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Existence Results of Noncompact Impulsive Delay Evolution Inclusions

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Abstract

This paper deals with a class of abstract Cauchy problems for impulsive delay evolution inclusions in the Banach spaces. By using measures of noncompactness, multi-valued analysis and fixed point theory, we establish the existence of mild solutions for the mentioned inclusions under the assumption that the semigroup generated by linear part is noncompact. Finally, an illustrating example is given.

Key words: Impulsive evolution inclusions; Delay; Weakly upper semi-continuous; Measures of noncompactness.

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INTRODUCTION

In recent years, there have been significant development in impulse theory. Especially in the area of impulsive differential equations and inclusions since their wide applications to problems arising in the population biology, the diffusion of chemicals, the spread of heat, the radiation of electromagnetic waves, etc.. These processes, for which the adequate mathematical models are impulsive differential equations and inclusions, are characterized by the sudden changes of states at certain moments of time and depend on their prehistory. At present, there are many researchers to address the impulsive differential equations, we may cite, among others (Ahmed, 2006; Benchohra, 2006; Fečkan, 2012; Lakshmikantham, 1989; Samoilenko, 1995). For a survey of results in the impulsive differential inclusions see, e.g., Benchohra (2003), Cardinali (2008), Chuong (2012), Djebali (2011), Gabor (2012) and the references therein.

In this paper, we deal with the abstract Cauchy problem of impulsive delay evolution inclusion in a Banach space X described in the form

$$\begin{cases} u'(t) - Au(t) \in F(t, u(t), u_t), & t \in J := [0,T], t \neq t_k, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), & k = 1, \dots, m, \\ u(t) = \varphi(t), & t \in [-h, 0], \end{cases} \tag{1}$$

where A is a closed linear operator generating a C0-semigroup $\{T(t)\}_{t\geq 0}$ on X; $F:J\times X\times PC_0\to 2^X$ is a multi-valued function with convex, closed values for which $F(t,\cdot,\cdot)$ is weakly upper semi-continuous for a.e. $t\in J$ and $F(\cdot,x,v)$ F has a strongly measurable selection for each $(x, v) \in X \times PC_0$, where $PC_0 = PC([-h,0;X])$ specified below; $u_t \in PC_0$ is defined by $u_t(s) = u(t+s)(s \in [-h,0])$ for all $s \in [-h,0]$; $\varphi \in C_h := C([-h,0];X)$; $I_k: X\to X$ $k=1,\dots,m$ are given functions to be specified later. On the one hand, most of the previous works established the existence theorem by assuming that

• The semigroup $\{T(t)\}_{t\geq 0}$ generated by the operator A is compact, or

• The evolution family $\{U(t, s)\}_{0\leq s\leq t\leq\infty}$ generated by the family $\{A(t)\}_{t\geq 0}$ is compact. This motivates us to study (1) by assuming that the semigroup $T(t)$ generated by the operator A is noncompact. The case when $A = -d$, it generates a uniformly continuous semigroup and this semigroup is noncompact (see, Wang, 2015). Since the lack of the compactness of the semigroup $T(t)$, the nonlinearity and impulsive functions need to satisfy the regularity conditions expressed by measures of noncompactness.

On the other hand, we also notice that for the study of the existence result of (1), the case when the multi-valued functions $F(t,\cdot)$ is weakly upper semi-continuous for $a.e.\ t\in J$ has not yet considered in the literature. This in fact is the other motivation of our work.

In this paper, we aim to investigate the existence of solutions for (1). Note that the key point of the existence result is that we should find a compact convex subset which is invariant under the multi- valued mapping $\mathcal{F}$ (see Theorem 3.1), and this mainly rely on the regularity condition on the nonlinearity and impulsive functions by measure of noncompactness. It is worth mentioning that the assumption of regularity with measure of noncompactness of the modulus of equicontinuity on impulsive functions (see Chuong, 2012) is not involved in our work.

1. PRELIMINARIES

Throughout, X is a Banach space with norm$\|\cdot\|$, 2^X stands for the collection of all nonempty subsets of X and $\mathcal{L}(X)$ denote the Banach space of all bounded linear operators from X to X. Let $C([a, b]; X)$ be the Banach space of all continuous functions from $[a, b]$ to X equipped with the sup-norm. $L(J; X)$ the Banach space consisting of all Bochner integrable functions from J to X with the norm $\|u\|=\int_0^T\|u(t)\|dt$, and the space $PC_t := PC([-h,t];X), 0\leq t\leq T<\infty$, denote by

$$PC_t=\{u:[-h,t]\to X: u \text{ is continuous on } [-h,t]\setminus\{t_k:k=1,\ldots,m\},$$
$$\text{and there exist } u(t_k^-),\ u(t_k^+) \text{ with } u(t_k)=u(t_k^-)\},$$

equipped with the norm

$$|u|_t=\sup_{t\in[-h,t]}\|u(t)\|.$$

Evidently, PC_t is a Banach space for $0\leq t\leq T<\infty$. Let $J_k=(t_k,t_k+1]$, $k=0,\ldots,m$, where $t_0=-h$ and $t_{m+1}=T$. For $u\in PC_T$, define

$$u_k\in C(\bar{J}_k;X), k=0,\ldots,m,$$

$$u_k(t)=\begin{cases}u(t), & t\in J_k,\\ u(t_k^+), & t=t_k.\end{cases}$$

Here, we define the maps $\mathcal{P}_k: PC_T\to C(\bar{J}_k;X), k=0,\ldots,m$ as follows:

$\mathcal{P}_k(u):=u_k$ for every $u\in PC(J;X)$ and $u_k\in C(\bar{J}_k;X)$.

$\mathcal{P}_k(D)$ donoes the restriction of $D\subset PC_T$ on $\bar{J}_k$.

Let Y and Z be metric spaces. Denote

$C(Y)=\{D\in 2^Y: D \text{ is closed}\}$,

$C_v(Y)=\{D\in C(Y): D \text{ is convex}\}$,

$K(Y)=\{D\in C(Y): D \text{ is compact}\}$.

Let $\psi: Y\to 2^Z$ be a multi-valued map and $Gra(\psi)$ the graph of ψ. Denote by $\psi^{-1}(D)=\{y\in Y:\psi(y)\cap D\neq\varnothing\}$ the complete preimage of D under ψ, where $D\subset Z$.

(i) ψ is called closed, if Gra(ψ) is closed in $Y\times Z$,

(ii) ψ is called qusi-compact, if $\psi(D)$ is relatively compact for each compact set $D\subset Y$,

(iii) ψ is called upper semi-continuous (shortly, *u.s.c.*), if $\psi^{-1}(D)$ is closed for each closed set $D\subset Z$, and lower semi-continuous (shortly, *l.s.c.*), if $\psi^{-1}(D)$ is open for each open set $D\subset Z$.

The following lemma (Theorem1.1.12 in Kamenskii, 2001) gives a sufficient condition for *u.s.c.* multi-valued maps.

Lemma 2.1. Let $\psi: Y\to K(Z)$be a closed and quasi-compact multi-valued map. Then ψ is *u.s.c.*

Furthermore, in the case when Y and Z are Banach spaces, a multi-valued map $\psi: D\subset Y\to 2^Z$ is called weakly upper semi-continuous (shortly, weakly *u.s.c.*), if $\psi^{-1}(B)$ is closed in D for every closed set $B\subset Z$.

It is easy to see that upper semi-continuity is stronger than weakly upper semi-continuity and weakly *u.s.c.* function with compact convex values may fail to be *u.s.c.* The following lemma gives an necessary and sufficient condition for weakly *u.s.c.* multi-valued maps (see, Lemma 2.2(ii) of Chen, 2013).

Lemma 2.2. Let $\psi: D\subset Y\to 2^Z$ be a multi-valued map with convex weakly compact values. Then ψ is weakly *u.s.c.* if and only if for each sequence $\{(y_m, z_m)\}_{m\geq 1}\subset D\times Z$ such that $y_m\to y$ *in*

Y and $z_m \in \psi(y_m), m \geq 1$, it follows that there exists a subsequence $\{z_{m_k}\}_{m_k \geq 1}$ of $\{z_m\}_{m \geq 1}$ and $z \in \psi(y)$ such that $z_{mk} \to z$ weakly in Z.

We here introduce some facts about the measure of noncompactness. For more information about the measure of noncompactness (see Kamenskii, 2001).

The Hausdorff MNC, defined by $\chi(\Omega)=\inf\{\varepsilon>0:\Omega$ has a finiteε-net$\}$, enjoys the property: for any $T \in \mathcal{L}(X)$ and $\Omega \subset X$ it follows that

$$\chi(T\Omega) \leq \| T \|_{\mathcal{L}(X)} \chi(\Omega). \tag{2}$$

We present the following assertion (see, O'Regan, 2001) which provides us with a basic MNC estimate.

Lemma 2.3. The sequence of functions $\{f_n\} \subset L(J;X)$ be integrably bounded, i.e.,

$$f_n(t) \leq \sigma(t) \text{ for a.e. } t \in J \text{ and all } n \geq 1,$$

where $\sigma \in L(J;R^+)$. Then the function $\chi(\{fn(t)\})$ belongs to $L(J;R^+)$ and satisfies that

$$\chi\left(\left\{\int_0^t f_n(s)\mathrm{d}s\right\}\right) \leq 2\int_0^t \chi(\{f_n(s)\})\mathrm{d}s$$

for each $y_0 \in D$.

Definition 2.1. A nonempty subset D of Y is called contractible if there exists a point $y \in D$.and a continuous function $h:[0,1]\times D \to D$ such that $h(0,y)=y_0$ and for every $y \in D$.

Below is a fixed point theorem for multi-valued maps (Lemma 1 of Bothe, 1998).

Lemma 2.4. Let D be a nonempty, compact and convex subset of a Banach space and $\psi:D \to 2^D$ an *u.s.c.* multi-valued map with contractible values. Then ψ has at least one fixed point.

Throughout this paper, A is a closed linear operator generating a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X, there exists a constant $M > 0$ such that $sup\{\|T(t)\|_{\mathcal{L}(X)},\ t \in J\} \leq M$.

For the linear Cauchy problem

$$\begin{cases} u'(t) - Au(t) = f(t), & t \in J, t \neq t_k, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), & k = 1,...,m, \\ u(t) = \varphi(t), & t \in [-h,0], \end{cases} \tag{3}$$

where $f \in L(J;X)$, we have the following definition of mild solution.

Definition 2.2. Given $\varphi \in C_h$, a function $u \in PC_T$ is called a mild solution of the problem (3), if $u(t) = \varphi(t)$ for all $t \in [-h, 0]$ and it satisfies

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s)\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k)), \quad t \in J.$$

By giving some suitable conditions, the existence and uniqueness of mild solution to the problem (3) can be obtained in a standard argument (e.g., Benchohra, 2006). Here, $u \in PC_T$ is a mild solution of (1), if u is a mild solution of the problem (3) with $f \in L(J;X)$ and $f(t) \in F(t,u(t),u_t)$ for $a.e.\ t \in J$.

At the end of this section, we present an approximation result whose proof is very closely related to the proof in lemma 2.4 of Wang (2013).

Lemma 2.5. If the two sequences $\{f_n\} \subset L(J;X)$ and $\{u_n\} \subset PC_T$, where u_n is a mild solution of the problem

$$\begin{cases} u_n'(t) - Au_n(t) = f_n(t), & t \in J, t \neq t_k, \\ u_n(t_k^+) = u_n(t_k) + I_k(u_n(t_k)), & k = 1,...,m, \\ u_n(t) = \varphi(t), & t \in [-h,0], \end{cases}$$

$\lim_{n\to\infty} f_n = f$ weakly in $L(J;X)$ and $\lim_{n\to\infty} u_n = u$ in PC_T then u is a mild solution of the limit problem

$$\begin{cases} u'(t) - Au(t) = f(t), & t \in J, t \neq t_k, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), & k = 1,...,m, \\ u(t) = \varphi(t), & t \in [-h,0]. \end{cases}$$

2. EXISTENCE RESULT

In this section, let us first introduce our basic assumptions.

For the multi-valued function $F:J\times X\times PC_0 \to C_v(X)$, we assume that

(H_1) $F(t,\cdot,\cdot)$ is weakly u.s.c. for a.e. $t \in J$ and $F(\cdot,x,v)$ has a strongly measurable selection for each $(x,v) \in X \times PC_0$;

(H_2) there exists a function $\eta \in L(J;R^+)$ such that $\| F(t,x,v)\| := \sup\{\| y\| : y \in F(t,x)\} \leq \eta(t)(1+\| x\| + | v |_0)$;

(H_3) there exists $\mu \in L(J;R^+)$ such that

$$\chi(F(t,\Omega,Q)) \leq \mu(t)\left(\chi(\Omega) + \sup_{s\in[-h,0]} Q(s)\right)$$

for $a.e.$ $t \in J$ and all bounded subsets $\Omega \subset X$ and $Q \subset PC_0$.

For the impulsive functions $I_k:X \to X, k=1,\dots,m$, we suppose that

(H_4) there exist constants $c_k \geq 0, k=1,\dots,m$, such that $\| I_k(x) \| \leq c_k$ or all $x \in X$;

(H_5) there exist constants $l_k \geq 0$, $k=1,\dots,m$, such that

$$\| I_k(x_1) - I_k(x_2)\| \leq l_k \| x_1 - x_2\|$$

for all $x_1, x_2 \in X$.

Remark 3.1. Under assumptions (H_1),(H_2) and let X be reflexive, it can be demonstrated from Lemma 3.1 of Chen (2013) that the multi-valued function F admits a Bochner integrable selection for each $u \in PC_T$.

Define a multi-valued map $Sel_F:PC_T \to 2^{L(J;X)}$ by

$Sel_F(u) := \{f \in L(J;X)$ and $f(t) \in F(t,u(t),u_t)$ for a.e. $t \in J\}$.

Then, we have the following assertion which provide us with a useful property of Sel_F.

Lemma 3.1. Let (H_1)- (H_2) be satisfied and let X be reflexive. Then Sel_F is weakly *u.s.c.* with nonempty, convex and weakly compact values.

Proof. Let $u \in PC_T$. Since discontinuity points are fixed, we can find a sequence $\{u_n\}$ of step functions which

