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Differential Invariants and First Integrals of the System of Two Linear Second-Order Ordinary Differential Equations

Yu. Yu. Bagderina[a],*

[a]Institute of Mathematics with Computer Center of Russian Academy of Sciences, Russia.

* Corresponding author.
Address: Institute of Mathematics with Computer Center of Russian Academy of Sciences, Russia; E-Mail: yulya@mail.rb.ru

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Abstract: In a recent paper the basis of algebraic invariants of the system of two linear second-order ordinary differential equations has been found. Now we obtain the differential invariants for this family of equations, which depend on the first-order derivatives. It is shown that the first integrals of such systems can be sought as the functions of the algebraic and differential invariants of a given system. Differential invariants can be useful also in constructing the transformation connecting two equivalent systems when their algebraic invariants are constant.

Key words: Differential invariant; Equivalence; Linear equations; First integral; Invariant

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1. INTRODUCTION

Systems of linear second-order ordinary differential equations (ODEs) have attracted a considerable interest since they find many applications in mechanics, physics, chemistry. A lot of papers have been devoted to the group classification of such systems [1]–[7], their equivalence [8,9], the dimension of an admitted Lie algebra [6,7,10]–[13]. The most studied is the case of the system of two linear equations of the second order

$$\begin{cases} x'' = p_1(t)x' + q_1(t)y' + l_1(t)x + m_1(t)y, \\ y'' = p_2(t)y' + q_2(t)x' + l_2(t)y + m_2(t)x. \end{cases} \tag{1.1}$$

This family of equations is closed with respect to a point change of variables

$$\tilde{t} = \theta(t), \quad \tilde{x} = \phi_{11}(t)x + \phi_{12}(t)y, \quad \tilde{y} = \phi_{21}(t)x + \phi_{22}(t)y, \quad \det\|\phi_{ij}(t)\| \neq 0. \tag{1.2}$$

Note that linear systems often have a nonlinear appearance in applications due to an unconvenient choice of the coordinates. One can linearize them using, for example, approaches of [14],[15]. In order to obtain the simplest form which is possible for a given system, one can use the invariants of linear systems. In [9] the equivalence problem has been solved for system (1.1) with respect to the transformations (1.2), which form the group E of the equivalence transformations of system (1.1). As (absolute) invariants of system (1.1) we name the invariants of the group E. As relative invariants of system (1.1) we name the invariants of a subgroup of E. If an invariant I is a function of the variables t, x, y only, then we name it an algebraic one. If an invariant depends on t, x, y, x', y', we name it a differential one. Differential invariants which depend on the derivatives of x, y of higher order are not considered, because the derivatives of higher order can be replaced by virtue of the system (1.1).

In [9] it has been found that system (1.1) can possess the algebraic invariants of two kinds. Namely, infinitely many invariants $I_j(t)$, $j \geq 1$, which depend on the variable t only, and an invariant $I_0(t, x, y)$ having the form of ratio of two homogeneous polynomials in x, y with the coefficients depending on t. In this set of the invariants there exists the finite basis. Arbitrary invariant $I(t)$ can be obtained from the basis invariants by algebraic operations and applying the operator $\mathcal{D}$ of invariant differentiation. The operator $\mathcal{D}$ is found from the condition that if $I_j(t)$ is an invariant of the system (1.1), then $\mathcal{D}I_j$ is its invariant too. Algebraic invariants of the form $I_j(t)$, $j \geq 1$ obtained in [9] in generic case of the system (1.1) coincide with the invariants constructed in [8]. Note that the invariants, which depend on x, y, have not been studied in [8]. Also the degenerate cases of system (1.1) have not been considered there. They were considered in detail in [9]. Differential invariants of system (1.1) have not been studied up to now.

In [9] it is shown how one can use the algebraic invariants in solving the equivalence problem. If two given systems (1.1) and

$$\begin{cases} \tilde{x}'' = \tilde{p}_1(\tilde{t})\tilde{x}' + \tilde{q}_1(\tilde{t})\tilde{y}' + \tilde{l}_1(\tilde{t})\tilde{x} + \tilde{m}_1(\tilde{t})\tilde{y}, \\ \tilde{y}'' = \tilde{p}_2(\tilde{t})\tilde{y}' + \tilde{q}_2(\tilde{t})\tilde{x}' + \tilde{l}_2(\tilde{t})\tilde{y} + \tilde{m}_2(\tilde{t})\tilde{x}, \end{cases} \tag{1.3}$$

are equivalent, then all their invariants coincide:

$$I_0(t, x, y) = \tilde{I}_0(\tilde{t}, \tilde{x}, \tilde{y}), \quad I_1(t) = \tilde{I}_1(\tilde{t}), \quad I_2(t) = \tilde{I}_2(\tilde{t}), \quad I_3(t) = \tilde{I}_3(\tilde{t}), \tag{1.4}$$

where I_0, I_j, $j \geq 1$ are the invariants of system (1.1) and $\tilde{I}_0$, $\tilde{I}_j$, $j \geq 1$ are the invariants of system (1.3). This means that the system of algebraic equalities (1.4) is consistent. Then the transformation (1.2) connecting two equivalent systems (1.1) and (1.3) can be found with the use of the relations (1.4). However, the equalities (1.4) turn out to be useless when all algebraic invariants of the systems (1.1), (1.3) are constant.

In a recent paper [16] the scalar second-order ODE with the cubic nonlinearity in the first-order derivative is considered. It is proposed there to construct the differential invariants in addition to algebraic ones. Then (1.4) can be supplemented by the equalities with the differential invariants. They allow one to find the change of variables in the case of constant algebraic invariants.

So, in Section 2 of the present paper using Lie's infinitesimal method [17,18] we obtain the basis of algebraic and differential invariants of system (1.1) and the operators of invariant differentiation as well. Differential invariants $J_i(t,x,y,x',y')$, $i=1,2$ have the form of ratio of two homogeneous polynomials in x, y, x', y' with the coefficients depending on t. Furthermore, one of the invariant differentiation operators, $\mathcal{D}_0$, up to a multiplier coincides with the operator

$$D_0=\partial_t+x'\partial_x+y'\partial_y+(p_1x'+q_1y'+l_1x+m_1y)\partial_{x'}+(p_2y'+q_2x'+l_2y+m_2x)\partial_y \quad (1.5)$$

of total differentiation by virtue of system (1.1).

For the invariants $I_0(t,x,y)$, $J_i(t,x,y,x',y')$, $i=1,2$ we show that the derived invariants $\mathcal{D}_0I_0$, $\mathcal{D}_0J_1$, $\mathcal{D}_0J_2$ are the functions of the invariants I_0, J_1, J_2 and the algebraic invariants $I_j(t)$, $\mathcal{D}_0I_j$, $j\geq 1$. In Section 3 we suggest to find the first integral of the system (1.1) $F(t,x,y,x',y')=C$ as a function of algebraic and differential invariants. Thus, we have either $F=F(I_a,I_0,J_1,J_2)$, where $I_a(t)$ is a nonconstant algebraic invariant of the system (1.1), or $F=F(I_0,J_1,J_2)$ when all invariants I_j, $j\geq 1$ are constant. This approach allows one to obtain some first integrals of the system (1.1).

In Section 4 example of apllication of the differential invariants in constructing the first integrals of linear systems is given. Also we provide the examples of finding the equivalent cases in the group classification results [1]–[5] when all algebraic invariants of the systems considered are constant. Concluding remarks are presented in Section 5.

2. DIFFERENTIAL AND ALGEBRAIC INVARIANTS OF LINEAR SYSTEM

Here we use the following notation from [9] for the relative invariants of system (1.1) of the first order

$$\alpha_0=\frac{1}{4}(p_1'-p_2')+\frac{1}{2}(l_2-l_1)+\frac{1}{8}(p_2^2-p_1^2),\quad \alpha_j=\frac{1}{2}q_j'-m_j-\frac{1}{4}(p_1+p_2)q_j,\quad j=1,2,$$

the second order

$$\begin{cases}\beta_0=\alpha_0'+\frac{1}{2}(q_2\alpha_1-q_1\alpha_2),\\ \beta_1=\alpha_1'+q_1\alpha_0+\frac{1}{2}(p_2-p_1)\alpha_1,\\ \beta_2=\alpha_2'-q_2\alpha_0+\frac{1}{2}(p_1-p_2)\alpha_2,\end{cases}$$

the third order

$$\begin{cases}\gamma_0=\beta_0'+\frac{1}{2}(q_2\beta_1-q_1\beta_2)-2\delta\alpha_0,\\ \gamma_1=\beta_1'+q_1\beta_0+\frac{1}{2}(p_2-p_1)\beta_1-2\delta\alpha_1,\\ \gamma_2=\beta_2'-q_2\beta_0+\frac{1}{2}(p_1-p_2)\beta_2-2\delta\alpha_2,\end{cases}$$

where $\delta = \frac{1}{2}(p_1' + p_2' - q_1q_2) - l_1 - l_2 - \frac{1}{4}(p_1^2 + p_2^2)$, the fourth order

$$\begin{cases} \varepsilon_0 = \gamma_0' + \frac{1}{2}(q_2\gamma_1 - q_1\gamma_2) - 5\delta\beta_0, \\ \varepsilon_1 = \gamma_1' + q_1\gamma_0 + \frac{1}{2}(p_2 - p_1)\gamma_1 - 5\delta\beta_1, \\ \varepsilon_2 = \gamma_2' - q_2\gamma_0 + \frac{1}{2}(p_1 - p_2)\gamma_2 - 5\delta\beta_2, \end{cases}$$

and their combinations

$$\begin{array}{lll} B_0 = \alpha_1\beta_2 - \alpha_2\beta_1, & B_1 = \alpha_2\beta_0 - \alpha_0\beta_2, & B_2 = \alpha_0\beta_1 - \alpha_1\beta_0, \\ G_0 = \alpha_1\gamma_2 + \alpha_2\gamma_1 - \frac{5}{2}\beta_1\beta_2, & \Gamma_k = \alpha_k\gamma_k - \frac{5}{4}\beta_k^2, \quad k = 0, 1, 2, \\ G_1 = \alpha_2\gamma_0 + \alpha_0\gamma_2 - \frac{5}{2}\beta_0\beta_2, & e_1 = \alpha_2\varepsilon_0 - \alpha_0\varepsilon_2 + \frac{9}{2}(\beta_0\gamma_2 - \beta_2\gamma_0), \end{array}$$

$$\begin{array}{lll} i_0 = \alpha_0^2 + \alpha_1\alpha_2, & j_4 = y^2\alpha_1 + 2xy\alpha_0 - x^2\alpha_2, & j_5 = y^2B_2 + xyB_0 - x^2B_1, \\ i_1 = B_0^2 + 4B_1B_2, & i_2 = 2\Gamma_0 + G_0, & i_3 = B_0\gamma_0 + B_1\gamma_1 + B_2\gamma_2, \end{array}$$

$$\begin{array}{lll} E_2 = \alpha_2^2\varepsilon_2 - \frac{9}{2}\alpha_2\beta_2\gamma_2 + \frac{15}{4}\beta_2^3, & K = \alpha_2G_1 - 2\alpha_0\Gamma_2, & j_0 = x\alpha_2 - y\alpha_0, \\ k_0 = K + 3yB_1^2 j_0^{-1}, & k_1 = \frac{1}{3}K^2 + B_1^2\Gamma_2, & k_2 = \alpha_2^2 e_1 - 3B_1\Gamma_2. \end{array}$$

We use also the relative invariants which depend on x', y'

$$\begin{cases} \delta_1 = x' - \frac{1}{2}(p_1x + q_1y), \\ \delta_2 = y' - \frac{1}{2}(p_2y + q_2x), \end{cases} \tag{2.1}$$

their combinations

$$j_1 = \alpha_2\delta_1 - \alpha_0\delta_2 + \frac{\beta_2 j_0}{4\alpha_2} - \frac{yB_1}{2\alpha_2}, \qquad j_2 = y\delta_1 - x\delta_2,$$
$$j_3 = (\alpha_0y - \alpha_2x)\delta_1 + (\alpha_0x + \alpha_1y)\delta_2 + \frac{i_0' j_4}{8i_0}$$

and the auxiliary operators (1.5) and

$$D_1 = x\partial_x + y\partial_y + x'\partial_{x'} + y'\partial_{y'}, \qquad D_3 = x\partial_{x'} + y\partial_{y'}, \qquad D_4 = \alpha_0\partial_{x'} + \alpha_2\partial_{y'},$$
$$D_2 = \alpha_0 j_0 \left(\partial_x + \frac{p_1}{2}\partial_{x'} + \frac{q_2}{2}\partial_{y'}\right) + \alpha_2 j_0 \left(\partial_y + \frac{q_1}{2}\partial_{x'} + \frac{p_2}{2}\partial_{y'}\right) + (\alpha_2\delta_1 - \alpha_0\delta_2)D_4.$$

The theorem below describes the invariants of the system (1.1) which depend on t, x, y, x', y'. Note that the algebraic invariant $I_0(t, x, y)$ and the differential invariants $J_k(t, x, y, x', y')$, $k = 1, 2$, for system (1.1) of the first and the second type are independent with respect to x, y, x', y'. This means that the condition

$$\operatorname{rank} \left\| \frac{\partial(I_0, J_1, J_2)}{\partial(x, y, x', y')} \right\| = 3$$

hold. But the invariants J_1, J_2 can be obtained by applying the operator $\mathcal{D}_0$ to the invariant I_0 (see (2.5), (2.9)). So, the invariants J_1, J_2 are not included to the basis and they are given as additional ones. Similarly, the invariant J_1 of system (1.1) of the third type can be obtained from J_2 (see (2.13)) and it is not included to the basis.

Theorem 1. *Every system (1.1) of two linear second-order ODEs belongs to one of five types of linear systems. For each type of the system the basis of algebraic and differential invariants, the operators $\mathcal{D}_0, \ldots, \mathcal{D}_4$ of invariant differentiation and additional independent invariants are defined by the following formulas:*

(I) Systems of the first type ($i_0 \neq 0$) have four basis invariants

$$I_0 = j_5 j_4^{-1} i_0^{-3/4}, \qquad I_1 = i_1 i_0^{-5/2}, \qquad I_2 = i_2 i_0^{-3/2}, \qquad I_3 = i_3 i_0^{-9/4}, \tag{2.2}$$

$$\mathcal{D}_0 = i_0^{-1/4} D_0, \qquad \mathcal{D}_1 = D_1, \qquad \mathcal{D}_4 = i_0^{-1/4}[(\alpha_0 x + \alpha_1 y)\partial_{x'} + (\alpha_2 x - \alpha_0 y)\partial_{y'}],$$

$$\begin{aligned} \mathcal{D}_2 = i_0^{-1/2}\Big[&(\alpha_0 x + \alpha_1 y)\left(\partial_x + \frac{p_1}{2}\partial_{x'} + \frac{q_2}{2}\partial_{y'}\right) \\ &+(\alpha_2 x - \alpha_0 y)\left(\partial_y + \frac{q_1}{2}\partial_{x'} + \frac{p_2}{2}\partial_{y'}\right) \\ &+(\alpha_0\delta_1 + \alpha_1\delta_2)\partial_{x'} + (\alpha_2\delta_1 - \alpha_0\delta_2)\partial_{y'}\Big], \\ \mathcal{D}_3 = i_0^{1/4} D_3,& \end{aligned} \tag{2.3}$$

additional invariants are

$$\begin{cases} J_1 = j_3 j_4^{-1} i_0^{-1/4}, \\ J_2 = j_2 j_4^{-1} i_0^{1/4}. \end{cases} \tag{2.4}$$

Nontrivial action of operators (2.3) on I_0, J_1, J_2 is given by

$$\begin{aligned} \mathcal{D}_0 J_1 = \frac{1}{8}I_2 - \frac{9}{64}I_1 - I_0 J_2 - J_2^2 - J_1^2, \qquad \mathcal{D}_0 J_2 = -1 - (J_0 + 2J_1)J_2, \\ \mathcal{D}_2 I_0 = -2J_0, \qquad \mathcal{D}_3 J_1 = 1, \qquad \mathcal{D}_4 J_2 = 1, \\ \mathcal{D}_0 I_0 = J_0\left(2\frac{I_3}{I_1} - I_0 - 2J_2\right) + \frac{I_0}{2I_1}\mathcal{D}_0 I_1, \qquad if \qquad I_1 \neq 0, \\ \mathcal{D}_0 I_0 = \kappa I_0(M - I_0 - 2J_2), \qquad if \qquad I_1 = 0, \qquad \alpha_2 \neq 0, \qquad B_1 \neq 0, \end{aligned} \tag{2.5}$$

where $J_0^2 = I_0^2 + \frac{1}{4}$, $M = i_0^{-3/4}\left(\frac{i_0\Gamma_2}{\alpha_2 B_1} - \frac{\alpha_2 i_2}{2B_1} + \frac{5B_1}{4\alpha_2}\right)$, $\kappa = \frac{\alpha_2 B_0 - 2\alpha_0 B_1}{2\sqrt{i_0}B_1}$.

(II) Systems of the second type ($i_0 = 0$, $\alpha_2 \neq 0$, $B_1 \neq 0$) have three basis invariants

$$I_0 = k_0\alpha_2^{-2/3}B_1^{-4/3}, \qquad I_1 = k_1\alpha_2^{-4/3}B_1^{-8/3}, \qquad I_2 = k_2\alpha_2^{-4/3}B_1^{-5/3}, \tag{2.6}$$

$$\begin{aligned} \mathcal{D}_0 = \alpha_2^{1/3}B_1^{-1/3}D_0, \qquad \mathcal{D}_1 = D_1, \qquad \mathcal{D}_2 = \alpha_2^{-1/3}B_1^{-2/3}D_2, \\ \mathcal{D}_3 = \alpha_2^{-1/3}B_1^{1/3}D_3, \qquad \mathcal{D}_4 = j_0\alpha_2^{-2/3}B_1^{-1/3}D_4, \end{aligned} \tag{2.7}$$

additional invariants are

$$\begin{cases} J_1 = j_1 j_0^{-1}\alpha_2^{1/3}B_1^{-1/3}, \\ J_2 = j_2 j_0^{-2}\alpha_2^{2/3}B_1^{1/3}. \end{cases} \tag{2.8}$$

Nontrivial action of operators (2.7) on I_0, J_1, J_2 is given by

$$\begin{aligned} &\mathcal{D}_0 I_0 = I_2 - 5I_1 + \frac{1}{3}I_0^2 - 3J_2, \quad \mathcal{D}_2 I_0 = 3, \\ &\mathcal{D}_0 J_1 = \frac{1}{4}I_1 - \frac{1}{12}I_0^2 - \frac{1}{3}I_0 J_1 + \frac{3}{2}J_2 - J_1^2, \\ &\mathcal{D}_0 J_2 = 1 + \frac{1}{3}I_0 J_2 - 2J_1 J_2, \\ &\mathcal{D}_2 J_1 = -\frac{1}{2}, \qquad \mathcal{D}_3 J_1 = 1, \qquad \mathcal{D}_4 J_2 = -1. \end{aligned} \tag{2.9}$$

(III) Systems of the third type ($i_0 = 0$, $\alpha_2 \neq 0$, $B_1 = 0$, $\Gamma_2 \neq 0$) have two basis invariants

$$\begin{cases} I_1 = E_2\Gamma_2^{-3/2}, \\ J_2 = j_2 j_0^{-2}\sqrt{\Gamma_2}, \end{cases} \tag{2.10}$$

$$\begin{aligned} &\mathcal{D}_0 = \alpha_2\Gamma_2^{-1/2}D_0, \qquad \mathcal{D}_1 = D_1, \qquad \mathcal{D}_2 = \alpha_2\Gamma_2^{-1}D_2, \\ &\mathcal{D}_3 = \alpha_2^{-1}\sqrt{\Gamma_2}D_3, \qquad \mathcal{D}_4 = j_0\Gamma_2^{-1/2}D_4, \end{aligned} \tag{2.11}$$

additional invariant is

$$J_1 = j_1 j_0^{-1}\alpha_2\Gamma_2^{-1/2}. \tag{2.12}$$

Nontrivial action of operators (2.11) on J_1, J_2 is given by

$$\begin{aligned} &\mathcal{D}_0 J_1 = \frac{1}{4} - \frac{1}{2}I_1J_1 - J_1^2, \qquad \mathcal{D}_0 J_2 = 1 + \frac{1}{2}I_1J_2 - 2J_1J_2, \\ &\mathcal{D}_3 J_1 = 1, \qquad \mathcal{D}_4 J_2 = -1. \end{aligned} \tag{2.13}$$

(IV) Systems of the fourth type ($i_0 = 0$, $\alpha_2 \neq 0$, $B_1 = 0$, $\Gamma_2 = 0$) have one basis invariant

$$J_1 = \alpha_2 j_1 j_2 j_0^{-3}, \tag{2.14}$$

$$\begin{aligned} &\mathcal{D}_0 = j_0 j_1^{-1}D_0, \qquad \mathcal{D}_1 = D_1, \qquad \mathcal{D}_2 = j_0^2\alpha_2^{-1}j_1^{-2}D_2, \\ &\mathcal{D}_3 = j_1 j_0^{-1}D_3, \qquad \mathcal{D}_4 = j_0^2\alpha_2^{-1}j_1^{-1}D_4. \end{aligned} \tag{2.15}$$

Nontrivial action of operators (2.15) on J_1 is given by

$$\mathcal{D}_0 J_1 = 1 - 3J_1, \qquad \mathcal{D}_3 J_1 = J_1, \qquad \mathcal{D}_4 J_1 = -1. \tag{2.16}$$

(V) Systems of the fifth type ($\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$) are equivalent to the simplest system $x'' = 0$, $y'' = 0$.

The proof of this statement is given in Appendix.

Remark. In items (I)–(IV) of Theorem 1 the systems with $\alpha_2 \neq 0$ are considered. Systems with $\alpha_2 = 0$, $\alpha_1 \neq 0$ are reduced to this case by a simple transformation

$$\tilde{t} = t, \qquad \tilde{x} = y, \qquad \tilde{y} = x,$$

while the transformation

$$\tilde{t} = t, \qquad \tilde{x} = \frac{1}{2}(x + y), \qquad \tilde{y} = \frac{1}{2}(x - y)$$

reduces similarly the systems with $\alpha_2 = 0$, $\alpha_1 = 0$, $\alpha_0 \neq 0$. So, we need not consider the cases $\alpha_2 = 0$, $\alpha_1 \neq 0$ and $\alpha_2 = 0$, $\alpha_1 = 0$, $\alpha_0 \neq 0$ separately.

3. FIRST INTEGRALS OF LINEAR SYSTEM

System of two second-order ODEs possesses four independent first integrals

$$F_i(t, x, y, x', y') = C_i, \quad C_i = \text{const}, \quad i = 1, 2, 3, 4.$$

Any linear system (1.1) admits Lie symmetry generator $X_0 = x\partial_x + y\partial_y + x'\partial_{x'} + y'\partial_{y'}$. Hence, three first integrals can be sought as a function of its invariants

$$F\left(t, \frac{y}{x}, \frac{x'}{x}, \frac{y'}{x}\right) = C \tag{3.1}$$

from the condition

$$D_0 F = 0 \tag{3.2}$$

with the operator (1.5). So, in order to find the integral (3.1), we need to solve the linear homogeneous first-order partial differential equation (PDE). Its characteristic system consists of three first-order ODEs.

Note that invariants I_0, J_1, J_2 defined by (2.2), (2.4), (2.6), (2.8), (2.10), (2.14) are the functions of the invariants t, yx^{-1}, $x'x^{-1}$, $y'x^{-1}$ of the generator X_0. And it is not difficult to see that the form of the operator $\mathcal{D}_0$ in (2.3), (2.7), (2.11), (2.15) allows us to find the first integral from the condition $\mathcal{D}_0 F = 0$. Thus, some first integrals of the system (1.1) can be sought in the form

$$F(I_a, I_0, J_1, J_2) = C \tag{3.3}$$

from the condition $\mathcal{D}_0 F = 0$. Here $I_a(t)$ is a nonconstant algebraic invariant of the system (1.1) and $\mathcal{D}_0 I_a$ is its invariant too. Expressions for $\mathcal{D}_0 I_0$, $\mathcal{D}_0 J_1$, $\mathcal{D}_0 J_2$ are given by (2.5), (2.9), (2.13), (2.16) for the corresponding types of the system (1.1). In general the characteristic system of the equation

$$\mathcal{D}_0 F \equiv \mathcal{D}_0 I_a \frac{\partial F}{\partial I_a} + \mathcal{D}_0 I_0 \frac{\partial F}{\partial I_0} + \mathcal{D}_0 J_1 \frac{\partial F}{\partial J_1} + \mathcal{D}_0 J_2 \frac{\partial F}{\partial J_2} = 0 \tag{3.4}$$

consists of three first-order ODEs as for the equation (3.2). However, the cases exist when the invariant form of the integral (3.3) implies that the number of equations in the characteristic system of PDE (3.4) reduces. And, therefore, this system becomes simpler for integration. Let us consider such cases in more detail.

N I.1. For arbitrary system (1.1) of the first type the characteristic system of PDE (3.4) consists of the equation

$$(\mathcal{D}_0 I_a)\frac{\mathrm{d}J_1}{\mathrm{d}I_a} = \frac{1}{8}I_2 - \frac{9}{64}I_1 - I_0 J_2 - J_2^2 - J_1^2$$

and the equations

$$(\mathcal{D}_0 I_a)\frac{\mathrm{d}J_2}{\mathrm{d}I_a} = -1 - (J_0 + 2J_1)J_2, \qquad (\mathcal{D}_0 I_a)\frac{\mathrm{d}I_0}{\mathrm{d}I_a} = J_0\left(2\frac{I_3}{I_1} - I_0 - 2J_2\right) + \frac{I_0}{2I_1}\mathcal{D}_0 I_1$$

when $I_1 \neq 0$, or the equations

$$(\mathcal{D}_0 I_a)\frac{\mathrm{d}J_2}{\mathrm{d}I_a} = -1 - (\kappa I_0 + 2J_1)J_2, \qquad (\mathcal{D}_0 I_a)\frac{\mathrm{d}I_0}{\mathrm{d}I_a} = \kappa I_0(M - I_0 - 2J_2)$$

when $I_1 = 0$, $\alpha_2 \neq 0$, $B_1 \neq 0$. It can be shown that the condition $I_1 = 0$ implies $I_3 = 0$. In this case the relative invariant i_1 becomes $i_1 = \alpha_2^{-2}[(\alpha_2 B_0 - 2\alpha_0 B_1)^2 - 4 i_0 B_1^2]$. Hence, parameter κ in (2.5) is equal to ± 1, when $I_1 = 0$. Below we consider the cases when the number of equations of the characteristic system can be reduced.

N I.2. For a system of the first type with the constant invariants I_1, I_2, I_3, where $I_1 \neq 0$, the first integral is sought in the form

$$F(I_0, J_1, J_2) = C \tag{3.5}$$

and the corresponding characteristic system consists of two equations

$$\frac{\mathrm{d}J_1}{\mathrm{d}I_0} = \frac{\frac{9}{64}I_1 - I_2/8 + I_0 J_2 + J_2^2 + J_1^2}{J_0(I_0 + 2J_2 - 2I_3 I_1^{-1})}, \qquad \frac{\mathrm{d}J_2}{\mathrm{d}I_0} = \frac{1 + (J_0 + 2J_1)J_2}{J_0(I_0 + 2J_2 - 2I_3 I_1^{-1})}.$$

N I.3. If the system of the first type has the invariants $I_1 = 0$, $I_2 = \mathrm{const}$, $I_3 = 0$ and its relative invariants satisfy the conditions $\alpha_2 \neq 0$, $B_1 \neq 0$, $M = \mathrm{const}$, then the first integral has the form (3.5) and the corresponding characteristic system consists of two equations

$$\frac{\mathrm{d}J_1}{\mathrm{d}I_0} = \frac{I_0 J_2 + J_2^2 + J_1^2 - I_2/8}{\kappa I_0(I_0 + 2J_2 - M)}, \qquad \frac{\mathrm{d}J_2}{\mathrm{d}I_0} = \frac{1 + (\kappa I_0 + 2J_1)J_2}{\kappa I_0(I_0 + 2J_2 - M)}.$$

N I.4. If the invariants of the system of the first type satisfy the conditions

$$I_1 = 0, \quad I_2 \neq \mathrm{const}, \quad I_3 = 0, \quad B_0 = 0, \quad B_1 = 0, \quad B_2 = 0$$

then we have $I_0 = 0$, $\mathcal{D}_0 J_1 = I_2/8 - J_2^2 - J_1^2$, $\mathcal{D}_0 J_2 = -1 - 2J_1 J_2$ from (2.5), the first integral is sought in the form

$$F(I_a, J_1, J_2) = C \tag{3.6}$$

and the characteristic system consists of two equations

$$\begin{cases} (\mathcal{D}_0 I_a)\frac{\mathrm{d}J_1}{\mathrm{d}I_a} = \frac{1}{8}I_2 - J_2^2 - J_1^2, \\ (\mathcal{D}_0 I_a)\frac{\mathrm{d}J_2}{\mathrm{d}I_a} = -1 - 2J_1 J_2. \end{cases} \tag{3.7}$$

Eliminating J_1 from equations (3.7) one obtains the second-order ODE

$$\frac{\mathrm{d}^2 J_2}{\mathrm{d}I_a^2} = \frac{3}{2J_2}\left(\frac{\mathrm{d}J_2}{\mathrm{d}I_a}\right)^2 + \frac{1}{\mathcal{D}_0 I_a}\left(\frac{2}{J_2} - \frac{\mathrm{d}(\mathcal{D}_0 I_a)}{\mathrm{d}I_a}\right)\frac{\mathrm{d}J_2}{\mathrm{d}I_a} + \frac{1}{(\mathcal{D}_0 I_a)^2}\left(2J_2^3 - \frac{I_2}{4}J_2 + \frac{1}{2J_2}\right). \tag{3.8}$$

If we choose such invariant I_a that $\mathcal{D}_0 I_a = \mathrm{const} \neq 0$, then the substitution

$$J_2 = -\frac{u}{u'\mathcal{D}_0 I_a}, \qquad u' = \frac{\mathrm{d}u}{\mathrm{d}I_a} \tag{3.9}$$

reduces (3.8) to the third-order ODE

$$2u'u''' - u''^2 - \frac{I_2}{2(\mathcal{D}_0 I_a)^2}u'^2 + \frac{4}{(\mathcal{D}_0 I_a)^4}u^2 = 0 \tag{3.10}$$

in $u(I_a)$, which linearizes on differentiation. Taking into account (3.9), the second equation (3.7) is solved for

$$J_1 = \frac{\mathcal{D}_0 I_a}{2}\frac{u''}{u'}. \tag{3.11}$$

N I.5. If the invariants of the system of the first type satisfy the conditions

$$I_1 = 0, \quad I_2 = \mathrm{const}, \quad I_3 = 0, \quad B_0 = 0, \quad B_1 = 0, \quad B_2 = 0$$

then the first integral has the form

$$F(J_1, J_2) = C \tag{3.12}$$

and we have a single characteristic equation, namely, the Abel equation

$$(1 + 2J_1J_2)\frac{\mathrm{d}J_1}{\mathrm{d}J_2} = J_1^2 + J_2^2 - \frac{1}{8}I_2.$$

N II.1. For arbitrary system (1.1) of the second type the characteristic system of PDE (3.4) consists of three equations

$$(\mathcal{D}_0 I_a)\frac{\mathrm{d}I_0}{\mathrm{d}I_a} = I_2 - 5I_1 + \frac{1}{3}I_0^2 - 3J_2, \qquad (\mathcal{D}_0 I_a)\frac{\mathrm{d}J_1}{\mathrm{d}I_a} = 1 + \frac{1}{3}I_0J_2 - 2J_1J_2,$$
$$(\mathcal{D}_0 I_a)\frac{\mathrm{d}J_2}{\mathrm{d}I_a} = \frac{1}{4}I_1 - \frac{1}{12}I_0^2 - \frac{1}{3}I_0J_1 + \frac{3}{2}J_2 - J_1^2.$$

The number of equations can be reduced in the following case.

N II.2. For a system of the second type with the constant invariants I_1, I_2 the first integral is sought in the form (3.5) and the corresponding characteristic system consists of two equations

$$\frac{\mathrm{d}J_1}{\mathrm{d}I_0} = \frac{1 + \frac{1}{3}I_0J_2 - 2J_1J_2}{I_2 - 5I_1 + \frac{1}{3}I_0^2 - 3J_2}, \qquad \frac{\mathrm{d}J_2}{\mathrm{d}I_0} = \frac{\frac{1}{4}I_1 - \frac{1}{12}I_0^2 - \frac{1}{3}I_0J_1 + \frac{3}{2}J_2 - J_1^2}{I_2 - 5I_1 + \frac{1}{3}I_0^2 - 3J_2}.$$

N III.1. For arbitrary system (1.1) of the third type the first integral has the form (3.6) and the characteristic system consists of two ODEs

$$(\mathcal{D}_0 I_a)\frac{\mathrm{d}J_1}{\mathrm{d}I_a} = \frac{1}{4} - \frac{1}{2}I_1J_1 - J_1^2, \qquad (\mathcal{D}_0 I_a)\frac{\mathrm{d}J_2}{\mathrm{d}I_a} = 1 + \frac{1}{2}I_1J_2 - 2J_1J_2,$$

where the first equation can be integrated separately.

N III.2. For a system of the third type with the constant invariant I_1 the first integral is sought in the form (3.12) and we have a single linear characteristic equation

$$\frac{\mathrm{d}J_2}{\mathrm{d}J_1} = \frac{2 + I_1J_2 - 4J_1J_2}{1/2 - I_1J_1 - 2J_1^2}. \tag{3.13}$$

N IV. For a system (1.1) of the fourth type the invariant form of the first integral is given by $F(J_1) = C$. The equality $\mathcal{D}_0 F \equiv (1 - 3J_1)F_{J_1} = 0$ has only the trivial solution $F \equiv \text{const}$. But if we suppose $1 - 3J_1 = 0$, then we obtain the relation

$$\Phi \equiv \alpha_2 j_1 j_2 - \frac{1}{3}j_0^3 = 0. \tag{3.14}$$

From the equality $D_0\Phi = \dfrac{3}{2}\left(\dfrac{3\beta_2}{2\alpha_2} + p_2 + q_2\dfrac{\alpha_0}{\alpha_2}\right)\Phi$ it follows $D_0\Phi\Big|_{\Phi=0} = 0$ and, therefore, the relation (3.14) defines an invariant manifold of the system (1.1) of the fourth type.

4. APPLICATIONS OF DIFFERENTIAL INVARIANTS

The first two examples below are devoted to the systems with constant algebraic invariants and to the linear systems of the fourth type which do not possess algebraic invariants. Of course, these systems are easily integrable. However, they arise in the group classification of linear systems, when one usually try to eliminate the equivalent cases. It is shown here how one can use the differential invariants for solving this problem. The last example illustrates application of the invariants of linear systems to finding the first integrals.

Example 1. In [2,3,5] the group classification of two linear equations with constant coefficients has been performed. In [5] the following linear system of the fourth type

$$x'' = 4x' - \frac{15}{4}x, \qquad y'' = b_{12}x + \frac{1}{4}y, \qquad b_{12} = \text{const} \neq 0 \tag{4.1}$$

has been found. Let us prove here that it is reducible to a simpler system of the fourth type

$$\tilde{x}'' = 0, \qquad \tilde{y}'' = \tilde{x} \tag{4.2}$$

obtained earlier in [2]. We suppose that systems (4.2), (4.1) are equivalent and equate the invariant

$$J_1 = \frac{2x' - 3x}{2b_{12}x^3}(xy' - yx' + 2xy)$$

of the system (4.1) to the invariant

$$\tilde{J}_1 = \frac{\tilde{x}'}{\tilde{x}^3}(\tilde{x}\tilde{y}' - \tilde{y}\tilde{x}')$$

of the system (4.2), where we substitute (1.2) and

$$\begin{cases} \tilde{x}' = \frac{1}{\theta'}(\phi_{11}x' + \phi_{12}y' + \phi'_{11}x + \phi'_{12}y), \\ \tilde{y}' = \frac{1}{\theta'}(\phi_{21}x' + \phi_{22}y' + \phi'_{21}x + \phi'_{22}y). \end{cases} \tag{4.3}$$

Equating like powers of x', y' in the relation $J_1 = \tilde{J}_1$ and studying further the conditions obtained we arrive at

$$\phi_{12} = 0, \qquad \left(\frac{\phi_{21}}{\phi_{11}}\right)' = 0, \qquad \frac{\phi'_{11}}{\phi_{11}} = -\frac{3}{2}, \qquad \frac{\phi'_{22}}{\phi_{22}} = \frac{1}{2}, \qquad \theta'^2 = b_{12}\frac{\phi_{22}}{\phi_{11}},$$

whence it follows

$$\begin{gathered} \tilde{t} = c_0\mathrm{e}^t + c_1, \qquad \tilde{x} = c_2x\mathrm{e}^{-3t/2}, \qquad \tilde{y} = c_3x\mathrm{e}^{-3t/2} + c_4y\mathrm{e}^{t/2}, \\ c_0, \ldots, c_4 = \text{const}, \qquad c_0^2c_2 = b_{12}c_4, \qquad c_0, c_2, c_4 \neq 0. \end{gathered} \tag{4.4}$$

It is readily verified that the change of variables (4.4) transforms (4.1) into (4.2).

Example 2. Consider an autonomous decoupled system obtained in [2,3]

$$\tilde{x}'' = \tilde{x}, \qquad \tilde{y}'' = k\tilde{y}, \qquad k = \text{const}, \qquad k \neq 1. \tag{4.5}$$

It is a system of the first type with the basis invariants

$$\tilde{I}_0 = 0, \qquad \tilde{I}_1 = 0, \qquad \tilde{I}_2 = \frac{8(k+1)}{k-1}, \qquad \tilde{I}_3 = 0. \tag{4.6}$$

Differential invariants (2.4) of the system (4.5) are given by

$$\begin{cases} \tilde J_1 = \frac{\tilde y\tilde x' + \tilde x\tilde y'}{\sqrt{2(k-1)\tilde x\tilde y}}, \\ \tilde J_2 = \frac{\tilde y\tilde x' - \tilde x\tilde y'}{\sqrt{2(k-1)\tilde x\tilde y}}. \end{cases} \tag{4.7}$$

In [1] the following system has been obtained (see the case I.9.2 there)

$$x'' = \frac{1}{t^2}(c_1x + c_2y), \qquad y'' = \frac{1}{t^2}(c_0x - c_1y), \qquad c_0, c_1, c_2 = \text{const} \neq 0. \tag{4.8}$$

Let $\Delta = c_0c_2 + c_1^2$ be nonzero, then (4.8) is a system of the first type with the invariants

$$I_0 = 0, \qquad I_1 = 0, \qquad I_2 = \frac{2}{\sqrt{\Delta}}, \qquad I_3 = 0, \tag{4.9}$$

$$\begin{cases} J_1 = \dfrac{t((c_1y - c_0x)x' + (c_1x + c_2y)y')}{\Delta^{1/4}(c_2y^2 + 2c_1xy - c_0x^2)}, \\ J_2 = \dfrac{\Delta^{1/4}t(xy' - yx')}{c_2y^2 + 2c_1xy - c_0x^2}. \end{cases} \tag{4.10}$$

Using the invariants one can find that systems (4.5) and (4.8) are equivalent when $\sqrt{\Delta} \neq 1/4$. In this case equalities (1.4) in algebraic invariants (4.6), (4.9) of systems (4.5), (4.8) are consistent. Then from (1.4) one can find the relation

$$k = \frac{1 + 4\sqrt{\Delta}}{1 - 4\sqrt{\Delta}} \tag{4.11}$$

only. In order to find the transformation (1.2) connecting (4.5) and (4.8), we substitute (1.2), (4.3) into invariants (4.7) and equate them to the corresponding invariants (4.10). Comparing like powers of x', y' in the equalities $J_1 = \tilde J_1$, $J_2 = \tilde J_2$ and equating then the coefficients of like powers of x, y, we obtain a number of relations. In particular, we have

$$\theta' = \frac{\sqrt{1 - 4\sqrt{\Delta}}}{2t}, \qquad \phi'_{ij} = -\frac{\phi_{ij}}{2t}, \qquad i, j = 1, 2. \tag{4.12}$$

Then the remaining relations become

$$\begin{aligned} &c_0(\phi_{11}\phi_{22} + \phi_{12}\phi_{21}) + 2c_1\phi_{11}\phi_{21} = 0, \quad c_0(\phi_{11}\phi_{22} - \phi_{12}\phi_{21}) - 2\sqrt{\Delta}\phi_{11}\phi_{21} = 0, \\ &c_2(\phi_{11}\phi_{22} + \phi_{12}\phi_{21}) - 2c_1\phi_{12}\phi_{22} = 0, \quad c_2(\phi_{11}\phi_{22} - \phi_{12}\phi_{21}) + 2\sqrt{\Delta}\phi_{12}\phi_{22} = 0, \\ &c_2\phi_{11}\phi_{21} + c_0\phi_{12}\phi_{22} = 0, \quad c_1(\phi_{11}\phi_{22} - \phi_{12}\phi_{21}) + \sqrt{\Delta}(\phi_{11}\phi_{22} + \phi_{12}\phi_{21}) = 0. \end{aligned} \tag{4.13}$$

Integration of (4.12), (4.13) provides the change of variables

$$\tilde t = \frac{1}{2}\sqrt{1 - 4\sqrt{\Delta}}\ln t,\ \tilde x = \frac{1}{\sqrt{t}}\left(c_0x - (c_1 + \sqrt{\Delta})y\right),\ \tilde y = \frac{1}{\sqrt{t}}\left((c_1 + \sqrt{\Delta})x + c_2y\right),$$

which transforms (4.8) to the system (4.5) with the parameter (4.11).

Example 3. In [19] the wave propagation on free surface of two-phase mixture has been studied. Solution of the form of damping wave reduces to the equations

$$(Nx')' - Nx = (My')' - My, \qquad -(Mx')' + Mx = (Ny')' - Ny \tag{4.14}$$

in $x(t)$, $y(t)$, where $M(t)$, $N(t)$ are real functions. If we denote $r = \sqrt{M^2+N^2}$, $\varphi = \arctan(N/M)$, $\Phi = \varphi'' + \varphi' r' r^{-1}$, then (4.14) become

$$\begin{cases} x'' = x - \frac{r'}{r}x' - \varphi' y', \\ y'' = y - \frac{r'}{r}y' + \varphi' x'. \end{cases} \tag{4.15}$$

For this system of the first type (when $\Phi \neq 0$) we have

$$B_0 = 0, \quad B_1 = 0, \quad B_2 = 0, \quad i_0 = -\frac{\Phi^2}{4}, \quad \mathcal{D}_0 = \sqrt{\frac{2}{i\Phi}} D_0, \quad i^2 = -1,$$
$$I_0 = 0, \quad I_1 = 0, \quad I_2 = \frac{i}{\Phi^3}\left[5\Phi'^2 - 4\Phi\Phi'' + 4\Phi^2\left(\varphi'^2 - 2\frac{r''}{r} + \frac{r'^2}{r^2} - 4\right)\right], \ I_3 = 0,$$
$$J_1 = \sqrt{\frac{2}{i\Phi}}\left(\frac{xx'+yy'}{x^2+y^2} + \frac{r'}{2r} + \frac{\Phi'}{4\Phi}\right), \quad J_2 = \sqrt{\frac{2i}{\Phi}}\left(\frac{xy'-yx'}{x^2+y^2} - \frac{\varphi'}{2}\right).$$

Hence, system (4.15) falls into the case *N I.4* described in the previous section.

Let, for instance, the parameters of system (4.15) be given by

$$r = \text{const} \neq 0, \qquad \varphi = \Phi = \lambda e^{\kappa t}, \qquad \kappa = \pm 1, \qquad \lambda = \text{const} \neq 0. \tag{4.16}$$

Then for system (4.15), (4.16) we have

$$\mathcal{D}_0 = \sqrt{\frac{2}{i\lambda}} e^{-\kappa t/2} D_0, \qquad I_2 = i\left(4\lambda e^{\kappa t} - \frac{15}{\lambda e^{\kappa t}}\right).$$

If we take $I_a = \lambda e^{\kappa t/2}$, then

$$\mathcal{D}_0 I_a = \kappa\sqrt{\frac{\lambda}{2i}}, \qquad I_2 = i\left(\frac{4I_a^2}{\lambda} - \frac{15\lambda}{I_a^2}\right)$$

and relations (3.9), (3.11) become

$$J_1 = \frac{\kappa}{2}\sqrt{\frac{\lambda}{2i}}\frac{u''}{u'}, \qquad J_2 = -\kappa\sqrt{\frac{2i}{\lambda}}\frac{u}{u'}. \tag{4.17}$$

Equation (3.10) takes the form

$$2u'u''' - u''^2 + \left(\frac{4I_a^2}{\lambda^2} - \frac{15}{I_a^2}\right)u'^2 - \frac{16}{\lambda^2}u^2 = 0. \tag{4.18}$$

Differentiating once we obtain a linear fourth-order ODE

$$u^{IV} + \left(\frac{4I_a^2}{\lambda^2} - \frac{15}{I_a^2}\right)u'' + \left(\frac{4I_a}{\lambda^2} + \frac{15}{I_a^3}\right)u' - \frac{16}{\lambda^2}u = 0$$

with the general solution

$$u = \frac{1}{I_a^2}\left(c_1 \sin(I_a^2/\lambda) + c_2 \cos(I_a^2/\lambda) + c_3\right) + c_4 I_a^2, \qquad c_1, c_2, c_3, c_4 = \text{const}. \tag{4.19}$$

Substitution of (4.19) into (4.18) leads to the relation

$$4\lambda^2 c_4(c_3 + \lambda^2 c_4) + c_1^2 + c_2^2 = 0$$

whence it follows

$$c_3 = -\lambda^2 c_4(1 + C_1^2 + C_2^2), \qquad C_1 = \frac{c_1}{2\lambda^2 c_4}, \qquad C_2 = \frac{c_2}{2\lambda^2 c_4}. \tag{4.20}$$

Function (4.19), (4.20) defines the general solution of ODE (4.18). Substituting it into (4.17) we obtain the relations

$$\frac{2\kappa(xy' - yx')}{\varphi(x^2 + y^2)} = 1 + \frac{1 + C_1^2 + C_2^2 - \varphi^2 - 2C_1\sin\varphi - 2C_2\cos\varphi}{1 + C_1^2 + C_2^2 + \varphi^2 - 2(C_2\varphi + C_1)\sin\varphi + 2(C_1\varphi - C_2)\cos\varphi},$$
$$-\frac{\kappa(xx' + yy')}{x^2 + y^2} = 1 + \frac{\varphi^2(C_1\sin\varphi + C_2\cos\varphi - 1)}{1 + C_1^2 + C_2^2 + \varphi^2 - 2(C_2\varphi + C_1)\sin\varphi + 2(C_1\varphi - C_2)\cos\varphi}.$$

Solving them with respect to C_1, C_2 one can find two first integrals of the system (4.15), (4.16). They can be supplemented by an integral

$$x'^2 + y'^2 - x^2 - y^2 = C_3$$

easily found from (4.15) when $r = \text{const}$.

5. CONCLUSION

Usually, to solve the equivalence problem for a family of ODEs it is sufficient to use the algebraic invariants of the family. Here we found the differential invariants for a system of two linear equations of the second order. Differential invariants are effective when either all algebraic invariants of the system are identically constant, or the system is of a degenerate type and does not possess algebraic invariants (as in Example 1 of the previous section).

Differential invariants find another application in constructing the first integrals. Using them one can find not more thah three integrals, while the system of two second-order ODEs has four independent first integrals. In general, the characteristic system of PDE (3.4) is three-dimensional being not simpler than that of PDE (3.2). But if we seek the first integral as a function of algebraic and differential invariants (3.3), then the dimension of the characteristic system of PDE (3.4) can be reduced. Simultaneously this leads to the reduction of the number of integrals which can be constructed by this way. All such cases are listed in Section 3. Namely, in the cases *N.I.2–N.I.4, N.II.2, N.III.1* one can find two first integrals of the form (3.5) or (3.6). In the cases *N.I.5* and *N.III.2* we have one first integral of the form (3.12) (cf. with the cases of the extension of admitted Lie symmetry algebra in [13]).

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APPENDIX. PROOF OF THE THEOREM 1

The proof of the statement of the Theorem 1 mainly repeats the proof of the Theorem 2.1 in [9]. The only difference is that the space of arguments of the differential invariant I is extended by two variables x', y' in comparison with the arguments of the algebraic invariant. Therefore, the set of independent algebraic and differential invariants of the system (1.1) is extended by two invariants $J_i = (t, x, y, x', y')$, $i = 1, 2$, in addition to the algebraic invariants $I_0(t, x, y)$, $I_j(t)$ constructed before in [9]. Differential invariant I is found from the invariance condition $\tilde{X}I = 0$. Here the operator X of the group E of the equivalence transformations of system (1.1) found in [9] is extended to x', y' by usual prolongation formulas [17]

$$
\begin{aligned}
X = \tau(t)\partial_t + [\rho_1(t)x + \sigma_1(t)y]\partial_x + [\rho_2(t)y + \sigma_2(t)x]\partial_y \\
+ [\rho_1' x + \sigma_1' y + (\rho_1 - \tau')x' + \sigma_1 y']\partial_{x'} \\
+ [\rho_2' y + \sigma_2' x + (\rho_2 - \tau')y' + \sigma_2 x']\partial_{y'}.
\end{aligned} \tag{6.1}
$$

Relation $\tilde{X}I = 0$, split by functions τ, ρ_1, ρ_2, σ_1, σ_2 and their derivatives, again leads to the system (A.6) from [9]. As is seen from (6.1) only nine operators X_2, $\dots$, X_6, X_8, $\dots$, X_{11} in the system (A.6) in [9] should be replaced by

$$
\begin{aligned}
\hat{X}_2(\tau') &= X_2 - x'\partial_{x'} - y'\partial_{y'}, \\
\hat{X}_3((\rho_1 + \rho_2)/2) &= X_3 + \frac{1}{2}(x'\partial_{x'} + y'\partial_{y'}), \\
\hat{X}_4((\rho_1 - \rho_2)/2) &= X_4 + \frac{1}{2}(x'\partial_{x'} - y'\partial_{y'}), \\
\hat{X}_5(\sigma_1) &= X_5 + y'\partial_{x'}, \\
\hat{X}_6(\sigma_2) &= X_6 + x'\partial_{y'}, \\
\hat{X}_8(\rho_1') &= X_8 + x\partial_{x'}, \\
\hat{X}_9(\rho_2') &= X_9 + y\partial_{y'}, \\
\hat{X}_{10}(\sigma_1') &= X_{10} + y\partial_{x'}, \\
\hat{X}_{11}(\sigma_2') &= X_{11} + x\partial_{y'}.
\end{aligned}
$$

The set of functionally independent solutions of such a modified system (A.6) defines the algebraic and differential invariants (2.1), (2.2) of the system (1.1) in generic case. And these invariants are given by (2.6), (2.8), (2.10), (2.14) for the corresponding degenerate types of the system (1.1). Analysis of the modified system (A.6) is similar to that performed in [9].

Systems of the fifth type possess neither algebraic, nor differential absolute invariants. Indeed, the relative invariants α_i, β_i, γ_i, ϵ_i, $i = 0, 1, 2$, vanish and six operators (A.9) from [9] act in the space of four variables x, y and δ_1, δ_2 defined by (2.1). Therefore, the subsystem of (A.6) with the operators (A.9) in [9] has only the trivial solution $I = \text{const}$ in this case.

The generator (6.1) of the equivalence transformation group E has the form

$$
X = \xi_0\partial_t + \xi_1\partial_x + \xi_2\partial_y + \xi_3\partial_{x'} + \xi_4\partial_{y'}.
$$

Then, according to [17], the components of an invariant differentiation operator

$$
\mathcal{D} = \lambda_0\partial_t + \lambda_1\partial_x + \lambda_2\partial_y + \lambda_3\partial_{x'} + \lambda_4\partial_{y'}
$$

satisfy the conditions

$$
X\lambda_i = \lambda_0\xi_{it} + \lambda_1\xi_{ix} + \lambda_2\xi_{iy} + \lambda_3\xi_{ix'} + \lambda_4\xi_{iy'}, \qquad i = 0, \dots, 4. \tag{6.2}
$$

The coefficients ξ_i in the operator X are defined by (6.1). Equalities (6.2) split by functions τ, ρ_1, ρ_2, σ_1, σ_2 and their derivatives give rise to a system which contains the nonhomogeneous equations

$$\begin{aligned}\hat{X}_2\lambda_0 = \lambda_0, \quad \hat{X}_3\lambda_1 = \tfrac{1}{2}\lambda_1, \quad \hat{X}_4\lambda_1 = \tfrac{1}{2}\lambda_1, \; \hat{X}_5\lambda_1 = \lambda_2, \quad \hat{X}_8\lambda_1 = x\lambda_0, \quad \hat{X}_{10}\lambda_1 = y\lambda_0,\\ \hat{X}_3\lambda_2 = \tfrac{1}{2}\lambda_2, \quad \hat{X}_4\lambda_2 = -\tfrac{1}{2}\lambda_2, \quad \hat{X}_6\lambda_2 = \lambda_1, \quad \hat{X}_9\lambda_2 = y\lambda_0, \quad \hat{X}_{11}\lambda_2 = x\lambda_0,\\ \hat{X}_2\lambda_3 = -\lambda_3, \quad \hat{X}_3\lambda_3 = \tfrac{1}{2}\lambda_3, \quad \hat{X}_4\lambda_3 = \tfrac{1}{2}\lambda_3, \quad \hat{X}_5\lambda_3 = \lambda_4, \quad \hat{X}_7\lambda_3 = -x'\lambda_0,\\ \hat{X}_8\lambda_3 = x'\lambda_0 + \lambda_1, \quad \hat{X}_{10}\lambda_3 = y'\lambda_0 + \lambda_2, \quad X_{12}\lambda_3 = x\lambda_0, \quad X_{14}\lambda_3 = y\lambda_0,\\ \hat{X}_2\lambda_4 = -\lambda_4, \quad \hat{X}_3\lambda_4 = \tfrac{1}{2}\lambda_4, \quad \hat{X}_4\lambda_4 = -\tfrac{1}{2}\lambda_4, \quad \hat{X}_6\lambda_4 = \lambda_3, \quad \hat{X}_7\lambda_4 = -y'\lambda_0,\\ \hat{X}_9\lambda_4 = y'\lambda_0 + \lambda_2, \quad \hat{X}_{11}\lambda_4 = x'\lambda_0 + \lambda_1, \quad X_{13}\lambda_4 = y\lambda_0, \quad X_{15}\lambda_4 = x\lambda_0,\end{aligned}$$

the remaining equations of the system being homogeneous. This system has five independent solutions which define the operators $\mathcal{D}_i$, $i = 0, \ldots, 4$, given by (2.3), (2.7), (2.11), (2.15) for the corresponding types of the system (1.1). Relations (2.5), (2.9), (2.13), (2.16) are found by direct calculation, i.e. applying the operators $\mathcal{D}_i$, $i = 0, \ldots, 4$, to the corresponding invariants I_0, J_1, J_2.