converge uniformly to u in PC_T. Subsequently, the lemma can be proved in an argument similar to Lemma 3.1 of Chen (2013).

Remark 3.2. From (H_5), it follows that there exist numbers $l_k \geq 0, k=1,...,m$, such that

$$\chi(I_k(D)) \leq l_k \chi(D)$$

for any bounded set $D \subset X$.

Theorem 3.1. Let the assumptions (H_1)-(H_5) hold. Assume further that X is reflexive and $T(t)$ is uniform operator topology continuous for $t > 0$, then for each $\varphi \in C_h$, problem (1) has at least one mild solution provided that

$$4M\|\mu\|_{L(J;\mathbb{R}^+)} + M\sum_{k=1}^{m} l_k < 1. \tag{4}$$

Proof. For $\varphi \in C_h$, let

$$\mathcal{D}_0 = \{u \in PC_T : u(t) = \varphi(t) \text{ for } t \in [-h,0], \text{ and } \sup_{s\in[0,t]} \|u(s)\| \leq \omega(t) \text{ for all } t \in J\},$$

where ω is the solution of the following integral equation

$$\omega(t) = N_1 + 2M\int_0^t \eta(s)\omega(s)\mathrm{d}s,$$

where

$$N_1 = M\|\varphi(0)\| + M\sum_{k=1}^{m} c_k + (M + M\|\varphi\|_{C_h})\|\eta\|_{L(J;R^+)}.$$

One can find that $\mathcal{D}_0 \subset PC_T$ is closed, bounded and convex.

Let us define the multi-valued map $\mathcal{F} : PC_T \to 2^{PC_T} T$ as follows:

$$\mathcal{F} := \mathcal{G} \circ Sel_F,$$

where $\mathcal{G}(f)$ is the unique mild solution of the problem (3) corresponding to $f \in L(J;X)$. In fact, (H_6) and (4) ensure the uniqueness of the mild solution of the problem (3).

We first claim that $\mathcal{F}(\mathcal{D}_0) \subset \mathcal{D}_0$. Indeed, taking $u \subset \mathcal{D}_0$ and $v \in \mathcal{F}(\mathcal{D}_0)$, there exists $f \in Sel_F(u)$ such that $v = \mathcal{G}(f)$. For each $t \in J$, it follows from (H_2) and (H_4) that

$$\begin{aligned}
\|v(t)\| &\leq M\|\varphi(0)\| + M\int_0^t \|f(s)\|\,\mathrm{d}s + M\sum_{0<t_k<t} \|I_k(u(t_k))\| \\
&\leq N_1 + 2M\int_0^t \eta(s) \sup_{\sigma\in[0,s]} u(\sigma)\mathrm{d}s \\
&\leq N_1 + 2M\int_0^t \eta(s)\omega(s)\mathrm{d}s \\
&= \omega(t),
\end{aligned}$$

here we have used

$$|u_s|_0 = \sup_{\sigma\in[-h,0]} \|u(s+\sigma)\| \leq \|\varphi\|_{C_h} + \sup_{\sigma\in[0,s]} \|u(\sigma)\|.$$

This implies $v \in \mathcal{D}_0$, and then one has $\mathcal{F}(\mathcal{D}_0) \subset \mathcal{D}_0$.

Let

$$\mathcal{D}_{n'+1} = \overline{\mathrm{conv}}\mathcal{F}(\mathcal{D}_{n'}), \quad n' = 0,1,2,\cdots,$$

then $D_{n'}$ is closed and convex. It is further easy to see that

$$\mathcal{D}_{n'+1} \subset \mathcal{D}_{n'} \subset \cdots \subset \mathcal{D}_0.$$

Define

$$\mathcal{D} := \bigcap_{n'=0}^{\infty} \mathcal{D}_{n'}.$$

Then $\mathcal{D}$ is nonempty and closed convex subset of PC_T, and $\mathcal{F}(\mathcal{D}) \subset \mathcal{D}$.

In the sequel, we will show that $\mathcal{D}$ is compact. Let us introduce the following MNC in PC_T : for a bounded set $\Omega \subset PC_T$,

$$\begin{aligned}
&\beta(\Omega) := \max_{D\in\Delta(\Omega)}(\xi(D), \max_{0\leq k\leq m} \mathrm{mod}_C(\mathcal{P}_k(D))), \\
&\xi(\Omega) = \sup_{t\in[a,b]} \chi(\Omega(t)), \Omega(t) = \{y(t) : y \in \Omega\}, \\
&\mathrm{mod}_C(\Omega) = \lim_{\delta\to 0}\sup_{y\in\Omega}\max_{|t_1-t_2|<\delta} \|y(t_1) - y(t_2)\|,
\end{aligned} \tag{5}$$

where $\Delta(\Omega)$ is the collection of all countable subsets of Ω and the maximum is taken in the sense of the partial order in the cone R_+^2. It is noted that β satisfies all usual properties of MNC, including the regularity (see e.g., Chuong, 2012; Obukhovskii, 2010).

By the definition of β, there exists a sequence $\{v_n\} \subset \mathcal{D}$ such that

$$\beta(\mathcal{D}) = (\xi(\{v_n\}), \max_{0\leq k\leq m} \mathrm{mod}_C(\{\mathcal{P}_k(v_n)\})).$$

For $t \in [-h,0]$, it is easy to see that $\chi(v_n(t)) = 0$. Let us take $f_n \in Sel_F(v_n)$ such that $v_n = \mathcal{G}(f_n)$. Then, it follows from (H_2) that for every $t \in J$ and $s < t$,

$$\|T(t-s)f_n(s)\| \leq MN_2\eta(s), \tag{6}$$

where $N_2 = 1 + \|\varphi\|_{C_h} + 2\omega(T)$. This yields that the set $\{T(t-\cdot)f_n(\cdot)\}$ is integral bounded in $L(J; X)$. Also, in view of (2) and (H_3), it yields that for every $t \in J$ and $s < t$,

$$\begin{aligned}
\chi(\{T(t-s)f_n(s)\}) &\leq M\mu(s)[\chi(\{v_n(s)\}) + \sup_{\sigma\in[-h,0]} \chi(\{v_n(s+\sigma)\}] \\
&\leq 2M\mu(s)\xi(\{v_n\}).
\end{aligned} \tag{7}$$

Then, by (2), (7), Lemma 2.3 and Remark 3.2, one obtains that for each $t \in J$,

$$\begin{aligned}
\chi(\{v_n(t)\}) &\leq \chi\left(\left\{\int_0^t T(t-s)f_n(s)\mathrm{d}s\right\}\right) + \chi\left(\left\{\sum_{0<t_k<t} T(t-t_k)I_k(v_n(t_k))\right\}\right) \\
&\leq \left(4M\|\mu\|_{L(J;\mathbb{R}^+)} + M\sum_{k=1}^{m} l_k\right)\xi(\{v_n\}).
\end{aligned}$$

