Hence, one can conclude that

$$\text{mod}_C(\{\mathcal{P}_k(v_n)\})=0, \quad k=0,\ldots,m. \tag{9}$$

Combining (5), (8) and (9), one has

$$\beta(\mathcal{D})=0,$$

which implies $\mathcal{D}$ is a relatively compact subset of PC_T. Moreover, $\mathcal{D}$ is compact because $\mathcal{D}$ is closed and relatively compact.

Now, we verify that F is *u.s.c.* on $\mathcal{D}$. Since, we have that $\mathcal{F}$ is qusi-compact. Let $\{(u_n,v_n)\}$ be a sequence in $Gra(\mathcal{F})$ such that

$$(u_n,v_n)\to(u,v) \quad \text{in } PC_T\times PC_T.$$

Then there exists a sequence $\{f_n\}\subset L(J;X)$ such that $f_n\in Sel_F(u_n)$ and $v_n\in\mathcal{G}(f_n)$. Observe that Sel_F is weakly *u.s.c.* with convex, weakly compact values due to Lemma 3.1. It follows from Lemma 2.2 that there exists $f\in Sel_F(u)$ and a subsequence of $\{f_n\}$, still denoted by $\{f_n\}$, such that $f_n\to f$ weakly in $L(J;X)$. Lemma 2.5 guarantees that $v\in\mathcal{G}(f)$ and thus $v\in\mathcal{F}(u)$, which implies that $\mathcal{F}$ is closed. Hence, it yields from Lemma 2.1 that $\mathcal{F}$ is *u.s.c.* on $\mathcal{D}$.

Finally, we process to prove that F has contractible values. Let $u\in\mathcal{D}$ and $\hat{f}\in Sel_F(u)$. Define a function $h:[0,\,1]\times\mathcal{F}(u)\to\mathcal{F}(u)$ by

$$h(\lambda,v)(t)=\begin{cases} v(t), & t\in[-h,\lambda T],\\ \tilde{u}(t;\lambda T,v(\lambda T),\hat{f}), & t\in(\lambda T,T],\end{cases}$$

where $\tilde{u}(t;\lambda T,v(\lambda T),\hat{f})$ is the unique mild solution of the following problem

$$\begin{cases} u'(t)-Au(t)=\hat{f}(t), & t\in[\lambda T,T],\ t\neq t_k,\\ u(t_k^+)=u(t_k)+I_k(u(t_k)), & k=1,\ldots,m,\\ u(\lambda T)=v(\lambda T).\end{cases}$$

Clearly, h is well defined, and for every $v\in\mathcal{F}(u)$,

$$h(0,v)=\mathcal{G}(\hat{f}) \text{ and } h(1,v)=v.$$

Below, we verify that h is continuous. Let $(\lambda_i,v_i)\in[0,\,1]\times\mathcal{F}(u), i=1,2$, with $\lambda_1\leq\lambda_1$, one can choose $f_i\in Sel_F(u)$ such that $h(\lambda_i,v_i)=\mathcal{G}(f_i)$ and $f_i(t)=f(t)$ for all $t\in[\lambda_i T,T]$. Write,

$$\rho(t)=h(\lambda_1,v_1)(t)-h(\lambda_2,v_2)(t), \quad t\in J.$$

For $t\in[-h,\lambda_1 T]$, we have

$$\begin{aligned}\|\rho(t)\|&=\|v_1(t)-v_2(t)\|\\ &\leq|v_1-v_2|_T.\end{aligned}$$

Also, for $t\in(\lambda_1 T,\lambda_2 T]$, it follows from (6) and (H4) that

$$\begin{aligned}\|\rho(t)\|&=\|\tilde{u}(t;\lambda_1 T,v_1(\lambda_1 T))-v_2(t)\|\\ &\leq M|v_1-v_2|_T+2MN_2\int_{\lambda_1 T}^{\lambda_2 T}\eta(s)\mathrm{d}s+2M\sum_{\lambda_1 T\leq t_k<\lambda_2 T}c_k.\end{aligned}$$

Moreover, in view of (H4), (H5) and the fact that $f_1(t)=f_2(t)$ for $t\in[\lambda_2 T,T]$, we obtain that for $t\in[\lambda_2 T,T]$,

$$\begin{aligned}\|\rho(t)\|&=\|\tilde{u}(t;\lambda_1 T,v_1(\lambda_1 T))-\tilde{u}(t;\lambda_2 T,v_2(\lambda_2 T))\|\\ &\leq M\|T(\lambda_2 T-\lambda_1 T)v_1(\lambda_1 T)-v_2(\lambda_2 T)\|+MN_2\int_{\lambda_1 T}^{\lambda_2 T}\eta(s)\mathrm{d}s\\ &\quad+M\sum_{\lambda_1 T\leq t_k<\lambda_2 T}c_k+M\sum_{k=1}^{m}l_k|\rho|_T.\end{aligned}$$

Therefore, we conclude that for each $t\in[-h,\,T]$,

$$\begin{aligned}\|\rho(t)\|&\leq M|v_1-v_2|_T+M\|T(\lambda_2 T-\lambda_1 T)v_1(\lambda_1 T)-v_2(\lambda_2 T)\|\\ &\quad+2MN_2\int_{\lambda_1 T}^{\lambda_2 T}\eta(s)\mathrm{d}s+2M\sum_{\lambda_1 T\leq t_k<\lambda_2 T}c_k+M\sum_{k=1}^{m}l_k|\rho|_T\end{aligned}$$

Note that

$$\begin{aligned}&M|v_1-v_2|_T+M\|T(\lambda_2 T-\lambda_1 T)v_1(\lambda_1 T)-v_2(\lambda_2 T)\|\\ &\quad+2MN_2\int_{\lambda_1 T}^{\lambda_2 T}\eta(s)\mathrm{d}s+2M\sum_{\lambda_1 T\leq t_k<\lambda_2 T}c_k\\ &\to 0 \text{ as } \lambda_1\to\lambda_2,\ v_1\to v_2\end{aligned}$$