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Ratio Testing for Changes in the Long Memory Indexes

CAO Wenhua[a],*; JIN Hao[a],[b]

[a]College of Science, Xi'an University of Science and Technology, Xi'an, China.
[b]Associate Professor, College of Management, Xi'an University of Science and Technology, Xi'an, China.
*Corresponding author.

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Abstract
This paper considers the problem of detecting for breaks in the long memory indexes of Gaussian observations having long-range dependence. Under the null hypothesis, the asymptotic distribution of the proposed ratio tests converges to a functional of fractional Brownian motion. Under the alternative hypothesis, the ratio tests diverge to infinity as the sample size grows. These results show that the reject rate seriously depends on the magnitude of change points. Finally, the Monte Carlo study presents that our test has reasonably good size and power properties.

Key words: Change point; Long memory series; Ratio tests; Asymptotic properties

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INTRODUCTION
The problem of testing for change points has an important issue in time series analysis since the change points are often interpreted as serious risks in econometrics and neglecting breaks can make radically misleading decisions. A vast amount of relevant articles including breaks in mean and variance have appeared in the literature. Gombay and Horvath (1990) derived asymptotic distribution of maximum likelihood tests for change point in the mean. Vogelsang (1994) detected a shift in the mean of a univariate time series and proved that the statistics are valid whether the errors are stationary or have a unit root. Jin, Tian, and Qin (2009) adopted subsampling tests for the mean change points with heavy-tailed innovations. Bai (2010) used the least squares method and the quasi maximum likelihood (QML) method to estimate breaks in means and in variances for panel data and found QML method was more efficient than the least squares even if there is no change in the variances. Qi, Duan, and Tian (2014) structured Bootstrap monitoring for mean changes of nonparametric regression models by wavelets and indicated that their procedure have good power and short detection delay in the monitoring of structural change of nonparametric regression models. The statistical literature on changes of variance started with (Hsu, Miller, & Wichern, 1974), they offered variance change point formulation as an alternative to the distribution to model stock returns. Wichern, Miller, and Hsu (1976) researched Changes of variance in first order autoregressive time series models with an application. Jin and Zhang (2011) employed the RCUSQ statistic to test variance changes in the linear autoregressive processes including AR(p) processes meanwhile autoregressive parameters shifts occur. Noorossana and Heydari (2012) considered a Maximum Likelihood Estimator (MLE) of estimating the time of a monotonic change in the variance of a normal quality characteristic, Numerical results revealed that the proposed estimator provides appropriate and robust estimation with regard to the magnitude and type of change. Li, Tian, Xiao, and Chen (2015) discussed variance change points detection in panel data models and proposed a CUSUM based statistic to test if there is a variance change point in panel data models.

On the other hand, many scholars already have studied the innovations which are long memory series for a long time and one of the focus is on estimating parameters and detecting change points. The phenomenon of long memory has appeared to be relatively common and widely raised the attention of people in the past decades, which had been observed in several areas of application a long time before stochastic model were known. Hidalgo, Peter, and Robinson (1996) tested structural changes in a long memory environment in the case of certain nonstochastic and stochastic regressors models. Wang (2009) utilized the GPH estimation of spatial long memory parameter to investigate stationary long memory random fields. Shao (2011) proposed a simple testing procedure to test for a change point in the mean of a possibly long range dependent time series and estimated memory indexes with Local Whittle method, the test can be used to discriminate between a stationary long memory and short range dependent time series with a change point in mean. Gustavo and Fotis (2015) adopted A Two-Stage Approach to analyse long memory series subject to structural change, which showed TSF methodology results in accurate and more robust forecasts when applied to long memory series with a break in the mean. These researches are in the case of constant indexes of long memory to analyse and study. In fact, it is possible to use models with long memory innovations including change points in indexes in a variety of practical problems.

In this paper, the goal of the article is to detect change points with statistics to show the existence of change points in the long memory indexes. Therefore, the primary contributions of this paper include three aspects. First, we derive the asymptotic distribution of the proposed ratio tests convergence to a functional of fractional Brownian motion under the null hypothesis. Second, under the alternative hypothesis, the ratio tests diverge to infinity as the sample size grows. Third, the Monte Carlo study shows that our test has reasonably good size and power properties.

The outline of the paper is organized as follows. Section 1 introduces some models, assumptions and test statistics. Section 2 contains the main results. Monte Carlo simulations are collected in Section 3. Then, draws a conclusion. Finally, all proofs are given in the appendix.

1. MODEL, ASSUMPTION AND STATISTIC

In the last few decades, we have witnessed a rapid development for statistical inference of long range dependent (or long memory) time series. Beran (1994), Robinson (2003) among others for book-length treatments of this topic. Let

$$(1-L)^d z_t = \varepsilon_t, t \in Z,$$

where L is the backward shift operator and $\{\varepsilon_t\}$ is a mean zero covariance stationary dependent process. We say that the process $\{z_t\}$ possesses long memory if $d \in (0,0.5)$ and short memory if $d \in (-0.5,0)$.

In order to study a stochastic process $\{y_t\}$ existing change points in indexes, we consider the following linear regression model given by:

$$y_t = \mu_0 + z_t, t = 1,2,\cdots,T,$$

where μ_0 is an arbitrary constant, and z_t is a stationary long memory series with index $d \in (0,0.5)$.

The null hypothesis can be described as

$$H_0 : d = d_0 \text{ for } t = 1,2,\cdots,T.$$

The alternative hypothesis is

$$H_1 : \begin{cases} d = d_0 & t = 1,2,\cdots,[T\tau^*] \\ d = d_1 & t = [T\tau^*]+1,\cdots,T. \end{cases}$$

where τ^* is unknown and $[T\tau^*]$ is the integer part of $T\tau^*$, $d_0 \neq d_1$. For the purpose of asymptotic analysis, we make the following assumption.

Assumption 1. There exists a $d \in (0,0.5)$, such that as $n \to \infty$, then

$$n^{-(0.5+d)} \sum_{t=1}^{[nr]} \{z_t - Ez_t\} \Rightarrow C_d B_d(r), r \in [0,1].$$

where the symbol $\Rightarrow$ signifies weak convergence of the associated probability measures, C_d is a positive and $B_d(\cdot)$ is the fractional Brownian motion. Marinucci and Robinson (1999) has given as follows:

$$B_d(\tau) \equiv \frac{1}{\Gamma(1+d)} \{ \int_0^\tau (\tau - s)^d W(s) + \int_{-\infty}^0 [(\tau-s)^d - (-s)^d] dW(s) \}$$

Where $\Gamma(\cdot)$ is the Gamma function and $W(s)$ is a standard Brownian motion. The assumption has been extensively studied in the literature; see, Davidson, Jame, De, and Robert (2000), Mandelbrot and Vanness (1968).

Before expressing the test statistics, let $\hat{z}_t$, $t=1,2,\ldots,T$ be the residuals from the regression of y_t on an constant. Then, let S_t be the following partial sum process:

$$S_t = \sum_{j=1}^{t} \hat{z}_j \text{ for } t = 1,2,\cdots,T,$$

Next, we can give some definitions about partial sum process respectively before and after break:

$$S_{1,t}(\tau) = \sum_{j=1}^{t} \hat{z}_j \text{ for } t = 1,2,\cdots,[T\tau],$$

$$S_{2,t}(\tau) = \sum_{j=[T\tau]+1}^{t} \hat{z}_j \text{ for } t = [T\tau]+1,\cdots,T,$$

The ratio test is defined as following:

$$\Xi_T(\tau) = \max \frac{[(1-\tau)T]^{-2} \sum_{[T\tau]+1}^{T} S_{2,t}(\tau)^2}{[T\tau]^{-2} \sum_{1}^{[T\tau]} S_{1,t}(\tau)^2}.$$

$$\Xi_T(\hat{\tau}) \equiv \max_{\tau\in\Lambda} \Xi_T(\tau)$$

where $\Lambda \in (0,1)$ and $\tau \in \Lambda$, $\hat{\tau}$ is estimation of τ .

2. MAIN RESULTS

Theorem 2.1. Suppose that Assumption 1 is true for z_t under null hypothesis, then

$$T^{-2d}\Xi_T(\tau) \Rightarrow \frac{(1-\tau)^{-2}\int_\tau^1 \psi_1(s,\tau)^2 ds}{\tau^{-2}\int_0^\tau \psi_2(s,\tau)^2 ds}.$$

where $\psi_1(s,\tau)=(B_d(s)-B_d(\tau)\ -\left(\frac{s-\tau}{1-\tau}\right)(B_d(1)-B_d(\tau)$

for $s\in[\tau,1]$; $\psi_2(s,\tau)=\ B_d(s)-\frac{s}{\tau}B_d(\tau)$ for $s\in[0,\tau]$,

$B_d(\cdot)$ is a fractional Brownian motion.

Remark 2.1. The result shows that the limiting distribution depends strongly on the long memory index *d*.

Theorem 2.2. Suppose that Assumption 1 is true for z_t under alternative hypothesis, then

(1) If $d_1 > d_0$, $\tau^* \geq \tau$, then $\Xi_T(\tau)=\infty$; $\tau^* < \tau$,

$\Xi_T(\tau)=o_P(1)$.

(2) If $d_0 > d_1$, $\tau^* \geq \tau$, then $\Xi_T(\tau)=o_P(1)$; $\tau^* < \tau$,

$\Xi_T(\tau)=o_P(1)$.

Then

$\Xi_T(\tau)=\ \max(\Xi_T(\tau)\ \Xi_T^{-1}(\tau)\ =\infty.$

Remark 2.2. These results show that statistics diverge to infinity as the sample size grows under the alternative hypothesis. It has also high power if the break is in both cases assumed to be from d_0 to d_1 or from d_1 to d_0.

3. MONTE CARLO STUDY

In this section we conduct a simulation study to evaluate the test in section 1 and section 2. All simulations are based on 1000 replications. We report empirical rejection frequencies of the tests with *T*=500,800,1000 for tests run at α=0.95.

We consider the data generating processes, henceforth DGP's, which satisfy:

$$y_t=\mu_0+z_t, t=1,2,\cdots,T,$$

where μ_0=0.5 and the innovations z_t is a stationary long memory series with indexes *d*. Subsequently, we consider the same model above allowing a change in index *d*:

$$y_t=\begin{cases}\mu_0+z_t^{d_0} & t\leq[T\tau^*]\\ \mu_0+z_t^{d_1} & t>[T\tau^*].\end{cases}$$

Where μ_0=0.5 and $d_0,d_1 \in$ { 0,0.1,0.2,0.3,0.4 } , the specific numerical simulations are expressed as follows.

Table 1
Empirical Size of Critical Value p=21.503

	T		
d	500	800	1000
0	0.039	0.046	0.058
0.1	0.047	0.057	0.069
0.2	0.058	0.052	0.055
0.3	0.040	0.043	0.045
0.4	0.036	0.041	0.053

We now discuss the main conclusions that can be drawn from three tables. First of all, the Table 1 shows the rejection rate are closed to 0.05 under null hypothesis and illustrates that ratio tests have a good size. Moreover, Tables 2-3 indicate the rejection rate are more greater with the larger distance between d_0 and d_1 under alternative hypothesis. If we set a value of d_0, the power increases with declining of d_1 in Table 2. Similarly, for a given value of d_0, the power increases as d_1 grows in Table 3. Meanwhile, it might be not intuitive that the power of breaks of equal distance, e.g. *T*=1000 and τ=0.3, the power is 0.243 when *d* is from 0.4 to 0.3, but the power is 0.171 when *d* is from 0.1 to 0 in Table 2, the same situation as shown in Table 3. On the whole, ratio tests depend on sample sizes and memory indexes, it is able to reject the null hypothesis and accept the alternative hypothesis to prove the existence of change points.

Table 2
Empirical Power of the Ratio Test (α=95%)

$d_0 \to d_1$	τ	*T*		
		500	800	1000
	0.3	0.278	0.302	0.243
0.4→0.3	0.5	0.267	0.245	0.254
	0.7	0.258	0.220	0.250
	0.3	0.352	0.328	0.322
0.4→0.2	0.5	0.316	0.319	0.306
	0.7	0.300	0.294	0.280
	0.3	0.520	0.470	0.479
0.4→0.1	0.5	0.486	0.455	0.528
	0.7	0.454	0.468	0.452
	0.3	0.663	0.638	0.633
0.4→0	0.5	0.625	0.609	0.622
	0.7	0.610	0.617	0.614
	0.3	0.235	0.248	0.238
0.3→0.2	0.5	0.233	0.216	0.230
	0.7	0.211	0.230	0.226
	0.3	0.328	0.334	0.320
0.3→0.1	0.5	0.324	0.328	0.331
	0.7	0.335	0.318	0.323
	0.3	0.430	0.429	0.432

To be continued

Continued

$d_0 \to d_1$	τ	T		
		500	800	1000
0.3→0	0.5	0.425	0.420	0.408
	0.7	0.422	0.429	0.401
	0.3	0.241	0.234	0.213
0.2→0.1	0.5	0.218	0.203	0.212
	0.7	0.200	0.237	0.205
	0.3	0.260	0.240	0.255
0.2→0	0.5	0.249	0.254	0.246
	0.7	0.240	0.245	0.237
	0.3	0.180	0.182	0.171
0.1→0	0.5	0.161	0.166	0.167
	0.7	0.131	0.172	0.166

Table 3
Empirical Power of the Ratio Test (α=95%)

$d_0 \to d_1$	τ	T		
		500	800	1000
	0.3	0.110	0.146	0.149
0→0.1	0.5	0.132	0.160	0.163
	0.7	0.154	0.166	0.145
	0.3	0.213	0.197	0.193
0→0.2	0.5	0.205	0.203	0.215
	0.7	0.230	0.214	0.238
	0.3	0.399	0.408	0.416
0→0.3	0.5	0.391	0.426	0.406
	0.7	0.407	0.435	0.426
	0.3	0.619	0.593	0.623
0→0.4	0.5	0.592	0.590	0.634
	0.7	0.614	0.639	0.636
	0.3	0.213	0.197	0.195
0.1→0.2	0.5	0.225	0.205	0.227
	0.7	0.221	0.228	0.191
	0.3	0.319	0.303	0.318
0.1→0.3	0.5	0.312	0.319	0.320
	0.7	0.323	0.307	0.329
	0.3	0.430	0.457	0.440
0.1→0.4	0.5	0.462	0.434	0.467
	0.7	0.465	0.464	0.469
	0.3	0.217	0.220	0.210
0.2→0.3	0.5	0.233	0.213	0.224
	0.7	0.221	0.302	0.240
	0.3	0.248	0.270	0.327
0.2→0.4	0.5	0.332	0.347	0.310
	0.7	0.335	0.353	0.321
	0.3	0.216	0.206	0.210
0.3→0.4	0.5	0.238	0.194	0.200
	0.7	0.250	0.216	0.235

Table 4 clearly provides simulation evidence of its estimation in finite samples. for all power experiments we consider three different locations of the breakpoints, at the beginning (τ=0.3), the middle (τ=0.5) and the end (τ=0.7) of the sample period. Obviously, estimates of τ depend on long memory series and different simple sizes. More precisely, the estimates are more accurate with the larger distance between d_0 and d_1, for example, T=1000 and τ=0.5, the estimate is 0.506 when d is from 0.4 to 0, but the estimate is 0.480 when d is from 0.4 to 0.3. The same situation at τ=0.3, and τ=0.7. In general, the estimators are close to the real values extremely.

Table 4
Estimating for Change Points

τ	$d_0 \to d_1$	T		
		500	800	1000
0.3	0.4→0	0.331	0.326	0.351
	0.4→0.1	0.298	0.314	0.307
	0.4→0.2	0.306	0.284	0.290
	0.4→0.3	0.257	0.277	0.301
0.5	0.4→0	0.501	0.490	0.506
	0.4→0.1	0.489	0.479	0.484
	0.4→0.2	0.487	0.482	0.507
	0.4→0.3	0.438	0.461	0.480
0.7	0.4→0	0.691	0.679	0.700
	0.4→0.1	0.693	0.643	0.710
	0.4→0.2	0.695	0.656	0.688
	0.4→0.3	0.637	0.651	0.686

CONCLUSION

In this paper, change points are considered in the long memory indexes detected by the ratio test in regression model. The asymptotic distribution of our test is Fractional Brownian motion under null hypothesis and is divergent as the sample size increases under alternative hypothesis. Moreover, the Monte Carlo studies have been conducted to investigate the performance of our test procedures and show the existence of change points in memory indexes may be unambiguous. Overall, the simulation results reveal the reject rate heavily depends on the magnitude of change points.

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APPENDIX

Proof of theorem 2.1. First, since $\hat{z}_t = y_t - \bar{y}$ $E(z_t) = 0$, it follows that

$$S_{2,[Ts]}(\tau) = \sum_{j=[T\tau]+1}^{[Ts]} \hat{z}_j = \sum_{j=[T\tau]+1}^{[Ts]} (y_j - \bar{y}) = \sum_{j=[T\tau]+1}^{[Ts]} (z_j - \bar{z}) = \sum_{j=[T\tau]+1}^{[Ts]} z_j - T[s-\tau]\bar{z},$$

Where $s \in [\tau, 1]$ and $[Ts]$ is the integer part of Ts. Then, by a functional central limit theorem, it follows that

$$\begin{aligned} T^{-\frac{1}{2}-d} \sum_{j=[T\tau]+1}^{[Ts]} z_j &= T^{-\frac{1}{2}-d} \left(\sum_{j=1}^{[Ts]} z_j - \sum_{j=1}^{[T\tau]} z_j\right) \\ &= T^{-\frac{1}{2}-d} \sum_{j=1}^{[Ts]} z_j - T^{-\frac{1}{2}-d} \sum_{j=1}^{[T\tau]} z_j \\ &\Rightarrow C_d (B_d(s) - B_d(\tau)), \end{aligned}$$

$$\begin{aligned} T^{-\frac{1}{2}-d} [T(s-\tau)\bar{z}] &= T^{-\frac{1}{2}-d} T(s-\tau) \left(\frac{1}{T(1-\tau)} \sum_{[T\tau]+1}^{T} z_j \right) \\ &= \left(\frac{s-\tau}{1-\tau}\right) T^{-\frac{1}{2}-d} \sum_{[T\tau]+1}^{T} z_j \\ &= \frac{s-\tau}{1-\tau} \left[T^{-\frac{1}{2}-d} \left(\sum_{1}^{T} z_j - \sum_{1}^{[T\tau]} z_j \right) \right] \\ &\Rightarrow \left(\frac{s-\tau}{1-\tau}\right) C_d (B_d(1) - B_d(\tau)), \end{aligned}$$

We have

$$\begin{aligned} T^{-\frac{1}{2}-d} S_{2,[Ts]}(\tau) &\Rightarrow C_d (B_d(s) - B_d(\tau)) - C_d \left(\frac{s-\tau}{1-\tau}\right) (B_d(1) - B_d(\tau)) \\ &= C_d \psi_1(s,\tau), \end{aligned}$$

where $\psi_1(s,\tau)$ is as defined in theorem 2.1. Kim (2000) is for reference in the proofs.

Let

$$T^{-2} \sum_{[T\tau]+1}^{T} S_{2,[Ts]}(\tau) = K_1(\tau, s),$$

We have

$$\begin{aligned} T^{-2d} K_1(\tau,s) &= T^{-2d} T^{-2} \sum_{[T\tau]+1}^{T} S_{2,t}(\tau)^2 \\ &= T^{-1} \sum_{[T\tau]+1}^{T} \left[T^{-\frac{1}{2}-d} S_{2,t}(\tau) \right]^2 \\ &= T^{-1} \sum_{[T\tau]+1}^{T} \left[C_d \psi_1(s,\tau) \right]^2 \\ &\Rightarrow C_d{}^2 \int_\tau^1 \psi_1(s,\tau)^2 ds. \end{aligned}$$

Thus for the nominator

$$T^{-2d}[(1-\tau)T]^{-2}\sum_{[T\tau]+1}^{t}S_{2,t}(\tau)^2 \Rightarrow (1-\tau)^{-2}C_d^{\,2}\int_\tau^1 \psi_1(s,\tau)^2\,ds\,.$$

In a similar way, we obtain the denominator

$$T^{-2d}[T\tau]^{-2}\sum_{1}^{t}S_{1,t}(\tau)^2 \Rightarrow \tau^{-2}C_d^{\,2}\int_0^\tau \psi_2(s,\tau)^2\,ds\,.$$

where $\psi_2(s,\tau)$ is as defined in theorem 2.1. Finally, combining the result for the nominator and the denominator gives the result, as follows

$$T^{-2d}\Xi_T(\tau) \Rightarrow \frac{(1-\tau)^{-2}\int_\tau^1 \psi_1(s,\tau)^2\,ds}{\tau^{-2}\int_0^\tau \psi_2(s,\tau)^2\,ds}\,.$$

The proof is completed.

Proof of theorem 2.2. First, we consider the case of $\tau^* \ge \tau$, under the alternative hypothesis, it follows that

$$\sum_{j=[T\tau]+1}^{[Ts]}(z_j-\bar{z}) = \sum_{j=[T\tau]+1}^{[T\tau^*]}(z_j-\bar{z}) + \sum_{j=[T\tau^*]+1}^{[Ts]}(z_j-\bar{z}),$$

Because a change point in the process, before the change point we use d_0 and after d_1, $d_0 \ne d_1$, we have

$$T^{-\frac{1}{2}-d_0}\sum_{j=[T\tau]+1}^{[T\tau^*]}(z_j-\bar{z}) = T^{-\frac{1}{2}-d_0}\sum_{j=[T\tau]+1}^{[T\tau^*]}z_j - T^{-\frac{1}{2}-d_0}T(\tau^*-\tau)\bar{z},$$

However

$$T^{-\frac{1}{2}-d_0}\sum_{j=[T\tau]+1}^{[T\tau^*]}z_j = T^{-\frac{1}{2}-d_0}\sum_{j=1}^{[T\tau^*]}z_j - T^{-\frac{1}{2}-d_0}\sum_{j=1}^{[T\tau]}z_j$$

$$\Rightarrow C_{d_0}(B_{d_0}(\tau^*)-B_{d_0}(\tau)),$$

$$T^{-\frac{1}{2}-d_0}T(\tau^*-\tau)\bar{z} = T^{-\frac{1}{2}-d_0}T(\tau^*-\tau)\frac{1}{T\tau^*}\sum_{j=1}^{[T\tau^*]}z_j$$

$$\Rightarrow \left(\frac{\tau^*-\tau}{\tau^*}\right)C_{d_0}B_{d_0}(\tau^*),$$

Hence, we have

$$T^{-\frac{1}{2}-d_0}\sum_{j=[T\tau]+1}^{[T\tau^*]}(z_j-\bar{z}) \Rightarrow C_{d_0}\left[(B_{d_0}(\tau^*)-B_{d_0}(\tau)) - \frac{\tau^*-\tau}{\tau^*}B_{d_0}(\tau^*)\right]$$

$$= C_{d_0}\Phi_0(s,\tau,\tau^*)\,.$$

Deducing in the similar way

$$T^{-\frac{1}{2}-d_1}\sum_{j=[T\tau^*]+1}^{[Ts]}(z_j-\bar{z}) = T^{-\frac{1}{2}-d_1}\sum_{j=[T\tau^*]+1}^{[Ts]}z_j - T^{-\frac{1}{2}-d_1}T(s-\tau^*)\bar{z}$$

$$T^{-\frac{1}{2}-d_1}\sum_{j=[T\tau^*]+1}^{[Ts]}z_j = T^{-\frac{1}{2}-d_1}\sum_{j=1}^{[Ts]}z_j - T^{-\frac{1}{2}-d_0}\sum_{j=1}^{[T\tau^*]}z_j$$

$$\Rightarrow C_{d_1}(B_{d_1}(s)-B_{d_1}(\tau^*)),$$

$$T^{-\frac{1}{2}-d_1} T(s-\tau^*)\bar{z} = T^{-\frac{1}{2}-d_1} T(s-\tau^*)\frac{1}{T(1-\tau^*)}\sum_{j=[T\tau^*]+1}^{T} z_j$$
$$= \frac{s-\tau^*}{1-\tau^*} T^{-\frac{1}{2}-d_1}(\sum_{1}^{T} z_j - \sum_{1}^{[T\tau^*]} z_j)$$
$$\Rightarrow \frac{s-\tau^*}{1-\tau^*} C_{d_1}(B_{d_1}(1) - B_{d_1}(\tau^*)),$$
$$T^{-\frac{1}{2}-d_1}\sum_{j=[T\tau^*]+1}^{[Ts]}(z_j - \bar{z}) \Rightarrow C_{d_1}\left[(B_{d_1}(s) - B_{d_1}(\tau^*)) - \frac{s-\tau^*}{1-\tau^*}(B_{d_1}(1) - B_{d_1}(\tau^*))\right]$$
$$= C_{d_1}\Phi_1(s,\tau,\tau^*).$$

Thus for the nominator

$$T^{-2d_0}[(1-\tau)T]^{-2}\sum_{t=[T\tau]+1}^{[T\tau^*]} S_{2,t}(\tau)^2 + T^{-2d_1}[(1-\tau)T]^{-2}\sum_{t=[T\tau^*]+1}^{T} S_{2,t}(\tau)^2$$
$$= (1-\tau)^{-2}T^{-1}\sum_{t=[T\tau]+1}^{[T\tau^*]}(T^{-\frac{1}{2}-d_0}S_{2,t}(\tau))^2 + (1-\tau)^{-2}T^{-1}\sum_{t=[T\tau^*]+1}^{T}(T^{-\frac{1}{2}-d_1}S_{2,t}(\tau))^2$$
$$\Rightarrow (1-\tau)^{-2}\int_{\tau}^{\tau^*} C_{d_0}{}^2\Phi_0(s,\tau,\tau^*)^2 ds + (1-\tau)^{-2}\int_{\tau^*}^{1} C_{d_1}{}^2\Phi_1(s,\tau,\tau^*)^2 ds.$$

For the denominator

$$T^{-\frac{1}{2}-d_0}\sum_{j=1}^{[Ts]}(z_j - \bar{z}) = T^{-\frac{1}{2}-d_0}\sum_{j=1}^{[Ts]} z_j - T^{-\frac{1}{2}-d_0} Ts\frac{1}{T\tau}\sum_{j=1}^{[T\tau]} z_j$$
$$\Rightarrow C_{d_0}(B_{d_0}(s) - \frac{s}{\tau}B_{d_0}(\tau))$$
$$= C_{d_0}\Phi_2(s,\tau,\tau^*),$$
$$T^{-2d_0}[T\tau]^{-2}\sum_{t=1}^{[T\tau]} S_{1,t}(\tau)^2 = \tau^{-2}T^{-1}\sum_{t=1}^{[T\tau]}(T^{-\frac{1}{2}-d_0}S_{1,t}(\tau))^2$$
$$\Rightarrow \tau^{-2}\int_{0}^{\tau} C_{d_0}{}^2\Phi_2(s,\tau,\tau^*)^2 ds,$$

Finally, we obtain

$$\Xi_T(\tau) = \frac{O_P(T^{2d_0}) + O_P(T^{2d_1})}{O_P(T^{2d_0})}.$$

Now, consider the other case of $\tau^* < \tau$, under the alternative hypothesis, for the nominator, it follows that

$$T^{-\frac{1}{2}-d_1}\sum_{j=[T\tau]+1}^{[Ts]}(z_j - \bar{z}) = T^{-\frac{1}{2}-d_1}\sum_{j=[T\tau]+1}^{[Ts]} z_j - T^{-\frac{1}{2}-d_1}T(s-\tau)\bar{z}$$
$$= T^{-\frac{1}{2}-d_1}(\sum_{j=1}^{[Ts]} z_j - \sum_{j=1}^{[T\tau]} z_j) - T^{-\frac{1}{2}-d_1}T(s-\tau)\frac{1}{T(1-\tau)}\sum_{j=[T\tau]+1}^{T} z_j$$
$$\Rightarrow C_{d_1}\left[(B_{d_1}(s) - B_{d_1}(\tau)) - \frac{s-\tau}{1-\tau}(B_{d_1}(1) - B_{d_1}(\tau))\right]$$
$$= C_{d_1}\Phi_3(s,\tau,\tau^*),$$
$$T^{-2d_1}[(1-\tau)T]^{-2}\sum_{[T\tau]+1}^{T} S_{2,t}(\tau)^2 \Rightarrow (1-\tau)^{-2}C_{d_1}{}^2\int_{\tau}^{1}\Phi_3(s,\tau,\tau^*)^2 ds.$$

For the denominator

$$\sum_{j=1}^{[Ts]}(z_j-\bar{z})=\sum_{j=1}^{[T\tau^*]}(z_j-\bar{z})+\sum_{j=[T\tau^*]+1}^{[Ts]}(z_j-\bar{z}),$$

$$T^{-\frac{1}{2}-d_0}\sum_{j=1}^{[T\tau^*]}(z_j-\bar{z})=T^{-\frac{1}{2}-d_0}\sum_{j=1}^{[T\tau^*]}z_j-T^{-\frac{1}{2}-d_0}T\tau^*\bar{z}$$

$$\Rightarrow C_{d_0}(B_{d_0}(\tau^*)-\frac{\tau^*}{\tau}B_{d_0}(\tau))$$

$$=C_{d_0}\Phi_4(s,\tau,\tau^*),$$

$$T^{-\frac{1}{2}-d_1}\sum_{j=[T\tau^*]+1}^{[Ts]}(z_j-\bar{z})=T^{-\frac{1}{2}-d_1}\sum_{j=[T\tau^*]+1}^{[Ts]}z_j-T^{-\frac{1}{2}-d_1}T(s-\tau^*)\bar{z}$$

$$\Rightarrow C_{d_1}\left[(B_{d_1}(s)-B_{d_1}(\tau^*))-\frac{s-\tau^*}{1-\tau^*}(B_{d_1}(1)-B_{d_1}(\tau^*))\right]$$

$$=C_{d_1}\Phi_5(s,\tau,\tau^*),$$

$$T^{-2d_0}[T\tau]^{-2}\sum_{t=1}^{[T\tau^*]}S_{1,t}(\tau)^2+T^{-2d_1}[T\tau]^{-2}\sum_{t=[T\tau^*]+1}^{T}S_{1,t}(\tau)^2$$

$$=\tau^{-2}T^{-1}\sum_{t=1}^{[T\tau^*]}(T^{-\frac{1}{2}-d_0}S_{1,t}(\tau))^2+\tau^{-2}T^{-1}\sum_{t=[T\tau^*]+1}^{T}(T^{-\frac{1}{2}-d_1}S_{1,t}(\tau))^2$$

$$\Rightarrow \tau^{-2}\int_0^{\tau^*}C_{d_0}{}^2\Phi_4(s,\tau,\tau^*)^2\,ds+\tau^{-2}\int_{\tau^*}^1 C_{d_1}{}^2\Phi_5(s,\tau,\tau^*)^2\,ds\,.$$

Finally, combining results above we can obtain

$$\Xi_T(\tau)=\frac{O_P(T^{2d_1})}{O_P(T^{2d_0})+O_P(T^{2d_1})}\,.$$

we should consider the size of d_0, d_1. If $d_1>d_0$, $\tau^*\ge\tau$, then $\Xi_T(\tau)=\infty$; $\tau^*<\tau$, $\Xi_T(\tau)=o_P(1)$; If $d_0>d_1$, $\tau^*\ge\tau$, then $\Xi_T(\tau)=o_P(1)$; $\tau^*<\tau$, $\Xi_T(\tau)=o_P(1)$, then, $\max(\Xi_T(\tau),\Xi_T{}^{-1}(\tau))=\infty$, that is to say, $\Xi_T(\tau)$ is divergent.

Hence, the proof of Theorem 2.2 is completed.

