

# OPTION PRICING UNDER JUMP-DIFFUSION MODEL WITH Q PROCESS VOLATILITY<sup>1</sup>

Su Jun<sup>2</sup>    Xu Genjiu<sup>3</sup>

**Abstract:** In this paper, a financial market model is presented, where the underlying asset price is given by the combination of a two state Q process volatility and a compound Poisson process. The formula of European call option price under this model is derived. It generalizes the results of Hull and White (1987). At last an empirical examination on the Shanghai Stock Exchange Index is done to prove that the volatility described by two states Q process satisfies the features of fat tails and volatility clustering of financial data. And the numerical simulation results show that the option price is related to the volatility of initial time.

**Key words:** European option, jump-diffusion model, compound Poisson process, finite state Q process

## 1. INTRODUCTION

A classical model for continuous time asset pricing models is of geometric Brownian motion. We will call this *the standard model*. An important feature of the standard model is that markets are complete. That is, a contingent claim that is measurable with respect to the filtration generated by the asset price process is a redundant claim. However, as has been long known ([1, 2]), empirical studies have shown that asset prices often have jumps. The most common model is jump-diffusion model, where Poisson jumps have been added to Brownian noise, modeled by a Wiener process. These models are useful to model asset prices whose jumps arise from exogenous events (such as natural disasters, interest rate announcements, etc., [2, 3]), rather than to model those for which the jumps are intrinsic to the trading noise. But it is clearly unrealistic that the volatility of the underlying asset prices is constant in these models.

In the standard model framework, the volatility smile is evident in option prices. To revise the result, researchers have tried to relax one of the crucial assumptions of the standard model, namely, that the volatility evolves deterministically. Many empirical studies reflect the fact that in reality the volatility is not constant but time depending. These models are often referred to as stochastic volatility (SV) models of option pricing and have been mostly developed in continuous time, in that trading is assumed to take place continuously through time. On the other hand, the GARCH models of volatility, developed in discrete time assume that the innovations driving the asset returns and volatility are the same. It is

---

<sup>1</sup> The project-sponsored by NNSF of China (No. 70571065)

<sup>2</sup> Department of Basic Courses, Xi'an University of Science and Technology, Xi'an, Shaanxi 710054, China.  
E-mail: junsu99@163.com

<sup>3</sup> Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China.  
E-mail: xugenjiu@nwpu.edu.cn

\* Received 2 September 2007; accepted 11 November 2007

interesting to note that Foster and Nelson ([4]) have shown that the discrete time GARCH models converge to continuous time stochastic volatility models, which implies that the two models are related. The issue of SV and its effects on option prices have been studied extensively. Some of the work in this area include Hull and White ([5]), Melino and Turnbull ([6]), Romano and Touzi([7]). Empirical studies have suggested that stochastic volatility model is more realistic. However, both jump-diffusion model and stochastic volatility model are deficient in option pricing separately ([8]). It points out that jump-diffusion processes with time-varying volatility provide a refined and accurate perspective on the option prices.

In this paper we present a financial market model that has jumps in dynamics of asset prices with stochastic volatility. Suppose that the volatility  $\sigma(t, \omega)$  is a two state  $Q$  process on the basis of a Poisson jump-diffusion model, asset returns and volatility are uncorrelated. This stochastic volatility satisfies the basic two features of fat tails and volatility clustering of financial data. We derive the formula for a call option price under this model. It generalizes the conclusion of Hull and White ([5]), which says that volatility is stochastic but uncorrelated with the asset price, and the price of a European option is the Black-Scholes price integrated over the probability distribution of the average variance rate during the life of the option. Furthermore, in Section 4 we conduct an empirical examination on the SSE (Shanghai Stock Exchange) Index to verify that the volatility described by two states  $Q$  process satisfies the features of financial data. We obtain the option price of Theorem 1 by the use of numerical simulation.

Section 2 describes the model; Section 3 describes the option prices formula under the model; Section 4 describes an empirical test and a numerical simulation.

## 2. THE MODEL

Let us consider a general model of a frictionless financial market, where investors are allowed to trade continuously up to some fixed time horizon  $T$ . We assume that there are two assets  $P(t)$  and  $S(t)$ .  $P(t)$  (the bond) has finite variation on  $[0, T]$ , and its price model is given by

$$dP(t) = P(t)r(t)dt \quad P(0) = 1 \quad (1)$$

The other asset is "risky". We assume the dynamic model for the price  $S(t)$  to be described by the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = (\mu(t) - \lambda(t)\theta)dt + \sigma(t, \omega)dB(t) + YdH(t) \quad (2) \text{ Let}$$

$$\beta(t) = \exp\left\{-\int_0^t r(s)ds\right\} \quad (3)$$

The discounted price process  $S^*(t)$  for the security is

$$S^*(t) = \beta(t)S(t) \quad (4)$$

In this model, stochastic volatility  $\{\sigma(t, \omega), 0 \leq t \leq T\}$  is assumed to be a two States  $Q$  process. And the coefficients  $r(t), \mu(t) > 0, \lambda(t) \geq 0$  are assumed to be predictable processes. The process  $\{B(t), 0 \leq t \leq T\}$  is a standard Brownian motion on its canonical space  $(\Omega^B, F^B, P^B)$ , and the filtration  $\{F_t^B, 0 \leq t \leq T\}$  is generated by  $\{B(t), 0 \leq t \leq T\}$ .  $H(t)$  is the number of random jumps in time  $[0, t]$ . And  $\{H(t), 0 \leq t \leq T\}$  is a Poisson process with intensity equal to  $\lambda(t)$ .  $B(t)$  and

$H(t)$  are independent processes. The last term in (2) gives rise to jumps in  $S(t)$  of random relative jump size  $Y$ , and  $Y$  is random variable.  $Y_i (i = 1, 2, \dots, H(t))$  are *i.i.d.* random variables with  $Y$ , which denotes jumps size at random time points  $\tau_1, \tau_2, \dots, \tau_{H(t)}$ .  $Y (Y > -1, a.s.)$  and  $H(t)$  is independent.  $\ln(1 + Y)$  is subject to normal distribution  $N(\ln(1 + \theta - \frac{1}{2}\sigma_j^2), \sigma_j^2)$ , and  $\theta$  is unconditional expectation of  $Y$ .

**Remark 1** When the process  $\{\sigma(t, \omega), 0 \leq t \leq T\}$  is constant (i.e. constant volatility), the model (2) is well-known as the jump-diffusion model of Merton ([2]). As an example of stochastic volatility model caused by finite state  $Q$  process, we take  $\lambda(t) = 0$  (i.e.  $H(t) \equiv 0$ ).

**Remark 2** Obviously, the market model above is generally incomplete. Thus the equivalent martingale measure  $P^*$  and the fair price are no longer unique, that is, we can not derive a unique price which is irrelevant to risk attitude of inventor.

It is known that the financial data have the two statistical features of fat tails and volatility clustering. Fat tails means that the kurtosis is greater than 3, i.e.,  $\kappa = E[\log S(t) - E(\log S(t))]^4 / [Var(\log S(t))]^2 > 3$ ; Volatility clustering denotes that a big wave is often followed immediately by a big wave. In the standard model the asset price is given by  $dS(t) = S(t)(\mu dt + \sigma dB(t))$ , and the probability distribution of the asset price  $S(t)$  at any time is lognormal, then  $\kappa = 3$ . We know that it does not satisfy the feature of fat tails. If  $t_1 < t_2 < t_3$ , for  $\log(S(t_2)/S(t_1)) - (\mu - \frac{\sigma^2}{2})(t_2 - t_1)$  and  $\log(S(t_3)/S(t_2)) - (\mu - \frac{\sigma^2}{2})(t_3 - t_2)$ , the nature of independent increment implies that it has no feature of volatility clustering.

In the following part, we aim to explain our assumption that the volatility  $\sigma(t, \omega)$  is a two states  $Q$  process in the model verifying these two features. With the usage a finite state  $Q$  process to describe volatility, the probability of the volatility keeping at the original state  $i$  is  $e^{-q_i \Delta t}$  after time  $\Delta t$ . Obviously, the feature of volatility clustering is satisfied.

Consider the kurtosis  $\kappa$ , it is given as:

$$\kappa = \frac{\iint x^4 \frac{1}{\sqrt{2\pi}\eta_t} \exp(-\frac{1}{2} \cdot \frac{x^2}{\eta_t^2}) dx dF_\eta}{\iint x^2 \frac{1}{\sqrt{2\pi}\eta_t} \exp(-\frac{1}{2} \cdot \frac{x^2}{\eta_t^2}) dx dF_\eta} = \frac{3 \int \eta_t^4 dF_\eta}{(\int \eta_t^2 dF_\eta)^2}$$

where  $x = \log S(t) - E(\log S(t))$ . By Schwarz inequality, we have  $\kappa > 3$ , which implies the feature of fat tails.

So stochastic volatility described by a two states  $Q$  process satisfies the basic features of financial data.

### 3. THE FORMULA OF A EUROPEAN CALL OPTION PRICE UNDER THE MODEL

In this section, the purpose is to derive the formula of an European call option price under our model. Before presenting our main result, we recite an approximate algorithm about the distribution of  $Q$  process integral in [9], firstly.

**Lemma 1** ([9]) Let  $\xi_t$  be a two state  $Q$  process and its matrix  $Q = \begin{pmatrix} -q_a & q_a \\ q_b & -q_b \end{pmatrix}$  is given. The denotation is  $\eta_T = \int_0^T \xi_s ds$ . Then its distribution function is defined as  $F(x) = P(\eta_T \leq x)$ .

Case 1. For initial state  $\xi_t = a$  :

If  $x \leq a(T-t)$  then  $F_n(x) = 0$  ; If  $x > b(T-t)$  then  $F_n(x) = 1$  ; If  $a(T-t) < x \leq b(T-t)$ , then

$$F_k(x) = \int_{\frac{b(T-t)-y}{b-a}}^{\infty} dP(\tau_1 - t \leq x_1) \\ + \int_0^{\frac{b(T-t)-y}{b-a}} dP(\tau_1 - t \leq x_1) \int_0^{\frac{y_1 - b(T-t_1)}{b-a}} dP(\tau_2 - t_1 \leq x_2) \int_{\frac{b(T-t_2)-y_2}{b-a}}^{\infty} dP(\tau_3 - t_2 \leq x_3) + \dots \\ + \int_0^{\frac{b(T-t)-y}{b-a}} dP(\tau_1 - t \leq x_1) \int_0^{\frac{y_1 - b(T-t_1)}{b-a}} dP(\tau_2 - t_1 \leq x_2) \dots \int_{\frac{b(T-t_{2k})-y_{2k}}{b-a}}^{\infty} dP(\tau_{2k+1} - t_{2k} \leq x_{2k+1})$$

where

$$P(\tau_{2k} - t_{2k-1} \leq x_{2k}) = 1 - \exp(-q_b x_{2k}) \\ P(\tau_{2k+1} - t_{2k} \leq x_{2k+1}) = 1 - \exp(-q_a x_{2k+1}) \\ t_0 = t; \quad t_i = t_{i-1} + x_i \\ y_0 = x; \quad y_{2k} = y_{2k-1} - b x_{2k}; \quad y_{2k+1} = y_{2k} - a x_{2k+1}$$

then  $F_n(x) \rightarrow F(x)$ .

Case 2. For initial state  $\xi_t = b$  :

$$F_n(x) = \begin{cases} 0 & x \leq a \cdot (T-t); \\ 1 - F_n^*(-x) & a \cdot (T-t) < x \leq b \cdot (T-t); \\ 1 & x > b \cdot (T-t), \end{cases}$$

Where  $F_n^*(x)$  is according to  $F_n(x)$  in Case 1, the corresponding integral functions is as followed:

$$P(\tau_{2k} - t_{2k-1} \leq x_{2k}) = 1 - \exp(-q_a x_{2k}) \\ P(\tau_{2k+1} - t_{2k} \leq x_{2k+1}) = 1 - \exp(-q_b x_{2k+1})$$

then  $F_n(x) \rightarrow F(x)$ .

The rate of convergence of the iterated algorithm is  $O(M^n/n!)$ ,  $M = \max(q_a, q_b)$ . For  $k$  states  $Q$  process, there is a similar iterated algorithm and its rate of convergence is  $O(A^n/n!)$ ,  $A = M \cdot (k-1)$ .

In order to price a European call option, we should compute the expectation  $E_{P^*} \left[ (S_T - K)^+ \exp(-\int_t^T r(s) ds) | F_t \right]$ , where  $P^*$  is a equivalent martingale measure of  $P$  and  $K$  is the exercise price of a European call option. How to select an appropriate martingale measure  $P^*$  is more delicate (see [10]). In the following part, we deduce a pricing formula of a call option on condition that  $P^*$  has been selected.

**Theorem 1** Let the risk asset price  $S(t)$  be the stochastic differential equation described by (2). Then the price at time zero  $c_0$  of the call option expiring at time  $T$  is given as

$$c_0 = E_{P^*} \left[ (S_T - K)^+ \exp\left(-\int_0^T r(s)ds\right) \right]$$

$$= \int \sum_{n=0}^{\infty} P\{H(T) = n\} E_{P^*} \left[ \left[ x e^{-\int_0^T \lambda(s)\theta ds} \exp\left(-\frac{1}{2}\eta_T + \sqrt{\eta_T} \cdot z + \sum_{i=1}^n \ln(1 + Y_i^*)\right) - K e^{-\int_0^T r(s)ds} \right] I_{\{\zeta - b \geq 0\}} \right] dF_{\eta}$$

where  $F_{\eta}$  is determined by Lemma 1.

**Proof:** Let  $\sigma_t$  be a two state  $Q$  process and its matrix  $Q = \begin{pmatrix} -q_a & q_a \\ q_b & -q_b \end{pmatrix}$  is given. Initial

conditions are  $S(0) = x > 0, \sigma_0 = a$ . The denotation is  $\eta_t = \int_0^t \sigma_s^2 ds, z \sim N(0,1)$ . By the Doléans-Dade exponential formula, from (2) we know that

$$S(t) = S(0) \exp\left\{ \int_0^t (\mu(s) - \lambda(s)\theta) ds - \frac{1}{2}\eta_T + \sqrt{\eta_T} \cdot z \right\} \cdot \prod_{i=1}^{H(t)} (1 + Y_i) \quad (5)$$

using (5) the discounted process  $S^*(t)$  can be written as

$$S^*(t) = S(0) \exp\left\{ -\frac{1}{2}\eta_T + \sqrt{\eta_T} \cdot z + \int_0^t \lambda(s)\theta ds \right\} \cdot \prod_{i=1}^{H(t)} (1 + Y_i^*) \quad (6)$$

where  $Y_i^* (i = 1, 2, \dots, H(t))$  are i.i.d. random variables and  $\sum_{i=1}^n \ln(1 + Y_i^*) \sim N(\ln(1 + \theta)^n - \frac{1}{2}n\sigma_j^2, n\sigma_j^2)$  under  $P^*$ ,  $N(x)$  is the cumulative distribution function of the standard normal distribution.

Denote

$$B = \left\{ S^*(T) \geq K \exp\left(-\int_0^T r(s)ds\right) \right\}$$

$$= \left\{ -\frac{1}{2}\eta_T + \sqrt{\eta_T} \cdot z + \sum_{i=1}^{H(T)} \ln(1 + Y_i^*) \geq \ln \frac{K}{x} + \int_0^T (\lambda(s)\theta - r(s)) ds \right\}$$

$$\zeta = -\frac{1}{2}\eta_T + \sqrt{\eta_T} \cdot z + \sum_{i=1}^n \ln(1 + Y_i^*), \quad b = \ln \frac{K}{x} + \int_0^T (\lambda(s)\theta - r(s)) ds$$

Then the price  $c_0$  of a call option for the security model (2) is given as

$$c_0 = E_{P^*} \left[ (S_T - K)^+ \exp\left(-\int_0^T r(s)ds\right) \right] = E_{P^*} \left[ (S_T^* - K \exp\left(-\int_0^T r(s)ds\right)) I_B \right]$$

$$= E_{P^*} \left\{ \left[ x \exp\left(-\int_0^T \lambda(s)ds\right) \exp\left(-\frac{1}{2}\eta_T + \sqrt{\eta_T} \cdot z + \sum_{i=1}^{H(T)} \ln(1 + Y_i)\right) - K \exp\left(-\int_0^T r(s)ds\right) \right] I_B \right\}$$

$$\begin{aligned}
 &= E_{P^*} \left\{ E_{P^*} \left[ \left( x e^{-\int_0^T \lambda(s)\theta ds} \exp \left( -\frac{1}{2} \eta_T + \sqrt{\eta_T} \cdot z + \sum_{i=1}^{H(T)} \ln(1 + Y_i) \right) - K e^{-\int_0^T r(s) ds} \right) I_B \right] \middle| H(T) \right\} \\
 &= \sum_{n=0}^{\infty} P \{ H(T) = n \} \cdot E_{P^*} \left\{ \left[ x e^{-\int_0^T \lambda(s)\theta ds} \exp \left( -\frac{1}{2} \eta_T + \sqrt{\eta_T} \cdot z + \sum_{i=1}^n \ln(1 + Y_i^*) \right) - K e^{-\int_0^T r(s) ds} \right] I_{\{\zeta - b \geq 0\}} \right\} \\
 &= \sum_{n=0}^{\infty} P \{ H(T) = n \} \cdot E_{P^*} \left\{ E_{P^*} \left[ \left( x e^{-\int_0^T \lambda(s)\theta ds} \exp \left( -\frac{1}{2} \eta_T + \sqrt{\eta_T} \cdot z + \sum_{i=1}^n \ln(1 + Y_i^*) \right) - K e^{-\int_0^T r(s) ds} \right) I_{\{\zeta - b \geq 0\}} \right] \middle| \eta_T \right\} \\
 &= \int \sum_{n=0}^{\infty} P \{ H(T) = n \} E_{P^*} \left[ \left( x e^{-\int_0^T \lambda(s)\theta ds} \exp \left( -\frac{1}{2} \eta_T + \sqrt{\eta_T} \cdot z + \sum_{i=1}^n \ln(1 + Y_i^*) \right) - K e^{-\int_0^T r(s) ds} \right) I_{\{\zeta - b \geq 0\}} \right] dF_{\eta}
 \end{aligned}$$

This completes the proof.

In [5], Hull and White derived that volatility is stochastic but uncorrelated with the asset price, and the price of an European option is the Black-Scholes price integrated over the probability distribution of the average variance rate during the life of the option.

Note that

$$\sum_{n=0}^{\infty} P \{ H(T) = n \} E_{P^*} \left[ \left( x e^{-\int_0^T \lambda(s)\theta ds} \exp \left( -\frac{1}{2} \eta_T + \sqrt{\eta_T} \cdot z + \sum_{i=1}^n \ln(1 + Y_i^*) \right) - K e^{-\int_0^T r(s) ds} \right) I_{\{\zeta - b \geq 0\}} \right]$$

is just the price of an European call option in the jump-diffusion model. By Theorem 1, we have the following corollary, which generalizes the result of Hull and White.

**Corollary 1** Let volatility be a two state  $Q$  process and uncorrelated to the risk asset price. Then the price of a European option is the option price in the jump-diffusion model integrated over the probability distribution of the average variance rate during the life of the option.

#### 4. AN EMPIRICAL EXAMINATION

In this section, an empirical examination is done on the SSE (Shanghai Stock Exchange) Index, for the confirmation of the validity of stochastic volatility described by a two state  $Q$  process. Moreover, we obtain the option price of Theorem 1 by the use of numerical simulation.

In reality, the matrix  $Q$  is unknown and the volatility  $\sigma(t, \omega)$  is not observable. At any time  $t$  we don't know that the volatility  $\sigma(t, \omega)$  is at state  $a$  or state  $b$ . Since the volatility  $\sigma(t, \omega)$  belongs to different states at before and after a stopping time  $\tau$ , the key problem is to determine stopping times  $\tau$  of state transformation. If these stopping times  $\tau$  are determined, the volatility  $\sigma(t, \omega)$  can be estimated in stages.

Next we recall a result in [11] for estimating the volatility  $\sigma(t, \omega)$  in stages.

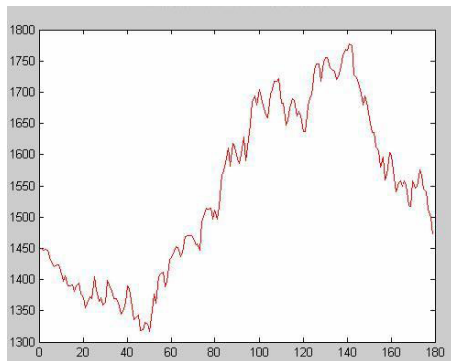
**Lemma 2** ([11]) Let  $\varepsilon_t$  be a white noise and  $\sigma_t$  be a two state  $Q$  process with initial state  $\sigma_0 = a$ , stopping time  $\tau = \min(t : \sigma_t = b)$ . For the stochastic process  $x_t = \sigma_t \cdot \varepsilon_t$ , there are even sample points  $\{x_k \mid k = 1, 2, \dots, n\}$  of  $x_t$  on  $(0, T)$ .

Denote

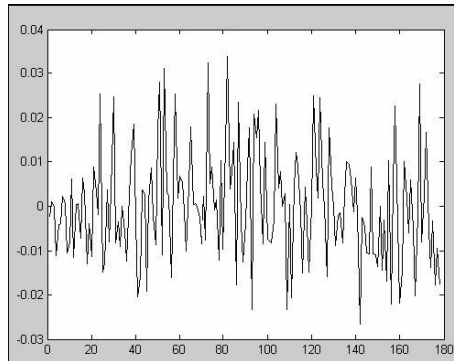
$$F_k = \frac{1}{k} \sum_{i=1}^k x_i^2, \quad B_k = \frac{1}{n-k} \sum_{i=k+1}^n x_i^2, \quad Z_k = F_k - B_k \quad (7)$$

If  $T \leq \tau$ , for any  $k \in \{1, 2, \dots, N\}$ , we have  $E(Z_k) = 0$ ; If  $T > \tau$ , for any  $k \in \{1, 2, \dots, N\}$ , we have  $\{[N \cdot \tau/T]\} = \{\tau_N | E(|Z_{\tau_N}|) = \max_k E(|Z_k|)\}$ .

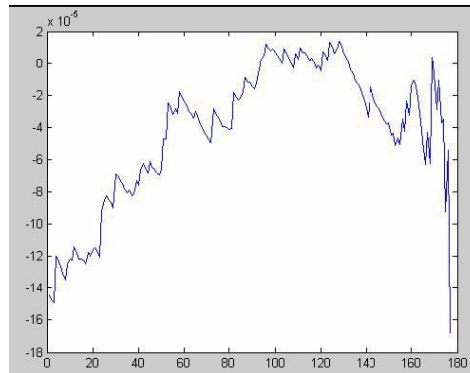
We test the features of fat tails and volatility clustering on the SSE Index from Sep. 1, 2003 to Jun. 9, 2004. There are 180 trading days in total as in Fig. 1. Its yield  $\mu_t = (S_{t+1} - S_t)/S_t$  is illustrated in Fig. 2 and the  $Z_t$  of the yield calculated by (7) is illustrated in Fig. 3.



**Fig. 1. SSE Index (2003.09.01-2004.06.09)**



**Fig. 2. The yield of SSE Index**



**Fig. 3. Zt diagram for the yield of SSE Index**

In these figures, it can be seen that there are two times of state transformation. They appeared at Nov. 18, 2003 and Apr. 6, 2004 respectively, i.e., the volatility of the yield falls into three stages as 50 trading days from Sep. 1, 2003 to Nov. 18, 2003, 92 trading days from Nov. 18, 2003 to Apr. 6, 2004, and 38 trading days from Apr. 6, 2004 to Jun. 9, 2004. The estimations of  $\sigma_t$  for these three stages are 0.1600, 0.2030, 0.1816. From this we know that the volatility in the second stage is obviously greater than that in the other two stages. It shows the feature of volatility clustering. In the standard model,  $\sigma_t$  is thought as a constant, and the estimation of  $\sigma_t$  should be 0.1943. It can not inflect the variation of the volatility.

And we have the kurtosis  $\kappa$  for different cases as follows

SSE Index data  $\kappa = 5.1306$

$\sigma_t$  is two states  $Q$  process  $\kappa = 6.0792$

$\sigma_t$  is constant  $\kappa = 2.7068$

From this empirical examination, it is easy to see that the volatility described by  $Q$  process satisfies the two features of fat tails and volatility clustering. Whereas the standard model does not satisfy these features.

At last, we calculate the option price of Theorem 1 by numerical simulation. Without loss of generality, set initial time  $t = 0$ , interest rate  $r = 0$ , parameter  $\lambda = 1$ , the security initial price  $S(0) = 1$ , and the exercise price of a European call option  $K = 1$ . When initial state is  $\sigma_0 = 0.1600$ , the option price of Theorem 1 is 0.3749 after 170 trading dates. When initial state is  $\sigma_0 = 0.2030$ , the option price is 0.5036 after 170 trading dates. Form simulation results, it can be seen that the option price is related to the volatility of initial time.

**Remark 3** *Although the distribution of  $Q$  process integral is not expressed by elementary function with finite item, by Lemma 1 the rate of convergence in approximate algorithm is well fast.*

## REFERENCES

- [1] Aase, K K.1988. Contingent claims valuation when the security price is a combination of an Itô process and a random point process. *Stochastic Processes and their Applications*, 28: 185-220.
- [2] Merton, R C. 1976. Option pricing when underling process of stock returns is discontinuous. *Journal of Financial Economics*, 3: 124-144.
- [3] Jeanblanc, P M. & Pontier, M. 1990. Optimal portfolio for a small investor in a market model with discontinuous prices. *Applied Mathematics and Optimization*, 22: 287-310.
- [4] Foster, D. & Nelson, D. 1994. Asymptotic filtering theory for univariate ARCH models. *Econometrica*, 62: 1-41.
- [5] Hull, J. & White, A. 1987. An analysis of the bias in option pricing caused by a stochastic volatility. *Journal of Finance*, 42: 281-300.
- [6] Melino, A. & Turnbull, S. 1990. The pricing of foreign currency options with stochastic volatility. *J. Econometrics*, 45: 239-265.
- [7] Romano, M. & Touzi, N. 1997. Contingent claims and market completeness in a stochastic volatility model. *Mathematical Finance*, 7: 399-412.
- [8] Das, S. R. & Sundaram, R. K. 1999. Of smiles and smirks: A term structure perspective. *Journal of Financial and Quantitative Analysis*, 34: 211-239.
- [9] Tian, J. & Zheng, L. 2003. Distribution of  $Q$  process integral and its application in the Black and Scholes equation. *Chinese J. Appl. Probab. Statist.*, 19: 303-312.
- [10] Föllmer, H. & Schweizer, M. 1990. Hedging of contingent claims under incomplete information. Davis M H A, Elliott R J (eds.) *Applied stochastic analysis*. London: Gordon and Breach.
- [11] Hull, J. 2000. *Options, Futures, and Other Derivatives* (Ed.), Prentice Hall, Inc.
- [12] Musiela, M. & Rutkowski, M. 1992. *Martingale Methods in Financial Modelling*. Springer, Heidelberg.