(H,Ω) CONJUGATE MAPS AND (H,Ω) DUALITY THEORY IN MULTIOBJECTIVE OPTIMIZATION[1]

Feng Junwen[2]

Abstract: This paper is devoted to develop a duality theory for the nonlinear multiobjective optimization problems which aim to find all the efficient solutions. The (H,Ω) conjugate maps of point-to-set maps are defined, and their properties and relationships are discussed. The multiobjective optimization problem called primal problem is embedded into a family of perturbed problems, and the dual problem with multiobjectives in a wide sense, called the (H,Ω) conjugate dual problem is defined with the help of its (H,Ω) conjugate maps. The theorems, such as weak, strong and inverse (H,Ω) duality, which describe the relationships between the primal and dual problems are developed by means of the (H,Ω)-stability. The concepts of (H,Ω)-Lagrangian map and saddle-point are provided, and it is shown that the solution of the primal and the corresponding solution of the dual provide a saddle-point of the (H,Ω)-Lagrangian map. Finally, several special cases for H and Ω are discussed.

Key words: conjugate map, subgradient, multiobjective optimization, efficiency, dual problem, Lagrangian map, saddle point, vector optimization.

1. INTRODUCTION

In recent years, the analysis of a programming problem with several incommensurable objectives conflicting with each other has been a focal issue. Such a multiobjective optimization problem reflects the complexity of the real world and is encountered in various fields. An optimal solution to such a problem is ordinarily chosen from the set of all Pareto optimal solutions to it. On the other hand, the duality theory in multiobjective optimization has been another focal issue for a long time, especially in multiobjective convex programming. It holds now a major position in the multiobjective programming due to not only its mathematical elegance but also its economic

[1] Supported by NSFC No.79870030

[2] School of Economics and Management, Nanjing University of Science and Technology, Nanjing, Jiangsu, 210094, P.R. China.
E-mail: jwfeng@jlonline.com

* Received 15 June 2007; accepted 9 August 2007

implications. The first work on duality in multiobjective optimization is the one by Gale-Kuhn-Tucher in 1951 [5], which treated "matrix optimization" including multiobjective optimization as a particular case. However, their result is far from a natural extension of that in traditional mathematical programming. Since then, there have been several developments in the duality theory of linear multiobjective optimization by, for example, Kornbluth [12], Isermann [6,7,8]. Above, all Isermann's work seems most elegant, because it keeps the parallel formulation to the traditional mathematical programming. For the nonlinear case, Tanino and Sawaragi [15,16,17] reported very attractive results on both Lagrange duality and conjugate duality in a framework of natural extension of traditional convex programming. Their main features are that (1) they used a vector-valued Lagrangian with vector multipliers, and (2) their duality is not "complete" in the sense that the dual solution set is not identical to the primal solution set. Especially, the point (1) is a big difference from the Isermann's formulation, in which matrix multipliers are used. After that, Bitran[1] tried to extend Isermann's result to nonlinear cases by using a linear approximation of nonlinear functions. Kawasaki [10,11] gave a complete duality with respect to weak efficiency for nonlinear cases. Brumelle [2] reported a conjugate duality for "pointwise infimum" solutions. Jahn [9] published a duality based on scalarization instead of using vector-valued Lagrangian. Recently, Nakayama [13] gave a geometric consideration of duality in multiobjective optimization. It clarifies a role of vector-valued Lagrangian as the supporting cone on behalf of the supporting hyperplane in traditional mathematical programming. Based on the geometrical meaning given in [13], this paper develops a new duality theory in multiobjective optimization by means of the generalization of conjugate maps. First, in section 2, the (H,Ω)conjugate maps of the vector-valued functions and point-to-set maps are defined. In section 3, the concept of (H,Ω) subgradient of point-to-set maps is provided. In section 4, a multiobjective minimization problem is embedded into a family of perturbed problems, and the dual problem in wide sense, called the (H,Ω) conjugate dual problem is defined with the help of their (H,Ω) conjugate maps. Especially, for a class of stable problems, the relationship between the primal and dual problems is made clear. In section 5, the (H,Ω) Lagrangian map and its saddle point are discussed. It is shown that the solution of the primal problem and the corresponding solution of the dual problem provide a saddle point of the (H,Ω) Lagrangian map. Finally, several special cases for H and Ω are considered.

Before we go further, for convenience, let us introduce the following notations. let $a=(a_1, a_2, ..., a_p)$, $b=(b_1, b_2, ..., b_p)$ be the vectors in p-dimensional Euclidean Space R^p. Then

$a \leq= b$ if and only if $a_i \leq b_i$ for all i=1,2,...,p.

$a \leq\neq b$ if and only if $a \leq= b$ is not true.

$a \geq= b$ if and only if $a_i \geq b_i$ for all i=1,2,...,p.

$a \geq\neq b$ if and only if $a \geq= b$ is not true.

$a \leq b$ if and only if $a_i \leq b_i$ for all i=1,2,...,p, and $a \neq b$.

$a < b$ if and only if $a_i < b_i$, for all i=1,2,...,p.

$R^p_+ = \{a \in R^p : a \geq= 0\}$, $R^p_- = -R^p_+$. Let A, $B \subseteq R^p$, if for any $a \in A$, and $b \in B$, $a \geq b$, we denote it by $A \geq B$. This denotation is true for the operations $\leq$, $\geq=$, $\geq\neq$, $\leq=$, $\leq\neq$, $>$, and $<$. Furthermore, we denote $P(R^n)$ to be the set of all possible sets in R^n, and for $F: R^n \to P(R^p)$, $X \subseteq R^n$, F(X) or $\cup_x F(x)$ to be $\cup_{x \in X} F(x)$. Sometimes, when the discussion universal of variable x is clear, we just denote $\cup_x F(x)$ to be the union of all possible F(x). Additionally, considering the length of the paper, we omit the proofs of all the theorems and corollaries.

2. (H,Ω) CONJUGATE MAPS

First, we define an efficient point of a set in the space R^p as follows. Given a set A in R^p, a point $a \in R^p$ is said to be a lower (respectively., an upper) efficient point of A if $a \in A$ and there is no $a' \in A$ such that

$a' \le a$ (respectively, $a' \ge a$), we denote this by $a \in \mathrm{Min}\ A$ (respectively, $a \in \mathrm{Max}\ A$). $A \subseteq R^p$ is called Max-complete if $A \subseteq \mathrm{Max}\ A + R^p_-$. Similarly, $A \subseteq R^p$ is called Min-complete if $A \subseteq \mathrm{Min}\ A + R^p_+$.

Definition 2.1 ((H,Ω) conjugate map and its inverse)

Let Ω be a set of functions ω: $R^n \to R^m$, and H be a family of vector-valued functions h: $R^m \to R^p$, where H is closed under Max-pointwise, that is, such that $H' \subseteq H$ implies $\mathrm{Max}\{h: h \in H'\} \subseteq H$. Given $F: R^n \to P(R^p)$ being the point-to-set map, a point-to-set map $F^c: \Omega \to P(H)$, defined by the formula: $\forall \omega \in \Omega$, $F^c(\omega) = \mathrm{Max}\{h: h \in H, h \otimes \omega \ge\ne F\} = \mathrm{Max}\{h: h \in H, h(\omega(x)) \ge\ne F(x) \text{ for } \forall x \in R^n\}$ i.e., for $\forall \omega \in \Omega$, $t \in R^m$, $F^c(\omega)(t) = \mathrm{Max}(h: h \in H, h(\omega(x)) \ge\ne F(x) \text{ for } \forall x \in R^n\}$. Inversely, given $G: \Omega \to P(H)$, its (H, R^n) conjugate map $G^*: R^n \to P(R^p)$ is defined by $G^* = \mathrm{Max} \cup_{\omega \in \Omega} G(\omega) \otimes \omega$ i.e., for $\forall x \in R^n$, $G^*(x) = \mathrm{Max} \cup_{\omega \in \Omega} G(\omega)(\omega(x))$ We call the (H,R^n) conjugate map of F^c the (H,Ω) biconjugate map of F, and denote it by F^{c*} , i.e., $F^{c*} = (F^c)^*$.

Theorem 2.1

For $F: R^n \to P(R^p)$, if $y \in F(x)$and $h' \in F^c(\omega)$ for any $x \in R^n$,$\omega \in \Omega$, then $y \le\ne h'(\omega(x))$, i.e., for $\forall x \in R^n$,and $\omega \in \Omega$, $F(x) \le\ne F^c(\omega)(\omega(x))$. Furthermore, $F(x) \le\ne F^{c*}(x)$ for $\forall x \in R^n$.

Now, we give the concept of the (H,Ω)-convexity.

Definition 2.2 ((H,Ω)-convexity)

A point-to-set map $F: R^n \to P(R^p)$ will be called weak (H,Ω)-convex, (H,Ω)-convex, strong (H,Ω)-convex and inverse (H,Ω)-convex at $x \in R^n$, respectively, if the following facts are true, respectively, $F(x) \cap F^0(x) \ne \emptyset$, $F(x) \subseteq F^0(x)$, $F(x) = F^0(x)$, $F(x) \supseteq F^0(x)$,where $F: R^n \to P(R^p)$, and for $\forall x \in R^n$, $F^0(x) = \mathrm{Max}\{h(\omega(x)): h \in H, \in \Omega, h \otimes \omega \ge\ne \omega F\}$. Furthermore, if F is weak (H,Ω)-convex,(H,Ω)-convex, strong (H,Ω)-convex and inverse (H,Ω)-convex at $\forall x \in X \subseteq R^n$, respectively, we will call F weak (H,Ω)-convex, (H,Ω-convex, strong (H,Ω)-convex, and inverse (H,Ω)-convex on X, respectively.

Remark. If H or Ω is taken broadly enough, for example, $H = H_L$ being the set of the affine linear functions, and Ω being the set of the all the functions $\omega: R^n \to R^p$, then any $F: R^n \to R^p$ (notice that m=p) is (H,Ω)-convex on R^n . Generally, given $F: R^n \to P(R^p)$, if $F(x) = \mathrm{Max}\ F(x)$ for $\forall x \in R^n$, then F is (H,Ω)-convex on R^n .

Assumption (A1). For $\forall \omega \in \Omega$, the set $\{h: h \in H, h \otimes \omega \ge\ne F\}$ is Max-complete, i.e., for

$\forall t \in R^m$, $\{h(t): h \in H, h \otimes \omega \ge\ne F\}$ is Max-complete.

Theorem 2.2

(1) $F^0 \subseteq F^{c*}$. If (A1) is true, then $F^0 = F^{c*}$.

(2) For $G: \Omega \to P(H)$, G^* be (H,Ω)-convex point-to-set map on R^n .

(3) $F^c \ge\ne F^{0c}$, i.e., $F^c(\omega) \ge\ne F^{0c}(\omega)$ for $\forall \omega \in \Omega$. If F is (H,Ω)-convex on R^n , then $F^c = F^{0c}$.

(4) F^0 is (H,Ω)-convex point-to-set map on R^n . Therefore, F^0 is the "greatest" (H,Ω)-convex point-to-set map of F with respect to the ordering "≥". F^0 will be called the (H,Ω)-convex map hull of F.

Corollary 2.2

If F is (H,Ω)-convex on R^n, then $F(x)\subseteq F^{c}*(x)$ for $\forall x\in R^n$. Especially, under (A1), $F(x)=F^{c}*(x)$ for $\forall x\in R^n$.

Dually, the convexity of $G:\Omega\to P(H)$ is introduced as follows.

Definition 2.3 ((H,Rn)-convexity)

The point-to-set map $G:\Omega\to P(H)$ is called weak (H,R^n)-convex, (H,R^n)-convex, strong (H,R^n)-convex, and inverse (H,R^n)-convex at $\omega\in\Omega$, respectively, if the following are true, respectively, $G(\omega)\cap G*^{c}(\omega)\neq\varnothing$, $G(\omega)\subseteq G*^{c}(\omega)$, $G(\omega)=G*^{c}(\omega)$, $G(\omega)\supseteq G*^{c}(\omega)$.Furthermore, if the above are true for any $\omega\in\Omega$, then we will called G weak (H,R^n)-convex, (H,R^n)-convex, strong (H,R^n)-convex, and inverse (H,R^n)-convex on Ω,respectively.

Theorem 2.3

Under the assumption (A1), if $F:R^n\to P(R^p)$ is (H,Ω)-convex on R^n, then F is strong (H,R^n)-convex.

3. (H,Ω)-SUBGRADIENT

Definition 3.1 ((H,Ω)-subgradient)

Let $F:R^n\to P(R^p)$, suppose $x\in R^n$, $y\in F(x)$. F is called (H,Ω)-subdifferentiable at (x;y), if there exist $\omega\in\Omega$ and $h\in H$ such that $h(\omega(x))=y$, and $h(\omega(x)\geq\neq F(x)$ for $\forall x\in R^n$ .where $\omega\in\Omega$ is called the (H,Ω)-subgradient of F at (x;y). The set of (H,Ω)-subgradients of F at (x;y) is called the (H,Ω)-subdifferential of F at (x;y) and is denoted by $\partial_\Omega F(x;y)$. Moreover, F is said to be (H,Ω)-subdifferentiable at x, if $\partial_\Omega F(x;y)\neq\varnothing$ for $\forall y\in F(x)$.

Let $\partial_\Omega F(x)=\cup_{y\in F(x)}\partial_\Omega F(x;y)$

Theorem 3.1

(1). If:F: $R^n\to P(R^p)$ is (H,Ω)-subdifferentiable at (x;y), then $y\in \text{Min } F(x)$. Hence, if F is (H,Ω)-subdifferentiable at x, then F(x)=Min F(x), i.e., F(x) is Min-complete.

(2).Suppose that $f:R^n\to R^p$, $f\in\Omega$, and $h\in H$ is the identical function, then f is (H,Ω)-subdifferentiable at any $x\in R^n$.

(3).Let $G:R^n\to P(R^p)$, and G(x)=Min F(x), $h\in H$ be the identical function, g be the induced function of G, i.e., $g:R^n\to R^p$, and $g(x)\in G(x)$ for $\forall x\in R^n$, then, as long as $g\in\Omega$, F is (H,Ω)-subdifferentiable at $\forall(x;g(x))$. Moreover, if for any induced function g of G, we have $g\in\Omega$, then G is (H,Ω)-subdifferentiable at $\forall x\in R^n$.

Theorem 3.2

(1). $F:R^n\to P(R^p)$ is weak (H,Ω)-convex at $x\in R^n$, if and only if there exists $y\in F(x)$ such that F is (H,Ω)-subdifferentiable at (x;y).

(2). $F:R^n \to P(R^p)$ is (H,Ω)-convex at $x \in R^n$, if and only if F is (H,Ω)-subdifferentiable at x.

Corollary 3.2

Let $F:R^n \to P(R^p)$, $\partial_\Omega F(x;y) \neq \emptyset$ if and only if $y \in F^{c*}(x)$. Therefore, F is (H,Ω)-subdifferentiable at x if and only if $F(x) \subseteq F^{c*}(x)$.

The relationship between the (H,Ω)-subgradient and the efficient solution of multiple objective minimization problem is given as follows.

Theorem 3.3

Suppose that Ω contains the zero element 0_Ω, i.e., $0_\Omega(x)=0$ for $\forall x \in R^n$, and H verifies the following properties: for $\forall \alpha \in R^n$, there exists $h_\alpha \in H$ such that $h_\alpha(0_\Omega)=\alpha$. Then for $F:R^n \to P(R^p)$, $y \in F(x)$ and $y \in \mathrm{Min} \cup F(x)$, if and only if $0_\Omega \in \partial_\Omega F(x;y)$. So, for $f:R^n \to R^p$, $f(x) \in \mathrm{Min} \cup_x f(x)\}=\mathrm{Min}\{f(x):x \in R^n\}$ if and only if $0_\Omega \in \partial_\Omega f(x)$.

The following theorem provides a relationship between the (H,Ω)-subgradient and (H,Ω) conjugate map.

Theorem 3.4

$\omega \in \Omega$ is the (H,Ω)-subgradient of $F:R^n \to P(R^p)$ at $(x;y)$, if and only if $y \in F(x)$ and $y \in F^c(\omega)(\omega(x))$.

Dually, the subgradient concept of $G:\Omega \to P(H)$ is defined as follows.

Definition 3.2 ((H,Rn)-subgradient)

Suppose that $G:\Omega \to P(H)$, $\omega \in \Omega$ and $h \in G(\omega)$, G is said to be the (H,R^n)-subdifferentiable at $(\omega; h)$ if there exists $x \in R^n$ such that $h(\omega(x)) \in G^*(x)$, and x is called the (H,R^n)-subgradient of G at $(\omega;h)$, and is denoted by $\partial_H G(\omega;h)$. Moreover, if $\partial_H G(\omega;h) \neq \emptyset$ for $\forall h \in G(\omega)$, then G is called the (H,R^n)-subdifferentiable at $\omega \in \Omega$. Denote $\partial_H G(\omega) = \cup_{h \in G(\omega)} \partial_H G(\omega;h)$

For the (H,Ω)-subgradient and (H,R^n)-subgradient, the following result describes their relationship.

Theorem 3.5

$F:R^n \to P(R^p)$. If $\omega \in \partial_\Omega F(x;y)$, then there exists $h \in F^c(\omega)$ such that $x \in \partial_H F^c(\omega;h)$. Inversely, if F is strong (H,Ω)-convex at x, and (A1) is true, and $x \in \partial_H F^c(\omega;h)$, then there must exist $y \in F(x)$such that $\omega \in \partial_\Omega F(x;y)$

4. (H,Ω) CONJUGATE DUALITY THEORY IN MULTIOBJECTIVE OPTIMIZATION

In this section, we are concerned with a so-called multiobjective minimization problem which aims to minimize several incommensurable objective point-to-set functions simultaneously. We define a dual problem in a wide sense and examine the relationship between the primal problem and the dual problem.

We are concerned with the following multiobjective set-valued minimization problem:

(MP) Min $\{F(x): x\in X\}$

where $F:R^n \to P(R^p)$, $X\subseteq R^n$. Problem (MP) will be termed the primal problem.

We analyze (MP) by embedding it into a family of perturbed problems. Consider the p-dimensional set-valued function $\Phi:X\times U\to P(R^p)$ such that $\Phi(x,0)=F(x)$ for $\forall x\in X$. where $U\subseteq R^l$ for some integer l, is the subset of R^l containing 0. U will be call the perturbed space. For every $u\in U$, the following minimization problem shall be considered:

(MP_u) Min $\{\Phi(x,u): x\in X\}$

Clearly, for u=0, (MP_u) is none other than (MP). Problems (MP_u) will be called perturbed problems of (MP) with respect to the given perturbation Φ.

Let Ω^k be a set of functions $\omega:R^k \to R^m$, where k can be any integer number. H is the same as section 2, i.e., it is a family of functions $h:R^m\to R^p$, closed under Max-pointwise. Especially, denote $\Omega^*=\Omega^n \oplus\Omega^l$, $\Omega=\Omega^l$, where $\omega^*\in\Omega^*$ if and only if there exist $\omega\in\Omega^n$, $\theta\in\Omega^l$ such that $\omega^*(x,u)=\omega(x)+\theta(u)$ for $\forall(x,u)\in X\times U$.

Taking (H,Ω) conjugate for $\Phi:X\times U\to P(R^p)$, we have by theorem 2.1:

$\Phi^c(\omega,\theta)(\omega(x)+\theta(u))\geq\neq\Phi(x,u)$ for $\forall(\omega,\theta)\in\Omega^n\times\Omega$

Suppose that Ω^n contains the zero element 0_Ω , that is, $0_\Omega(x)=0$ for $\forall x\in R^n$. Setting $\omega=0_\Omega$, u=0, we obtain

$\Phi^c(0_\Omega,\theta)(\theta(0)) \geq\neq \Phi(x,0)$ for $\forall x\in X$, $\theta\in\Omega$.

This motivates the following definition:

Definition 4.1

The following problem

(MD) Max $\cup_{\theta\in\Omega} \Phi^c(0_\Omega,\theta)(\theta(0))$

will be called the (H,Ω) conjugate dual problem of (MP) with respect to the perturbation Φ, briefly called the dual problem. Moreover, the following perturbed set-valued optimization problem:

(MD_u) Max $\cup_{\theta\in\Omega} \Phi^c(0_\Omega,\theta)(\theta(u))$

will be called the (H,Ω) conjugate dual problem of the perturbed problem (MP).

Definition 4.2

$x\in X$ is said to be the efficient solution of (MP), if

$\Phi(x,0)\cap \text{Min} \cup_x \Phi(x,0)\neq\varnothing$

Similarly, $\theta\in\Omega$ is said to be the efficient solution of (MD), if

$\Phi^c(0_\Omega,\theta)(\theta(0))\cap \text{Max} \cup_{\theta\in\Omega} \Phi^c(0_\Omega,\theta)(\theta(0))\neq\varnothing$

Let Min(MP)=Min $\cup_x \Phi(x,0)$

Max(MD)=Max $\cup_{\theta\in\Omega} \Phi^c(0_\Omega,\theta)(\theta(0))$

The first dual result between (MP) and (MD) is the following theorem.

Theorem 4.1 ((H,Ω) weak duality theorem)

(1). For $\forall x\in R^n$, and $\theta\in\Omega$, we have $\Phi^c(0_\Omega,\theta)(\theta(0))\geq\neq\Phi(x,0)$

(2). If $x\in X,\theta\in\Omega$,and $\Phi^c(0_\Omega,\theta)(\theta(0))\cap\Phi(x,0)\neq\varnothing$, then x and θ are the efficient solutions of (MP) and (MD), respectively, and Min(MP)∩Max(MD)≠∅ .

(3). If there exist $x\in X$ such that $\Phi(x,0)\cap$ Max(MD)≠∅, then x is the efficient solution to (MP) and there exists the common efficient point for (MP) and (MD).

(4). Suppose that $\theta\in\Omega$ satisfies Min(MP)$\cap\Phi^c(0_\Omega,\theta)(\theta(0))\neq\varnothing$, then θ is the efficient solution to (MD), and there exists the common efficient point for (MP) and (MD).

Definition 4.3

The (H,Ω) conjugate dual gap between (MP) and (MD) is said to be weakly zero, zero, strongly zero, and inversely zero, respectively, if, respectively, Min(MP)∩Max(MD)≠∅, Min(MP)⊆Max(MD), Min(MP)=Max(MD), and Min(MP) ⊇ Max(MD).

Definition 4.4

The primal perturbation map $P:U\to P(R^p)$ and the dual perturbation map $D:\Omega^n\to P(H)$ are defined by

$P(u)=\mathrm{Min}\ \cup_x \Phi(x,u)$ for $\forall u\in U$

$D(\omega)(t)=\mathrm{Max}\ \cup_{\theta\in\Omega}\Phi^c(0_\Omega,\theta)(t+\theta(0))$ for $\forall\omega\in\Omega$, $t\in R^m$.

Assumption (A2). The set $\cup_x \Phi(x,u)$ is Min-complete for $\forall u\in U$.

Assumption (A3). The set $\cup_{\theta\in\Omega}\Phi^c(\omega,\theta)(\omega(x)+\theta(0))$ is Max-complete for $\forall\omega\in\Omega$,$x\in X$.

Assumption (A4). The set $\{h: h\in H, h\otimes\theta\geq\neq P\}$ is Max-complete for $\forall\theta\in\Omega$, i.e., for $\forall\theta\in\Omega,t\in R^m$, $\{h(t): h\in H, h\otimes\theta\geq\neq P\}$ is Max-complete.

Associating with P its (H,Ω) conjugate map P^c , D its (H,R^n) conjugate map D* , we have the following result.

Theorem 4.2

(1). $\Phi c(0\Omega,\theta)\geq\neq Pc(\theta)$ for $\forall\theta\in\Omega$. Under assumption (A2), $\Phi c(0\Omega,\theta)=Pc(\theta)$ for $\forall q\in\Omega$.

(2). $\Phi c^*(x,0)\subseteq D^*(x)$ for $\forall x\in X$. Under Assumption (A3), $\Phi c^*(x,0)=D^*(x)$ for $\forall x\in X$.

According to this theorem, under (A2), (MD) can be rewritten as

(MD) $\quad \mathrm{Max}\ \cup\theta\in\Omega\ Pc(\theta)(\theta(0))$

Hence, Max(MD)=Pc*(0). On the other hand, by the definition of P, we have Min(MP) =P(0). If we define the (H,Rn) dual problem of (MD) to be

(MP') $\quad \mathrm{Min}\ \cup x\psi^*(x)$

where $\psi:\Omega n\to P(H)$ is defined by

$\psi(\omega)(t)=\ \cup\theta\in\Omega\ \Phi c(\omega,\theta)(t+\theta(0))$

then under (A3), (MP') can be rewritten as

(MP') Min $\cup$x D*(x).

It is easily shown that $\psi^*(x)=\Phi c^*(x,0)$, for $\forall x\in X$.

Corollary 4.2

Under (A4), (A2), the (H,Ω) conjugate dual gap between (MP) and (MD) is weakly zero, zero, strongly zero, and inversely zero, respectively, if and only if the primal perturbation map P is weak (H,Ω)-convex, (H,Ω)-convex, strong (H,Ω)-convex at $0\in X$, and $P^0(0)\subseteq P(0)$, respectively.

Definition 4.5 (stability)

(MP) is called (H,Ω)-stable, if P is (H,Ω)-subdifferentiable at $0\in U$. Correspondingly, (MD) is called (H,R^n)-stable, if D is (H,R^n)-subdifferentiable at $0_\Omega \in\Omega$.

Theorem 4.3

Under the complete conditions A2 and A4, Eff(MD)$\subseteq\cup\ \partial_\Omega\ P^0(0;y)$, where Eff(MD) denotes the set of efficient solutions to (MD). Moreover, if P is (H,Ω)-convex map on U, then Eff(MD)= $\cup_y\partial_\Omega\ P^0(0;y)$.

Based on the (H,Ω)-stability and (H,Ω)-subgradient, we have the following strong duality.

Theorem 4.4 ((H,Ω) strong duality theorem)

Under the assumption (A2),

(1) (MP) is (H,Ω)-stable, if and only if for every efficient solution $x\in X$ to (MP) and $y\in\Phi(x,0)\cap P(0)$, there exists $\theta\in\Omega$ as the efficient solution to (MD), such that $y\in\Phi^c(0_\Omega,\theta)(\theta(0))$. Therefore, for every $y\in P(0)\cap\Phi(x,0)$, there exists $\theta\in\Omega$ such that $y\in P^c(\theta)(\theta(0))$, i.e., $\theta\in\partial_\Omega\ P(0;y)$.

(2) Suppose that $P(0)\neq\varnothing$, that is, Eff(MP)$\neq\varnothing$. Then, (MD) has the efficient solution, i.e., Eff(MD)$\neq\varnothing$, and the dual gap is zero between (MP) and (MD), if and only if $\partial_\Omega P(0;y)\neq\varnothing$ for $\forall y\in P(0)$. In this time, Eff(MD)$\supseteq\ \cup_y\partial_\Omega P(0;y)$. Furthermore, if the dual gap between (MP) and (MD) is strongly zero, than Eff(MD)= $\cup_y\ \partial_\Omega\ P(0;y)$.

Theorem 4.5

Under the assumption (A2), suppose that there exists $h_0\in H$ being an identical element, i.e., $h_0(t)=t$ for $\forall t\in R^m$, and for every induced function θ of P, $\theta\in\Omega$, then P is (H,Ω)-subdifferentiable at $0\in U$, hence, (MD) has the efficient solution and $P(0)\subseteq P^{c*}(0)$, that is, the dual gap between (MP) and (MD) is zero.

This theorem shows that, taking $H=\{h_b: h_b(t)=t+b\ (\forall t\in R^m)\ ,\ b\in R^p\ \}(p=m)$, as long as Ω is broad enough, then (MD) has the efficient solution and dual gap between (MP) and (MD) is zero. The way to take Ω can be implemented based on the primal problem (MP).

Assumption (A5). {h: $h\in H$, $h(\omega(x)+\theta(u))\geq\neq\Phi(x,u)$ for $\forall x\in X$, and $u\in U$}is Max-complete for $\forall\omega\in\Omega^n$,$\theta\in\Omega$.

Theorem 4.6 ((H,Ω) strong duality theorem)

Suppose that for $\forall x\in X$, Φ is strong (H,Ω)-convex or inverse (H,Ω)-convex at $(x,0)\in X\times U$, and the assumption (A5) is true.

(1). Eff(MD)$\neq\varnothing$ and the dual gap is weakly zero, if and only if there exists $z\in\psi(0_\Omega)$ such that $\partial_H\psi(0_\Omega,z)\neq\varnothing$, so Eff(MP)$\neq\varnothing$. In this time, Eff(MP)$\supseteq\cup_z\partial_H\psi(0_\Omega;z)$. Furthermore, if the dual gap is zero, then Eff(MP)$=\cup_z\partial_H\psi(0_\Omega;z)$. Under the assumption (A3), substituting ψ for D, the above conclusion is still true.

(2). Eff(MD)$\neq\varnothing$ and the dual gap is inversely zero, i.e., Max(MD)$\subseteq$Min(MP), if and only if for every $z\in$Max $\psi(0_\Omega)$, therefore $z\in\psi(0_\Omega)$, $\partial_H\psi(0_\Omega;z)\neq\varnothing$. In this time, Eff(MP)$\neq\varnothing$ and Eff(MP)$\supseteq\cup_z\partial_H\psi(0_\Omega;z)$. Furthermore, if the dual gap is zero, than Eff(MP)= $\cup_z\partial_H\psi(0_\Omega;z)$. Under the assumption (A3), substituting ψ for D, the above conclusion is still true. Notice that D(ω)=Max $\psi(\omega)$ for $\forall\omega\in\Omega$.

(3). Under assumptions (A2) and (A3), (MP) is (H,Ω)-stable and (MD) is (H,R^n)-stable, if and only if Eff(MP)$\neq\varnothing$, Eff(MD)$\neq\varnothing$, and the dual gap is strongly zero.

(4). If the dual gap is strongly zero, then Eff(MP)$\neq\varnothing$ if and only if Eff(MD)$\neq\varnothing$.

Definition 4.6 (normality)

If P is weak (H,Ω)-convex, (H,Ω)-convex, strong (H,Ω)-convex and inverse (H,Ω)-convex at $0\in U$, respectively, then (MP) is said weak (H,Ω)-normal,(H,Ω)-normal,, strong (H,Ω)-normal and inverse (H,)-normal, respectively. Dually, if D is weak (H,R^n)-convex, (H,R^n)-convex, strong (H,R^n)-convex and inverse (H,R^n)-convex, respectively, then (MD) is said weak (H,R^n)-normal,(H,R^n)-normal, strong (H,R^n)-normal and inverse (H,R^n)-normal.

Corollary

Under the assumptions (A2) and (A3), suppose that for $\forall x\in X$, Φ is strong (H,Ω)-convex at $(x,0)\in X\times U$, then the dual gap is weakly zero, zero, strongly zero and inversely zero, if and only if (MP) is weak (H,Ω)-normal or equivalently (MD) is weak (H,R^n)-normal, (MP) is (H,Ω)-normal or equivalently (MD) is inverse (H,R^n)-normal, (MP) is strong (H,Ω)-normal or equivalently (MD) is (H,R^n)-normal, and (MP) is inverse (H,Ω)-normal or equivalently (MD) is (H,R^n)-normal, respectively.

5. (H,Ω)-LAGRANGIAN MAP AND SADDLE POINTS

In this section, we will define the (H,Ω)-Lagrangian map of (MP) and investigate the properties of its saddle-points.

Definition 5.1 ((H,Ω)-Lagrangian map) The point-to-set map $L:X\times U\to P(Rp)$ defined by the formula:

$$L(x,\theta)=\Phi_x(\theta)(\theta(0)) \text{ for } \forall(x,\theta)\in X\times\Omega$$

is called the (H,Ω)-Lagrangian map of (MP) relative to the given perturbation Φ, where $\Phi_x:U\to P(R^p)$ for $\forall x\in X$ is defined by $\Phi_x(u)=\Phi(x,u)$ for $\forall u\in U$.

First, we will discuss the expressions of (MP) and (MD) in terms of the map L.

Assumption (A6). For $\forall x\in X$, $\{h: h\in H, h\otimes\theta\geq\neq\Phi_x\}$ is Max-complete.

Assumption (A7). For $\forall x\in X$, Φ_x is (H,Ω)-convex at $0\in U$.

Assumption (A8). For $\forall\theta\in\Omega,x\in X,u\in U$, and $y\in\Phi_x(u)$, there exists $h\in\Phi_x^c(\theta)$ such that $h(\theta(u))\leq\neq y$.

Assumption (A9). For $\forall\theta\in\Omega$, $y\in\Phi_x^c(0_\Omega,\theta)(\theta(0))$, there exists no $x\in X$ such that $y\leq\neq\Phi_x^c(\theta)(\theta(0))$.

Theorem 5.1

(1). Under the assumptions (A5) and (A7), $F(x,0)=\mathrm{Max}\ \cup_{\theta\in\Omega} L(x,\theta)$,and (MP) can be expressed as $\mathrm{Min}\cup_x\{\mathrm{Max}\ \cup_\theta L(x,\theta)\}$.

(2). Under the assumptions (A6) and (A7), for $\forall\theta\in\Omega$,we have $\Phi_x^c(0_\Omega,\theta)(\theta(0))\subseteq\mathrm{Min}\ \cup_x L(x,\theta)$. Furthermore, if H is closed Min-pointwise and (A5),(A8) are true, then $\Phi_x^c(0_\Omega,\theta)(\theta(0))=\mathrm{Min}\cup_x L(x,\theta)$ for $\forall\theta\in\Omega$, and (MD) can be expressed as $\mathrm{Max}\ \cup_\theta\{\mathrm{Min}\ \cup_x L(x,\theta)\}$.

Definition 5.2 (saddle-point)

$(x,\theta)\in X\times\Omega$ is called the saddle-point of map L, if

$$L(x,\theta)\cap(\mathrm{Max}\cup_\theta L(x,\theta))\cap(\mathrm{Min}\cup_x L(x,\theta))\neq\varnothing.$$

The relationship between the saddle-points of map L and the efficient solutions to (MP) and (MD) is given as follows.

Theorem 5.2

Under the assumptions (A5)---(A9), suppose that H is closed Min-pointwise. Then

(1) $x\in\mathrm{Eff(MP)}$, $\theta\in\mathrm{Eff(MD)}$ and $\Phi(x,0)\cap\Phi_x^c(0_\Omega,\theta)(\theta(0))\neq\varnothing$ if and only if (x,θ) is the saddle-point of map L.

(2) under the assumption (A2), if (MP) is (H,Ω)-stable or the dual gap is zero, then $x\in\mathrm{Eff(MP)}$ if and only if there exists $\theta\in\Omega$ such that (x,θ) is the saddle-point of map L.

(3) under the assumption (A3), if (MD) is (H,R^n)-stable or the dual gap is inversely zero, then $\theta\in\mathrm{Eff(MD)}$ if and only if there exists $x\in X$ such that (x,θ) is the saddle-point of map L.

6. SOME SPECIAL CASES FOR H AND Ω

Case 1

Consider the following multiobjective convex programming problem:

(MCP) $\mathrm{Min}\{f(x):x\in X\}$, $X=\{x\in X': g(x)\leq=Q\ 0,\ X'\subseteq Rn\}$

where

(1) Q is the pointed closed convex cone with the nonempty interior $\mathrm{int}(Q)\neq\varnothing$ in Rm ;

(2) $f:Rn\rightarrow Rp$, continuous;

(3) $g:Rn\rightarrow Rm$, continuous and Q-convex.

$H=\{hb : hb(t)=-t+b\ (\forall t\in Rp),\ b\in Rp\}$

$\Omega=\{\Lambda\in Rp\times m : \Lambda Q\subseteq Rp\}$

and perturbation map Φ is vertical, given by $\Phi(x,u)=f(x)$ for $x\in X'$ and $g(x)\leq=Q$, $u\in Rm$, otherwise

$\Phi(x,u)=+\infty$.

Under some proper conditions, the (H,Ω) conjugate dual problem (MCD) of (MCP) is the same as the dual problem discussed in [13] with the corresponding domination cone being positive Rm+ . Therefore, the dual results of [13] can be considered as the special case of this paper. Especially, for the linear case, the duality results of [8] are the special case in this paper.

Case 2

Consider the following nonlinear programming problem:

(NP) min $\{f(x): x\in X\}$ $X=\{x: g(x)\leq 0, x\in X'\subseteq Rn\}$.

where $f:Rn \to R$, $g:Rn \to Rm$. Let p=1, then the duality results of this paper are the same as [3,4].

7. CONCLUDING REMARKS

In this paper, we have developed a duality theory for multiobjective optimization, and that for multiobjective programming is the special case for special Φ, with the help of the generalized conjugate map. The approach taken in this paper is based on efficiency (Pareto optimality) and some interesting results parallel with the well-known ordinary conjugate duality are obtained. For the multiobjective convex optimization problem, the form of H and Ω discussed in section 6 is enough for developing the necessary duality results. But for the multiobjective nonconvex optimization problem, such a form of H and Ω is not enough, and other families of H and Ω need to be developed. How to select properly the form of H and Ω such that the duality can be used to make decision analysis for multiobjective problems is a program to be researched further.

However, there are some unsatisfactory points for the (H,Ω) duality theory, since the concept of "vector supremum" or "vector infimum" based on efficiency is not well defined. One possible way to define "vector supremum" is given by Sup A=Max clA, where $A\subseteq R^p$, clA denotes the topological closure of set A. As yet, there has been no development of Sup or Inf playing an effective role in duality for efficient solutions. This should be the subject of further research. Kawasaki [10,11] provided some interesting results by defining conjugate and subgradients via weak supremum. His approach is, however, artificial and so very difficult to understand, and weak Pareto optimality is not a good solution concept as Pareto optimality. Hence, we have adopted the more intuitive approach in this paper. The interested readers may refer to his papers or Sawaragi et al [14].

There are some possible generalizations which perhaps can be useful in multiobjective nonconvex duality theory.

(a). A first generalization can be made by replacing the family of vector-valued functions H or Ω by more general family of set-valued functions.

(b). Another possible generalization can be obtained by taking $H=\cup H_\omega$, where H_ω is a family of vector-valued functions depending upon $\omega\in\Omega$, and Ω is given as in this paper.

(c). Finally, we have discussed the duality in finite-dimensional Euclidean space. Similarly, we can extend the discussion to the case in some topological space, which will lead the conclusion of duality theory to its applications in the optimal control problem and variational problem.

REFERENCES

[1] Bitran,G.R.. Duality for nonlinear multi-criteria optimization problems. *Journal of Optimization Theory and Applications*, 1981, 35, 367-401.

[2] Brumelle, S., Duality for multiple objective convex programs, *Mathematics of Operations Research*, 1982,7,95-110.

[3] Feng, Junwen. (H,Ω) conjugate function theory. *System Science and Mathematical Science*, 1990, 10(1).

[4] Feng, Junwen. (H,Ω) duality theory for nonlinear programming. *System Science and Mathematical Science*, 1990, 10(2),.

[5] Gale, D.. H.W.Kuhn and A.W.Tucker, Linear programming and the theory of games, in T.C.Koopmans (ed.). *Activity Analysis of Production and Allocation*, John Wiley, 1951, 317-329.

[6] Isermann,H.. The relevance of duality in multiobjective linear programming, in: Starr,M.K. and M.Zeleny (eds.). *Multiple Criteria Decision Making*. Amsterdam: North-Holland, 1977.

[7] Isermann,H.. On some relations between a dual pair of multiple objective linear programs. *Zeitschrift fur Operations Research*, 1978, 22,33-41.

[8] Isermann, H.. Duality in multiple objective linear programming, in:S.Zionts (ed.). *Multiple Criteria Problem Solving*. New York: Springer-Verlag, 1978.

[9] Jahn, J.. Duality in vector optimization. *Mathematical Programming*, 1983, 25,343-353.

[10] Kawasaki, H.. Conjugate relations and weak subdifferentials. *Mathematics of Operations Research*, 1981, 6,593-607.

[11]. Kawasaki,H.. A duality theory in Multiobjective nonlinear programming. *Mathematics of Operations Research*, 1982, 7,95-110.

[12] Kornbluth,J.S.H., Duality. indifference and sensitivity analysis in multiple objective linear programming. *Operational Research Quarterly*, 1974, 25,599-614.

[13] Nakayama,H., A geometric consideration on duality in vector optimization, *Journal of Optimization Theory and Applications*, 1984, 44(4).

[14]. Sawaragi,Y.. H.Nakayama and T.Tanino. *Theory of Multiobjective Optimization*. New York: Academic Press, 1986.

[15] Tanino,T.. Saddle points and duality in multiobjective programming, *International Journal of Systems Science*, 1982, 13,323-335.

[16] Tanino,T. and Y.Sawaragi. Duality theory in multiobjective programming, *Journal of Optimization Theory and Applications*, 1979, 27,509-529.

[17] Tanino,T, and Y.Sawaragi. Conjugate maps and duality in multiobjective optimization, *Journal of Optimization Theory and Applications*,1980, 31, 473-499.