2 q_1 q_2}{\sqrt{m_1 m_2}}, \qquad I_3 = -\frac{(q_1 m_2 + q_2 m_1)(q_1 m_2 - q_2 m_1)^2}{4(m_1 m_2)^{9/4}}. \end{aligned} \tag{21}$$

If $m_1 = 0$, $q_1 \neq 0$ and $m_2 \neq 0$, system (20) is of the second type. Its basis invariants are

$$I_0 = -\frac{3}{2}\left(\frac{2q_1^2}{m_2}\right)^{1/3}\frac{y}{x}, \qquad I_1 = 9q_2\left(\frac{q_1}{2m_2^2}\right)^{1/3}, \qquad I_2 = \frac{3}{2} I_1.$$

Case III: System

$$x'' = p_1 x' + m_1 y, \qquad y'' = p_2 y' + m_2 x, \tag{22}$$

has following relative invariants

$$\begin{aligned} &\alpha_0 = (p_2^2 - p_1^2)/8, \qquad \alpha_1 = -m_1, \qquad \alpha_2 = -m_2, \qquad i_0 = (p_2^2 - p_1^2)^2/64 + m_1 m_2, \\ &B_1 = -m_2(p_1 - p_2)^2(p_1 + p_2)/16, \qquad \Gamma_2 = m_2^2(7p_1^2 + 2p_1 p_2 + 7p_2^2)/16. \end{aligned}$$

If $i_0 \neq 0$, it is a system of the first type with invariants

$$I_0 = \frac{(p_1 - p_2)(16m_1m_2xy + (p_1^2 - p_2^2)(m_2x^2 - m_1y^2))}{4i_0^{3/4}((p_2^2 - p_1^2)xy + 4(m_2x^2 - m_1y^2))},$$
$$I_1 = \frac{m_1m_2(p_1 - p_2)^2}{i_0^{3/2}}, \qquad I_2 = \frac{9}{8}I_1 + \frac{p_1^2 + p_2^2}{\sqrt{i_0}}, \qquad I_3 = \frac{m_1m_2}{32i_0^{9/4}}(p_1 - p_2)^4(p_1 + p_2). \tag{23}$$

When $i_0 = 0$, $m_2 \neq 0$, $p_1 \pm p_2 \neq 0$, it is a system of the second type with invariants

$$I_0 = \frac{3}{2}\left(\frac{2(p_1 - p_2)}{p_1 + p_2}\right)^{1/3} \frac{(p_1^2 - p_2^2)y + 8m_2x}{(p_1^2 - p_2^2)y - 8m_2x},$$
$$I_1 = \frac{3}{2}\left(\frac{2(p_1 - p_2)}{p_1 + p_2}\right)^{2/3} \frac{5p_1^2 - 2p_1p_2 + 5p_2^2}{(p_1 - p_2)^2}, \qquad I_2 = \frac{3}{4}\left(\frac{2(p_1 - p_2)}{p_1 + p_2}\right)^{2/3} - \frac{5}{3}I_1.$$

3.2 System with a Separating Equation

System

$$x'' = p_1(t)x' + l_1(t)x, \qquad y'' = p_2(t)y' + q_2(t)x' + l_2(t)y + m_2(t)x, \tag{24}$$

has relative invariants

$$\alpha_1 = 0, \qquad i_0 = \alpha_0^2, \qquad B_1 = \alpha_2\alpha_0' - \alpha_0\beta_2, \qquad \Gamma_0 = \alpha_0\alpha_0'' - 5/4\alpha_0'^2 - 2\delta\alpha_0^2,$$

where $\alpha_0, \alpha_2, \delta, \beta_2$ are calculated by formulas (5), (7), (8). When $\alpha_0 \neq 0$ it is a system of the first type with invariants

$$I_0 = \frac{B_1x}{\alpha_0^{3/2}(x\alpha_2 - 2y\alpha_0)}, \qquad I_1 = 0, \qquad I_2 = \frac{2\Gamma_0}{\alpha_0^3}, \qquad I_3 = 0. \tag{25}$$

When $i_0 = 0$ system (24) falls into the third type ($\Gamma_2 \neq 0$), the fourth type ($\Gamma_2 = 0$) or the fifth type ($\alpha_2 = 0$) of linear systems. It is readily verified that in any linear system (1) of the third or the fourth type (i.e. when $i_0 = 0$, $B_1 = 0$) an equation

$$\bar{x}'' = 2\rho\bar{x}' + (\rho' - \rho^2 - \delta/2)\bar{x}, \qquad \rho = p_2/2 + (\beta_2 + q_1\alpha_0/2)/\alpha_2,$$

separates for a function $\bar{x} = x\alpha_2 - y\alpha_0$. Linear system of the second type cannot be reduced to the form (24).

3.3 Decoupled System

System

$$x'' = p_1(t)x' + l_1(t)x, \qquad y'' = p_2(t)y' + l_2(t)y, \tag{26}$$

belongs to the first type, provided $\alpha_0 \neq 0$. Its basis invariants are

$$I_0 = 0, \qquad I_1 = 0, \qquad I_2 = \frac{2\alpha_0\alpha_0'' - 5/2\alpha_0'^2 - 4\delta\alpha_0^2}{\alpha_0^3}, \qquad I_3 = 0. \tag{27}$$

If $\alpha_0 = 0$, system (26) is of the fifth type. Linear systems (1) of the second, third or fourth types cannot be reduced to decoupled equations (26).

System, which often arises in applications,

$$x'' = p(t)x' + q(t)y' + l(t)x + m(t)y, \qquad y'' = p(t)y' - q(t)x' + l(t)y - m(t)x, \tag{28}$$

has the following relative invariants (5), (7)

$$\alpha_0 = 0, \qquad \alpha_1 = (q' - pq)/2 - m, \qquad \alpha_2 = -\alpha_1, \qquad \delta = p' - 2l + (q^2 - p^2)/2.$$

When $\alpha_1 \neq 0$ it is a system of the first type having the invariants

$$I_0 = 0, \qquad I_1 = 0, \qquad I_2 = i(5/2\alpha_1'^2 - 2\alpha_1\alpha_1'' + 4\delta\alpha_1^2)/\alpha_1^3, \qquad I_3 = 0,$$

where $i^2 = -1$. Comparing them with invariants (27) we infer that system (28) is equivalent to a decoupled system. Indeed, in variables

$$z = x + i\,y, \qquad \bar{z} = x - i\,y,$$

system (28) takes the form

$$z'' = (p - iq)z' + (l - im)z, \qquad \bar{z}'' = (p + iq)\bar{z}' + (l + im)\bar{z}.$$

When $\alpha_1 = 0$ system (28) is mapped to the simplest equations $\bar{x}'' = 0$, $\bar{y}'' =$ by the change of variables

$$\bar{t} = \theta(t), \qquad \bar{x} = \varphi_1(t)x + \psi_1(t)y, \qquad \bar{y} = \varphi_2(t)x + \psi_2(t)y,$$

where $\theta(t)$ satisfies the equation

$$2\theta'\theta''' - 3\theta''^2 = 2\delta\theta'^2,$$

and $(\varphi_j(t), \psi_j(t))$, $j = 1, 2$ is the fundamental system of solutions of equations

$$2\varphi' = (\theta''/\theta' - p)\varphi + q\psi, \qquad 2\psi' = (\theta''/\theta' - p)\psi - q\varphi.$$

Note that system (28) involves the equations of harmonic oscillator with time-dependent frequency

$$x'' + \omega(t)x = 0, \qquad y'' + \omega(t)y = 0.$$

The change of variables, which transforms them to the form (4) is given in [3].

3.4 System which Admits Lagrangian Representation

Following [14] one can find which systems (1) multiplied by a non-singular matrix $g_{ij}(t)$ become a set of Euler-Lagrange equations for a quadratic Lagrangian

$$L(t, x, y, x', y') = \frac{1}{2}g_{11}(t)x'^2 + g_{12}(t)x'y' + \frac{1}{2}g_{22}(t)y'^2 + \frac{1}{2}h_{11}(t)x^2 + h_{12}(t)xy + \frac{1}{2}h_{22}(t)y^2. \tag{29}$$

Namely, system (1) is represented in the form of Euler-Lagrange equations iff there exists non-singular symmetric matrix $g_{ij}(t)$ satisfying algebraic and differential equations

$$m_1 g_{11} + (l_2 - l_1)g_{12} - m_2 g_{22} = 0, \tag{30}$$

$$q_1 g_{11} + (p_2 - p_1)g_{12} - q_2 g_{22} = 0, \tag{31}$$

$$g_{11}' + p_1 g_{11} + q_2 g_{12} = 0, \quad 2g_{12}' + q_1 g_{11} + (p_1 + p_2)g_{12} + q_2 g_{22} = 0, \quad g_{22}' + p_2 g_{22} + q_1 g_{12} = 0. \tag{32}$$

Coefficients $h_{ij}(t)$ in Lagrangian (29) are equal to

$$h_{11} = l_1 g_{11} + m_2 g_{12}, \quad 2h_{12} = m_1 g_{11} + (l_1 + l_2)g_{12} + m_2 g_{22}, \quad h_{22} = l_2 g_{22} + m_1 g_{12}.$$

System (30)–(32) is overdetermined. Its compatibility conditions arise after differentiating equation (31), eliminating derivatives g_{ij}' by virtue of (32) and repeating this procedure for an algebraic equation obtained. Linear combinations of these algebraic equations with previous ones and equations (30), (31) give rise to the system of linear homogeneous equations

$$A_{j1}g_{11} + A_{j2}g_{12} + A_{j3}g_{22} = 0, \tag{33}$$

where $A_{11} = m_1$, $A_{12} = l_2 - l_1$, $A_{13} = -m_2$, $A_{21} = q_1$, $A_{22} = p_2 - p_1$, $A_{23} = -q_2$, $A_{31} = \alpha_1$, $A_{32} = -2\alpha_0$, $A_{33} = -\alpha_2$ and

$$\begin{aligned} A_{j+1,1} &= A'_{j1} - \tfrac{1}{2}q_1A_{j2} + \tfrac{1}{2}(p_2 - p_1)A_{j1} - \tfrac{1}{2}j(j-3)\delta A_{j-1,1}, \\ A_{j+1,2} &= A'_{j2} - q_2A_{j1} - q_1A_{j3} - \tfrac{1}{2}j(j-3)\delta A_{j-1,2}, \\ A_{j+1,3} &= A'_{j3} - \tfrac{1}{2}q_2A_{j2} + \tfrac{1}{2}(p_1 - p_2)A_{j3} - \tfrac{1}{2}j(j-3)\delta A_{j-1,3}, \quad j \geq 3. \end{aligned}$$

Comparing with (8)–(10) it is readily verified that $A_{41} = \beta_1$, $A_{42} = -2\beta_0$, $A_{43} = -\beta_2$, $A_{51} = \gamma_1$, $A_{52} = -2\gamma_0$, $A_{53} = -\gamma_2$, etc. System (33) has nontrivial solution if and only if

$$\operatorname{rank} A < 3. \tag{34}$$

It is not difficult to see that condition (34) holds, for example, for all systems (1) of the fifth type (in this case $A_{jk} = 0$, $k = 1, 2, 3$, $j \geq 3$).

4. EXAMPLES OF EQUIVALENT SYSTEMS

Now consider a few examples, which illustrate application of the invariants of linear systems to solving equivalence problem.

Example 1. Consider the system from [15] (a, c are nonzero constants)

$$u'' = c^2(3\cos^2 at - 1)u + \frac{3}{2}c^2 v \sin 2at, \qquad v'' = c^2(3\sin^2 at - 1)v + \frac{3}{2}c^2 u \sin 2at, \tag{35}$$

which is of the first type, since in this case we have $i_0 = 9c^4/4$. Comparing its invariants

$$\tilde{I}_0 = \frac{a}{c}\sqrt{\frac{2}{3}}\frac{2(u^2+v^2)}{(u^2-v^2)\sin 2at - 2uv\cos 2at}, \qquad \tilde{I}_1 = -\frac{32a^2}{3c^2}, \qquad \tilde{I}_2 = \frac{8}{3} - 12\frac{a^2}{c^2}, \qquad \tilde{I}_3 = 0, \tag{36}$$

with invariants (25), (27) we conclude that system (35) is not reducible to a decoupled system or a system with separating equation, because $\tilde{I}_1$ is nonzero. However, invariants $\tilde{I}_1$, $\tilde{I}_2$, $\tilde{I}_3$ of system (35) are constant. Therefore, it may be equivalent to a system with constant coefficients. Comparing invariants (36) with invariants (19), (21) and (23) we see that system (35) is reducible to an autonomous system of the form (18). Equating invariants (19) I_1, I_2 to the corresponding invariants (36) $\tilde{I}_1$, $\tilde{I}_2$ one can find

$$l_1 = \frac{a^2 - c^2}{a^2 + 2c^2}l_2, \qquad q_1q_2 = -\frac{4a^2}{a^2 + 2c^2}l_2. \tag{37}$$

Taking into account (37) and multiplying equality $I_0 = \tilde{I}_0$ by x/y one obtains quadratic equation in x/y, which has the solutions

$$\frac{x}{y} = -\frac{2a}{q_2}\sqrt{\frac{l_2}{a^2 + 2c^2}}\frac{u\cos at + v\sin at}{u\sin at - v\cos at}, \qquad \frac{x}{y} = -\frac{2a}{q_2}\sqrt{\frac{l_2}{a^2 + 2c^2}}\frac{u\sin at - v\cos at}{u\cos at + v\sin at}.$$

Taking, for example,

$$x = u\sin at - v\cos at, \qquad y = u\cos at + v\sin at, \tag{38}$$

one obtains the transformation, which relates system (35) and an autonomous system of the form (18). In variables (38) equations (35) become

$$x'' = 2ay' + (a^2 - c^2)x, \qquad y'' = -2ax' + (a^2 + 2c^2)y.$$

Example 2. In [16] all systems of two second-order ODEs admitting real four-dimensional Lie algebras are classified. For instance, system

$$x'' + f(y)(x'^2 + 1)y' = 0, \qquad y'' + (f(y)x' + g(y))y'^2 = 0, \tag{39}$$

admits Lie algebra $\mathcal{L}^2_{4.22}$ of operators

$$X_1 = \partial_x, \qquad X_2 = \partial_t, \qquad X_3 = t\partial_t + x\partial_x, \qquad X_4 = x\partial_t - t\partial_x.$$

Moreover, we find that system (39) admits also the symmetries

$$\begin{array}{l} X_5 = a(y)\partial_t + b(y)\partial_x, \qquad X_6 = b(y)\partial_t - a(y)\partial_x, \\ a'(y) = \sin(\int f dy)\exp(\int g dy), \qquad b'(y) = \cos(\int f dy)\exp(\int g dy). \end{array}$$

Operators X_1, X_2, X_5, X_6 form the basis of an abelian Lie algebra and hence system (39) satisfies one of the linearization criteria established in [5]. In [6] it is established that transformation

$$\bar{t} = y, \qquad \bar{x} = x, \qquad \bar{y} = t,$$

turns (39) into a linear system of the form (28)

$$\bar{x}'' = g(\bar{t})\bar{x}' - f(\bar{t})\bar{y}', \qquad \bar{y}'' = g(\bar{t})\bar{y}' + f(\bar{t})\bar{x}'. \tag{40}$$

Therefore, system (40) reduces to a decoupled form

$$z'' = (g(\bar{t}) + if(\bar{t}))z', \qquad \bar{z}'' = (g(\bar{t}) - if(\bar{t}))\bar{z}',$$

by the change of variables $z = \bar{x} + i\bar{y}$, $\bar{z} = \bar{x} - i\bar{y}$.

Example 3. Linear ODEs of higher order arise in theory of hydrodynamic stability. In particular, linear stability problem for a parallel flow can be reduced to the solution of Orr-Sommerfeld equation[17]

$$\varphi^{IV} - 2a^2\varphi'' + a^4\varphi = iaR[(U(t) - c)(\varphi'' - a^2\varphi) - U''(t)\varphi], \tag{41}$$

for the disturbance amplitude $\varphi(t)$. In equation (41) $U(t)$ is the velocity profile of the undisturbed flow which is a known function. Parameters a and R are positive constants representing, respectively, the disturbance wave number and a flow Reynolds number, $c = c_R + ic_I$ is a complex constant. This linear fourth-order ODE allows representation in the form of two linear equations of the second order

$$\begin{array}{l} \chi'' = (a^2 + V(t))\chi - V''(t)\varphi, \\ \varphi'' = a^2\varphi + \chi, \qquad V(t) = iaR(U(t) - c). \end{array} \tag{42}$$

Assume $V''(t) \neq 0$ and find when system (42) is reducible to linear system with separating equation in one of unknown functions. System (42) has relative invariants

$$\alpha_0 = -\frac{1}{2}V, \qquad \alpha_1 = V'', \qquad \alpha_2 = -1, \qquad i_0 = \frac{1}{4}V^2 - V'', \qquad B_1 = \frac{1}{2}V'.$$

Hence it cannot belong to linear systems of the third, fourth or fifth type. As it follows from Subsection 4.2 it should be of the first type and have vanishing invariants I_1, I_3. Relative invariants of system (42)

$$i_1 = V'''^2 - VV'V''' + V'^2V'', \qquad i_3 = (V'V^{IV} - V''V''')/2,$$

are equal to zero provided function $V(t)$ satisfies the condition

$$V'^2 = c_0 - 2c_1^2 V + c_1 V^2, \qquad c_0, c_1 = \text{const}, \qquad c_1 \neq 0. \tag{43}$$

If $c_1 = aRc_I$, then for the function $U(t)$ relation (43) takes the form

$$U'^2 = aRc_I(U - c_R)^2 + aRc_I^3 - c_0/(aR)^2.$$

In this case system (42) has invariants $I_1 = 0$, $I_3 = 0$ and

$$I_0 = \frac{V'(c_1\varphi - \chi)}{2(V/2 - c_1)^{3/2}((c_1 - V)\varphi + \chi)}, \quad I_2 = \frac{(V/2 - c_1)V'' - 5/8V'^2 + (2a^2 + V)(V - 2c_1)^2}{(V/2 - c_1)^3}.$$

Equating invariant I_0 of system (42) to invariant I_0 of system (24) it is readily seen that it is worthwhile to take x proportional to $c_1\varphi - \chi$. Indeed, when $i_0 \neq 0$ and function $V(t)$ satisfies condition (43), equations (42) become

$$x'' = (a^2 + V(t) - c_1)x, \qquad y'' = (a^2 + c_1)y + x,$$

in variables $x = \chi - c_1\varphi$, $y = \varphi$. Thus, if function $U(t)$ in (41) is given by

$$U(t) = c_R + U_0 \sin \sqrt{-aRc_I}(t + t_0), \qquad c_I < 0,$$
$$U(t) = c_R + U_0 \exp(\sqrt{aRc_I}t) + U_1 \exp(-\sqrt{aRc_I}t), \qquad c_I > 0, \qquad t_0, U_0, U_1 = \text{const},$$

then solution of Orr-Sommerfeld equation reduces to integrating separated system of two second-order ODEs.

5. CONCLUSION

In the present paper we study the equivalence problem for systems of two linear second-order ODEs. Differential invariants of this class of equations were constructed using the classical Lie approach. Invariants provide a simple way of finding the equations, which may be equivalent to a given system, and the transformation connecting two equivalent systems. Their applications are demonstrated with a number of examples. Here we use an autonomous linear system, system with a separating equation and a decoupled system as canonical ones, because the procedure that yields their solution is evident. However, one can consider as a canonical system any other linear system with known solution.

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APPENDIX: PROOF OF THEOREM 2.1

To prove the statement of Theorem 2.1 we use Lie's infinitesimal method [1, Chapter 7]. Invariants of system (1) are found from the condition of their invariance under infinitesimal operator

$$X = \xi_0\partial_t + \xi_1\partial_{x_1} + \xi_2\partial_{x_2} + \sum_{i=1}^{8}\eta_i^{(0)}\partial_{f_i}, \tag{A.1}$$

corresponding to group E of equivalence transformations of system (1). Here we denote $(x_1, x_2) = (x, y)$, $(f_1, \ldots, f_8) = (p_1, p_2, q_1, q_2, l_1, l_2, m_1, m_2)$ and, following [1], assume that coordinates $\xi_0, \xi_1, \xi_2, \eta_i^{(0)}$ depend on t, x_1, x_2, f_i, $i = 1, \ldots, 8$. Operator (A.1) extended to the derivatives of f_i with respect to x_j and to x'_j, x''_j, $j = 1, 2$,

$$X = \xi_0\partial_t + \sum_{j=1}^{2}\left(\xi_j\partial_{x_j} + \xi_j^{(1)}\partial_{x'_j} + \xi_j^{(2)}\partial_{x''_j}\right) + \sum_{i=1}^{8}\left(\eta_i^{(0)}\partial_{f_i} + \zeta_{i1}\partial_{f_{ix_1}} + \zeta_{i2}\partial_{f_{ix_2}}\right),$$

should leave invariant the system

$$x''_j = f_j x'_j + f_{2+j}x'_{3-j} + f_{4+j}x_j + f_{6+j}x_{3-j}, \quad f_{ix_j} = 0, \quad (j = 1, 2,\ i = 1, \ldots, 8). \tag{A.2}$$

Coordinates $\xi_j^{(1)}, \xi_j^{(2)}$ are calculated by the standard prolongation formulas[1]

$$\begin{aligned}&\xi_j^{(1)} = D\xi_j - x'_j D\xi_0, \qquad \xi_j^{(2)} = D\xi_j^{(1)} - x''_j D\xi_0,\\ &D = \partial_t + \textstyle\sum_{j=1}^{2}\left(x'_j\partial_{x_j} + x''_j\partial_{x'_j} + \ldots\right) + \sum_{i=1}^{8}\left(f_{it} + x'_1 f_{ix_1} + x'_2 f_{ix_2}\right)\partial_{f_i} + \ldots\end{aligned}$$

In order to calculate ζ_{ij} we regard t, x_1, x_2 as independent variables and f_i as dependent ones:

$$\begin{aligned}&\zeta_{ij} = D_{x_j}\eta_i^{(0)} - f_{it}D_{x_j}\xi_0 - f_{ix_1}D_{x_j}\xi_1 - f_{ix_2}D_{x_j}\xi_2,\\ &D_{x_j} = \partial_{x_j} + \textstyle\sum_{i=1}^{8}\left(f_{ix_j}\partial_{f_i} + f_{itx_j}\partial_{f_{it}} + f_{ix_kx_j}\partial_{f_{ix_k}} + \ldots\right).\end{aligned}$$

Invariance condition $Xf_{ix_j}\big|_{(A.2)} = 0$ leads to relations $\eta^{(0)}_{ix_1} = 0$, $\eta^{(0)}_{ix_2} = 0$. Action of X on first two equations (A.2) provides two determining equations. Equating coefficients of like powers of x'_1, x'_2, f_{itt} in these equations one obtains

$$\xi_{0x_j} = 0,\ \xi_{0f_i} = 0,\ \xi_{jx_1x_1} = 0,\ \xi_{jx_1x_2} = 0,\ \xi_{jx_2x_2} = 0,\ \xi_{jf_i} = 0 \quad (j = 1, 2,\ i = 1, \ldots, 8).$$

Taking into account these conditions one can find from the determining equations the final form of operator X

$$\begin{aligned}X = {} & \tau(t)\partial_t + (\rho_1(t)x + \sigma_1(t)y)\partial_x + (\rho_2(t)y + \sigma_2(t)x)\partial_y\\ & + \textstyle\sum_{j=1}^{2}\Big[(2\rho'_j - \tau'' - p_j\tau' - q_j\sigma_{3-j} + q_{3-j}\sigma_j)\partial_{p_j}\\ & + (2\sigma'_j + (p_{3-j} - p_j)\sigma_j + q_j(\rho_j - \rho_{3-j} - \tau'))\partial_{q_j}\\ & + (\rho''_j - p_j\rho'_j - q_j\sigma'_{3-j} - 2l_j\tau' + m_{3-j}\sigma_j - m_j\sigma_{3-j})\partial_{l_j}\\ & + (\sigma''_j - p_j\sigma'_j - q_j\rho'_{3-j} + (l_{3-j} - l_j)\sigma_j + m_j(\rho_j - \rho_{3-j} - 2\tau'))\partial_{m_j}\Big].\end{aligned} \tag{A.3}$$

Fourth-order invariants of system (1), which depend on variables

$$t,\ x,\ y,\ p_j,\ q_j,\ l_j,\ m_j,\ p'_j,\ q'_j,\ l'_j,\ m'_j,\ p''_j, \ldots, p'''_j, \ldots, m_j^{IV} \quad (j = 1, 2), \tag{A.4}$$

are found from the invariance condition $\tilde{X}I = 0$. Here $\tilde{X}$ is the fourth-order extension of operator (A.3)

$$\begin{aligned}&\tilde{X} = X + \textstyle\sum_{i=1}^{8}\left(\eta_i^{(1)}\partial_{f'_i} + \eta_i^{(2)}\partial_{f''_i} + \eta_i^{(3)}\partial_{f'''_i} + \eta_i^{(4)}\partial_{f_i^{IV}}\right),\\ &\eta_i^{(n)} = D_t\eta_i^{(n-1)} - f_i^{(n)}\tau'(t) \quad (n = 1, 2, 3, 4), \quad D_t = \partial_t + \textstyle\sum_{i=1}^{8}\left(f'_i\partial_{f_i} + f''_i\partial_{f'_i} + \ldots\right).\end{aligned} \tag{A.5}$$

From (A.3), (A.5) it is not difficult to see that operator $\tilde{X}$ depends linearly on arbitrary functions τ, ρ_1, ρ_2, σ_1, σ_2 and their derivatives up to the sixth order. Invariant I does not depend on them. Hence, according to the theory of invariants of infinite transformation groups[1], the relation $\tilde{X}I = 0$ should be split by these functions and their derivatives. This gives rise to a homogeneous system of linear first-order PDEs

$$X_1(\tau)I = 0, \quad \dots, \quad X_{34}(\sigma_1^{VI})I = 0, \quad X_{35}(\sigma_2^{VI})I = 0, \tag{A.6}$$

where every operator X_i is a coefficient of certain derivative in $\tilde{X}$ that we label in parentheses. Functionally independent solutions of system (A.6) provide all independent differential invariants of system (1) up to the fourth order.

The solution of system (A.6) is found in several steps. First we consider the subsystem of equations (A.6) with 10 operators

$$\begin{aligned}
&X_1(\tau) = \partial_t, \qquad X_{26+j}(\rho_j^V) = \partial_{l_j'''} + 2\partial_{p_j^{IV}} - p_j\partial_{l_j^{IV}} - q_{3-j}\partial_{m_{3-j}^{IV}},\\
&X_{28+j}(\sigma_j^V) = \partial_{m_j'''} + 2\partial_{q_j^{IV}} - p_j\partial_{m_j^{IV}} - q_{3-j}\partial_{l_{3-j}^{IV}}, \qquad X_{31}(\tau^{VI}) = -\partial_{p_1^{IV}} - \partial_{p_2^{IV}},\\
&X_{31+j}(\rho_j^{VI}) = \partial_{l_j^{IV}}, \qquad X_{33+j}(\sigma_j^{VI}) = \partial_{m_j^{IV}}, \qquad (j = 1, 2).
\end{aligned}$$

In 43-dimensional space of variables (A.4) it has 33 functionally independent solutions

$$x,\ y,\ p_j,\ q_j,\ l_j,\ m_j,\ p_j',\ q_j',\ l_j',\ m_j',\ p_j'',\ q_j'',\ l_j'',\ m_j'',\ p_j''',\ q_j''',\ \varepsilon_0,\ \varepsilon_1,\ \varepsilon_2, \quad (j = 1, 2), \tag{A.7}$$

where ε_0, ε_1, ε_2 are defined by (10). In these variables next 10 operators in system (A.6) become

$$\begin{aligned}
&X_{16+j}(\rho_j''') = \partial_{l_j'} + 2\partial_{p_j''} - p_j\partial_{l_j''} - q_{3-j}(\partial_{m_{3-j}''} + \partial_{q_{3-j}'''}) + q_j\partial_{q_j'''},\\
&X_{18+j}(\sigma_j''') = \partial_{m_j'} + 2\partial_{q_j''} - p_j\partial_{m_j''} + q_{3-j}(\partial_{p_j'''} - \partial_{p_{3-j}'''} - \partial_{l_{3-j}''}) + (p_{3-j} - p_j)\partial_{q_j'''},\\
&X_{21}(\tau^{IV}) = -\partial_{p_1''} - \partial_{p_2''} - p_1\partial_{p_1'''} - p_2\partial_{p_2'''} - q_1\partial_{q_1'''} - q_2\partial_{q_2'''}, \qquad X_{21+j}(\rho_j^{IV}) = \partial_{l_j''} + 2\partial_{p_j'''},\\
&X_{23+j}(\sigma_j^{IV}) = \partial_{m_j''} + 2\partial_{q_j'''} \qquad (j = 1, 2), \qquad X_{26}(\tau^V) = -\partial_{p_1'''} - \partial_{p_2'''}.
\end{aligned}$$

In the space of variables (A.7) the subsystem of equations (A.6) $X_{17}(\rho_1''')I = 0, \ \dots, \ X_{26}(\tau^V)I = 0$ has 23 independent solutions. As these solutions we can take x, y, p_j, q_j, l_j, m_j, p_j', q_j', $j = 1, 2$ and β_i, γ_i, ε_i, $i = 0, 1, 2$, defined by (8)–(10). In these variables the subsystem of equations (A.6) with 9 operators

$$\begin{aligned}
&X_{7+j}(\rho_j') = 2\partial_{p_j} - p_j\partial_{l_j} - q_{3-j}(\partial_{m_{3-j}} + \partial_{q_{3-j}'}) + q_j\partial_{q_j'},\\
&X_{9+j}(\sigma_j') = 2\partial_{q_j} - p_j\partial_{m_j} + q_{3-j}(\partial_{p_j'} - \partial_{p_{3-j}'} - \partial_{l_{3-j}}) + (p_{3-j} - p_j)\partial_{q_j'},\\
&X_{11+j}(\rho_j'') = \partial_{l_j} + 2\partial_{p_j'}, \quad X_{13+j}(\sigma_j'') = \partial_{m_j} + 2\partial_{q_j'} \quad (j = 1, 2), \quad X_{16}(\tau''') = -\partial_{p_1'} - \partial_{p_2'},
\end{aligned}$$

has 14 independent solutions

$$x, \quad y, \quad \alpha_i, \quad \beta_i, \quad \gamma_i, \quad \varepsilon_i, \quad (i = 0, 1, 2). \tag{A.8}$$

In variables (A.8) the remainig six operators in system (A.6) take the form

$$\begin{aligned}
&X_2(\tau') = -\textstyle\sum_{i=0}^2 (2\alpha_i\partial_{\alpha_i} + 3\beta_i\partial_{\beta_i} + 4\gamma_i\partial_{\gamma_i} + 5\varepsilon_i\partial_{\varepsilon_i}), \qquad X_3(\rho_1 + \rho_2) = \tfrac{1}{2}(x\partial_x + y\partial_y),\\
&X_4(\rho_1 - \rho_2) = \tfrac{1}{2}(x\partial_x - y\partial_y) + \alpha_1\partial_{\alpha_1} - \alpha_2\partial_{\alpha_2} + \beta_1\partial_{\beta_1} - \beta_2\partial_{\beta_2} + \gamma_1\partial_{\gamma_1} - \gamma_2\partial_{\gamma_2} + \varepsilon_1\partial_{\varepsilon_1} - \varepsilon_2\partial_{\varepsilon_2},\\
&X_5(\sigma_1) = y\partial_x + \alpha_2\partial_{\alpha_0} - 2\alpha_0\partial_{\alpha_1} + \beta_2\partial_{\beta_0} - 2\beta_0\partial_{\beta_1} + \gamma_2\partial_{\gamma_0} - 2\gamma_0\partial_{\gamma_1} + \varepsilon_2\partial_{\varepsilon_0} - 2\varepsilon_0\partial_{\varepsilon_1},\\
&X_6(\sigma_2) = x\partial_y - \alpha_1\partial_{\alpha_0} + 2\alpha_0\partial_{\alpha_2} - \beta_1\partial_{\beta_0} + 2\beta_0\partial_{\beta_2} - \gamma_1\partial_{\gamma_0} + 2\gamma_0\partial_{\gamma_2} - \varepsilon_1\partial_{\varepsilon_0} + 2\varepsilon_0\partial_{\varepsilon_2},\\
&X_7(\tau'') = -\textstyle\sum_{i=0}^2 (2\alpha_i\partial_{\beta_i} + 5\beta_i\partial_{\gamma_i} + 9\gamma_i\partial_{\varepsilon_i}).
\end{aligned} \tag{A.9}$$

(1) Operator X_7 has invariants x, y, α_0, α_1, α_2, (11) and

$$\begin{aligned}
&e_0 = \alpha_1\varepsilon_2 - \alpha_2\varepsilon_1 + \tfrac{9}{2}(\beta_2\gamma_1 - \beta_1\gamma_2), \qquad e_2 = \alpha_0\varepsilon_1 - \alpha_1\varepsilon_0 + \tfrac{9}{2}(\beta_1\gamma_0 - \beta_0\gamma_1),\\
&i_5 = (\alpha_0^2 + \alpha_1\alpha_2)(2\alpha_0\varepsilon_0 + \alpha_2\varepsilon_1 + \alpha_1\varepsilon_2 - \tfrac{3}{2}(2\beta_0\gamma_0 + \beta_2\gamma_1 + \beta_1\gamma_2))\\
&\qquad -\tfrac{3}{2}(2\alpha_0\beta_0 + \alpha_2\beta_1 + \alpha_1\beta_2)(2\Gamma_0 + G_0), \qquad G_2 = \alpha_0\gamma_1 + \alpha_1\gamma_0 - \tfrac{5}{2}\beta_0\beta_1.
\end{aligned} \tag{A.10}$$

Only eight variables (11), (A.10) are independent: B_0, B_1, Γ_0, Γ_1, Γ_2, e_1, e_2, i_5, another ones are given for the symmetry. In these variables operators X_4, X_5, X_6 become

$$\begin{aligned}
X_4 &= \tfrac{1}{2}(x\partial_x - y\partial_y) + \alpha_1\partial_{\alpha_1} - \alpha_2\partial_{\alpha_2} - B_1\partial_{B_1} + B_2\partial_{B_2} + 2\Gamma_1\partial_{\Gamma_1} - 2\Gamma_2\partial_{\Gamma_2}\\
&\qquad -G_1\partial_{G_1} + G_2\partial_{G_2} - e_1\partial_{e_1} + e_2\partial_{e_2},\\
X_5 &= y\partial_x + \alpha_2\partial_{\alpha_0} - 2\alpha_0\partial_{\alpha_1} + 2B_1\partial_{B_0} - B_0\partial_{B_2} + G_1\partial_{\Gamma_0} - 2G_2\partial_{\Gamma_1} - 2G_1\partial_{G_0}\\
&\qquad +2\Gamma_2\partial_{G_1} + (G_0 - 4\Gamma_0)\partial_{G_2} + 2e_1\partial_{e_0} - e_0\partial_{e_2},\\
X_6 &= x\partial_y - \alpha_1\partial_{\alpha_0} + 2\alpha_0\partial_{\alpha_2} - 2B_2\partial_{B_0} + B_0\partial_{B_1} - G_2\partial_{\Gamma_0} + 2G_1\partial_{\Gamma_2} + 2G_2\partial_{G_0}\\
&\qquad +(4\Gamma_0 - G_0)\partial_{G_1} - 2\Gamma_1\partial_{G_2} - 2e_2\partial_{e_0} + e_0\partial_{e_1}.
\end{aligned}$$

System $X_4I = 0$, $X_5I = 0$, $X_6I = 0$ has 10 functionally independent solutions

$$\begin{aligned}
\tilde{\imath}_3 &= \frac{B_0}{\alpha_0}\Gamma_0 + \frac{B_1}{\alpha_1}\Gamma_1 + \frac{B_2}{\alpha_2}\Gamma_2 - \frac{5}{4}\frac{B_0B_1B_2}{\alpha_0\alpha_1\alpha_2},\\
i_4 &= \frac{2B_0 i_0^2}{\alpha_0\alpha_1\alpha_2}(\alpha_1G_1 - \alpha_2G_2) + (\alpha_2B_2 - \alpha_1B_1)\left(\frac{4i_0}{\alpha_0}\Gamma_0 - \frac{2\alpha_0 i_0}{\alpha_1\alpha_2}G_0 - \frac{5B_1B_2i_0}{\alpha_0\alpha_1\alpha_2} - \frac{5}{2}\frac{i_1}{\alpha_0}\right),\\
i_6 &= \tfrac{1}{2}((\alpha_2B_2 - \alpha_1B_1)e_0 + (\alpha_1B_0 - 2\alpha_0B_2)e_1 + (2\alpha_0B_1 - \alpha_2B_0)e_2),\\
i_7 &= 2i_0^2(B_0e_0 + 2B_2e_1 + 2B_1e_2 - 5\tilde{\imath}_3B_0/\alpha_0) + \frac{2i_0^2}{\alpha_0^2}((\alpha_1G_1 - \alpha_2G_2)^2 + 5B_0^2\Gamma_0)\\
&\quad -13i_0i_1i_2 + \frac{75}{8}i_1^2 + \frac{4i_0^2}{\alpha_1\alpha_2}(2(\alpha_0G_0 - \alpha_1G_1)(\alpha_2G_2 - \alpha_0G_0) + 5B_1B_2G_0),
\end{aligned}$$

and j_0, j_1, i_0, i_1, i_2, i_5 defined by (12), (A.10). It is readily verified that $\tilde{\imath}_3$ coincides with i_3 defined by (12) and relations

$$i_4 = i_0i_1' - \frac{5}{2}i_1i_0', \qquad i_5 = i_0i_2' - \frac{3}{2}i_2i_0', \qquad i_6 = i_0i_3' - \frac{9}{4}i_3i_0', \qquad i_7 = i_0i_4' - \frac{15}{4}i_4i_0', \tag{A.11}$$

hold. In variables j_0, j_1, $i_0, \ldots, i_7$ remaining operators (A.9) take the form

$$\begin{aligned}
X_2 &= -2j_0\partial_{j_0} - 5j_1\partial_{j_1} - 4i_0\partial_{i_0} - 10i_1\partial_{i_1} - 6i_2\partial_{i_2} - 9i_3\partial_{i_3} - 15i_4\partial_{i_4} - 11i_5\partial_{i_5} - 14i_6\partial_{i_6} - 20i_7\partial_{i_7},\\
X_3 &= j_0\partial_{j_0} + j_1\partial_{j_1},
\end{aligned}$$

and have eight independent invariants (14) and

$$I_4 = i_4 i_0^{-15/4}, \qquad I_5 = i_5 i_0^{-11/4}, \qquad I_6 = i_6 i_0^{-7/2}, \qquad I_7 = i_7 i_0^{-5}. \tag{A.12}$$

Applying approach of [1] to system (1), we seek the invariant differentiation operator $\mathcal{D} = fD_t$ from the relation $\tilde{X}f = f\tau'$. It should be split by τ, ρ_1, ρ_2, σ_1, σ_2 and their derivatives, which yields the system of linear first-order PDEs

$$X_1f = 0, \qquad X_2f = f, \qquad X_kf = 0, \qquad k = 3, \ldots, 35. \tag{A.13}$$

Therefore, f is a function of variables $j_1j_0^{-1}$, $i_0, \ldots, i_7$ and one can take $f = i_0^{-1/4}$ as a solution of this system.

As it is seen from (A.11), invariants (A.12) satisfy the relations $I_4 = \mathcal{D}I_1$, $I_5 = \mathcal{D}I_2$, $I_6 = \mathcal{D}I_3$, $I_7 = \mathcal{D}I_4$ with the operator $\mathcal{D} = i_0^{-1/4}D_t$. Three fourth-order invariants I_5, I_6, I_7 are obtained by invariant differentiation of three third-order invariants I_2, I_3, I_4. If we further extend operator (A.3) to the n-th order derivatives of functions p_j, q_j, l_j, m_j, $j = 1, 2$, we should split the invariance condition $\tilde{X}I = 0$ by functions τ, ρ_1, ρ_2, σ_1, σ_2 and their derivatives up to $(n+2)$-th order. This yields the system of $5(n+3)$ linear first-order PDEs $X_1I = 0, \ldots, X_{5n+15}I = 0$. In $(8n+11)$-dimensional space of variables t, x, y, $p_1, \ldots, m_2^{(n)}$ its solution provides $3n - 4$ independent invariants up to n-th order. Similarly, at the next step one can obtain

$3(n+1)-4$ independent invariants up to $(n+1)$-th order. Hence, at every step one obtains three n-th order invariants. This coincides with the number of invariants, which one can find by invariant differentiation of three $(n-1)$-th order invariants obtained at the previous step. Therefore, invariants (14) form the basis for a linear system (1) of the first type (with $i_0 \neq 0$).

(2) Operators (A.9) leave invariant ten equations

$$\begin{array}{lllll} j_0 = 0, & j_1 = 0, & i_0 = 0, & i_1 = 0, & i_2 = 0, \\ i_3 = 0, & i_4 = 0, & i_5 = 0, & i_6 = 0, & i_7 = 0, \end{array} \tag{A.14}$$

as well as four systems of equations

$$\begin{array}{ll} \alpha_0,\ \alpha_1,\ \alpha_2 = 0; & \alpha_0,\ \alpha_1,\ \alpha_2,\ \beta_0,\ \beta_1,\ \beta_2,\ \gamma_0,\ \gamma_1,\ \gamma_2 = 0; \\ \alpha_0,\ \alpha_1,\ \alpha_2,\ \beta_0,\ \beta_1,\ \beta_2 = 0; & \alpha_i,\ \beta_i,\ \gamma_i,\ \varepsilon_i = 0, \quad i = 0,1,2. \end{array} \tag{A.15}$$

The invariance condition of equations (A.14) and systems (A.15) under remaining operators from system (A.6) holds identically. Linear systems can be separated into different equivalence classes with respect to invariant manifolds defined by (A.14), (A.15). For example, if i_0 is nonzero for a given system (1), then no change of variables (2) can transform it to a linear system (3) with $i_0 = 0$. To find invariants in degenerate case $i_0 = 0$, we use the approach proposed in [10]. Following it we restrict the action of operators in system (A.6) on the corresponding invariant manifold ($i_0 = 0$).

Let $i_0 = 0$ and one or more relative invariants α_i, $i = 0,1,2$ be nonzero. Both α_1, α_2 cannot be equal to zero, since from $\alpha_1 = 0$, $\alpha_2 = 0$, $i_0 = 0$ it follows $\alpha_0 = 0$. Assume $\alpha_2 \neq 0$, then the relations

$$\begin{aligned} &\alpha_1 = -\frac{\alpha_0^2}{\alpha_2}, \qquad \beta_1 = \frac{\alpha_0^2}{\alpha_2^2}\beta_2 - \frac{2\alpha_0}{\alpha_2}\beta_0, \qquad \gamma_1 = \frac{\alpha_0^2}{\alpha_2^2}\gamma_2 - \frac{2\alpha_0}{\alpha_2}\gamma_0 - \frac{2}{\alpha_2^3}(\alpha_0\beta_2 - \alpha_2\beta_0)^2, \\ &\varepsilon_1 = \frac{\alpha_0^2}{\alpha_2^2}\varepsilon_2 - \frac{2\alpha_0}{\alpha_2}\varepsilon_0 + \frac{6}{\alpha_2^3}(\alpha_0\beta_2 - \alpha_2\beta_0)(\alpha_2\gamma_0 - \alpha_0\gamma_2) + \frac{6\beta_2}{\alpha_2^4}(\alpha_0\beta_2 - \alpha_2\beta_0)^2, \end{aligned} \tag{A.16}$$

follow from $i_0 = 0$ and (8)–(10). Hence, operators (A.9) act on variables

$$x, \quad y, \quad \alpha_0, \quad \alpha_2, \quad \beta_0, \quad \beta_2, \quad \gamma_0, \quad \gamma_2, \quad \varepsilon_0, \quad \varepsilon_2,$$

and we should substitute (A.16) into the corresponding coordinates of operators (A.9). In this space of variables operator X_7 has nine invariants

$$x, \quad y, \quad \alpha_0, \quad \alpha_2, \quad B_1, \quad \Gamma_2, \quad G_1, \quad e_1, \quad E_2. \tag{A.17}$$

In variables (A.17) operators X_4, X_5, X_6 become

$$\begin{aligned} X_4 &= \tfrac{1}{2}(x\partial_x - y\partial_y) - \alpha_2\partial_{\alpha_2} - B_1\partial_{B_1} - 2\Gamma_2\partial_{\Gamma_2} - G_1\partial_{G_1} - e_1\partial_{e_1} - 3E_2\partial_{E_2}, \\ X_5 &= y\partial_x + \alpha_2\partial_{\alpha_0} + 2\Gamma_2\partial_{G_1}, \\ X_6 &= x\partial_y + \frac{\alpha_0^2}{\alpha_2}\partial_{\alpha_0} + 2\alpha_0\partial_{\alpha_2} + \frac{2\alpha_0}{\alpha_2}B_1\partial_{B_1} + 2G_1\partial_{\Gamma_2} + \frac{3}{\alpha_2^2}(2\alpha_0(\alpha_2G_1 - \alpha_0\Gamma_2) - B_1^2)\partial_{G_1} \\ &\quad + \left(\frac{2\alpha_0}{\alpha_2}e_1 + \frac{6B_1}{\alpha_2^3}(\alpha_2G_1 - 2\alpha_0\Gamma_2)\right)\partial_{e_1} + \frac{2}{\alpha_2}(3\alpha_0E_2 + \alpha_2^2e_1 - 9B_1\Gamma_2)\partial_{E_2}, \end{aligned}$$

and have six independent invariants

$$J_0 = (x\alpha_2 - y\alpha_0)\alpha_2^{-1/2}, \quad J_1 = B_1\alpha_2^{-1}, \quad K_0 = k_0\alpha_2^{-2}, \quad K_1 = k_1\alpha_2^{-4}, \quad K_2 = k_2\alpha_2^{-3}, \quad K_3 = k_3\alpha_2^{-6},$$

where $k_3 = B_1^3E_2 + \frac{2}{3}K(B_1k_2 - 4k_1 - 2B_1^2\Gamma_2)$ and k_0, k_1, k_2 are defined by (13). In these variables the remaining operators (A.9) take the form

$$X_2 = -J_0\partial_{J_0} - 3J_1\partial_{J_1} - 4K_0\partial_{K_0} - 8K_1\partial_{K_1} - 5K_2\partial_{K_2} - 12K_3\partial_{K_3}, \qquad X_3 = \frac{1}{2}J_0\partial_{J_0},$$

and have four independent invariants (15) and $I_3 = k_3\alpha_2^{-2}B_1^{-4}$ when $B_1 \neq 0$. It is readily seen that operators $X_2, \ldots, X_7$ leave invariant the equation $B_1 = 0$.

As a solution of system (A.13) one can take $f = J_1^{-1/3}$. Then with the operator $\mathcal{D} = \alpha_2^{1/3}B_1^{-1/3}D_t$ the relation $I_3 = \mathcal{D}I_1$ hold. Note that in this case n-th order extension of operator (A.3) leads again to the system of $5(n+3)$ PDEs $X_1 I = 0, \ldots, X_{5n+15}I = 0$. In restricted space of $7n+11$ variables it has $2n-4$ independent solutions. Hence, at every step one obtains two n-th order invariants of system (1). Invariant differentiation of two fourth-order invariants I_2, I_3 provides two fifth-order invariants and so forth. Therefore, invariants (15) form the basis for a linear system (1) of the second type.

(3) If $i_0 = 0$, $B_1 = 0$, $\alpha_2 \neq 0$, then we should substitute

$$\alpha_1 = -\frac{\alpha_0^2}{\alpha_2}, \qquad \beta_0 = \frac{\alpha_0}{\alpha_2}\beta_2, \qquad \beta_1 = -\frac{\alpha_0^2}{\alpha_2^2}\beta_2,$$
$$\gamma_0 = \frac{\alpha_0}{\alpha_2}\gamma_2, \qquad \gamma_1 = -\frac{\alpha_0^2}{\alpha_2^2}\gamma_2, \qquad \varepsilon_0 = \frac{\alpha_0}{\alpha_2}\varepsilon_2, \qquad \varepsilon_1 = -\frac{\alpha_0^2}{\alpha_2^2}\varepsilon_2,$$

into the coordinates of operators (A.9). In the space of seven variables $x, y, \alpha_0, \alpha_2, \beta_2, \gamma_2, \varepsilon_2$ they have one invariant (16) provided $\Gamma_2 \neq 0$. As a solution of system (A.13) we take $f = \alpha_2\Gamma_2^{-1/2}$. Extending operator (A.3) to the higher derivatives one can find relative invariants of the fifth order

$$\lambda_0 = \varepsilon_0' + \frac{1}{2}(q_2\varepsilon_1 - q_1\varepsilon_2) - 9\delta\gamma_0, \quad \lambda_j = \varepsilon_j' + (-1)^j(\frac{1}{2}(p_1 - p_2)\varepsilon_j - q_j\varepsilon_0) - 9\delta\gamma_j \quad (j = 1, 2),$$

of the sixth order

$$\mu_0 = \lambda_0' + \frac{1}{2}(q_2\lambda_1 - q_1\lambda_2) - 14\delta\varepsilon_0, \quad \mu_j = \lambda_j' + (-1)^j(\frac{1}{2}(p_1 - p_2)\lambda_j - q_j\lambda_0) - 14\delta\varepsilon_j \quad (j = 1, 2),$$

etc. Similarly to relative invariants (8)–(10), λ_0, λ_1 and μ_0, μ_1 are expressible in terms of α_0, α_2, λ_2, μ_2. It is not difficult to see that at every step operators

$$\begin{aligned}
&X_2 = -2\alpha_0\partial_{\alpha_0} - 2\alpha_2\partial_{\alpha_2} - 3\beta_2\partial_{\beta_2} - 4\gamma_2\partial_{\gamma_2} - 5\varepsilon_2\partial_{\varepsilon_2} - 6\lambda_2\partial_{\lambda_2} - 7\mu_2\partial_{\mu_2} - \ldots,\\
&X_3 = \tfrac{1}{2}(x\partial_x + y\partial_y), \qquad X_5 = y\partial_x + \alpha_2\partial_{\alpha_0},\\
&X_4 = \tfrac{1}{2}(x\partial_x - y\partial_y) - \alpha_2\partial_{\alpha_2} - \beta_2\partial_{\beta_2} - \gamma_2\partial_{\gamma_2} - \varepsilon_2\partial_{\varepsilon_2} - \lambda_2\partial_{\lambda_2} - \mu_2\partial_{\mu_2} - \ldots,\\
&X_6 = x\partial_y + \alpha_0\alpha_2^{-1}(\alpha_0\partial_{\alpha_0} + 2\alpha_2\partial_{\alpha_2} + 2\beta_2\partial_{\beta_2} + 2\gamma_2\partial_{\gamma_2} + 2\varepsilon_2\partial_{\varepsilon_2} + 2\lambda_2\partial_{\lambda_2} + 2\mu_2\partial_{\mu_2} + \ldots),\\
&X_7 = -2\alpha_2\partial_{\beta_2} - 5\beta_2\partial_{\gamma_2} - 9\gamma_2\partial_{\varepsilon_2} - 14\varepsilon_2\partial_{\lambda_2} - 20\lambda_2\partial_{\mu_2} - \ldots
\end{aligned}$$

have one n-th order invariant. Namely, fourth-order invariant (16), an invariant $I_2 = \alpha_2^2\Gamma_2^{-2}(\alpha_2\lambda_2 - 7\beta_2\varepsilon_2 + \frac{63}{10}\gamma_2^2)$ of the fifth order, an invariant $I_3 = \alpha_2^2\Gamma_2^{-5/2}(\alpha_2^2\mu_2 - 10\alpha_2\beta_2\lambda_2 + \frac{28}{5}\alpha_2\gamma_2\varepsilon_2 + 28\beta_2^2\varepsilon_2 - \frac{126}{5}\beta_2\gamma_2^2)$ of the sixth order, etc. These invariants can be obtained from invariant I_1, since they satisfy the relations $I_2 = \frac{3}{2}I_1^2 + \frac{54}{5} + \mathcal{D}I_1$, $I_3 = 2I_1I_2 + \mathcal{D}I_2$ with the operator $\mathcal{D} = \alpha_2\Gamma_2^{-1/2}D_t$. Therefore, invariant (16) forms the basis for a linear system (1) of the third type.

(4) When $i_0 = 0$, $B_1 = 0$, $\Gamma_2 = 0$, $\alpha_2 \neq 0$ one can express in terms of α_0, α_2, β_2 all other relative invariants (8)–(10),

$$\alpha_1 = -\frac{\alpha_0^2}{\alpha_2}, \qquad \beta_0 = \frac{\alpha_0}{\alpha_2}\beta_2, \qquad \beta_1 = -\frac{\alpha_0^2}{\alpha_2^2}\beta_2,$$
$$\gamma_0 = \frac{5}{4}\frac{\alpha_0}{\alpha_2^2}\beta_2^2, \qquad \gamma_1 = -\frac{5}{4}\frac{\alpha_0^2}{\alpha_2^3}\beta_2^2, \qquad \gamma_2 = \frac{5}{4}\frac{\beta_2^2}{\alpha_2}, \qquad \varepsilon_i = \frac{3}{2}\frac{\beta_2}{\alpha_2}\gamma_i,$$

as well as all relative invariants of higher order,

$$\lambda_i = \frac{21}{8}\frac{\beta_2^2}{\alpha_2^2}\gamma_i, \qquad \mu_i = \frac{21}{4}\frac{\beta_2^3}{\alpha_2^3}\gamma_i \qquad (i = 0, 1, 2),$$

etc. Thus, whatever the extension of operator (A.3) we calculate, operators (A.9) will act on the space of five variables x, y, α_0, α_2, β_2. In this space of variables operators X_4, X_5, X_6, X_7 have one invariant $J_0 = (x\alpha_2 - y\alpha_0)\alpha_2^{-1/2}$ and then operators X_2, X_3 become $X_2 = -J_0\partial_{J_0}$, $X_3 = \frac{1}{2}J_0\partial_{J_0}$. It is readily seen that system $X_2 I = 0$, $X_3 I = 0$ has only trivial solution $I = \text{const}$ and equation $J_0 = 0$ is invariant under operators $X_2, \ldots, X_7$.

(5) Statement of the theorem concerning systems (1) with $\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$ follows immediately from results of [4]. This concludes the proof.