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Tests for the Variance Changes in Presence of Breaks in Mean

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Abstract
In this paper we consider the detection problem of variance change in presence of breaks in mean. The limiting distribution is derived under the null and the alternative hypothesis and the consistence of the tests is also established. We find that the size and power of the cumulative sum of squares (CUSSQ) tests suffers severe distortions due to neglected of shifts in mean. The numerical simulation results are consistent with our theoretical analysis.

Key words: Variance change; Break in mean; CUSSQ tests; Asymptotic distribution

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INTRODUCTION

Change point studies is an important part of statistics, economic and financial analysis. Since the paper of (Page, 1955), a vast amount of relevant articles have appeared in the literature, especially the paper in regard to the mean and variance changes. For example, Brown, Durbin, and Evans (1975) proposed the cumulative sum test (CUSUM) for mean changes based on recursive residuals and the cumulative sum of squares test (CUSSQ) for variance change point; Horvath and Kokoszka (1997) gave an estimator of the change point in a long-range dependent series; Bai and Perron (1998) considered the detection of the mean change point in linear processes; Jin, Tian, and Qin (2009) extend the CUSUM test for the mean change point with heavy-tailed innovations; Horváth, Kokoszka, and Zhang (2006) investigated change point in unconditional variance in a conditionally heteroskedastic time series. Recent contributions include: (Wang & Wang, 2006; Zhao, Xia, & Tian, 2010; Jin & Zhang, 2011; Bucchia, 2014) as well as the literature by (Amado & Teräsvirta, 2014).

All of the works above are concentrated on the case where the series only exists shifts in mean or variance. There is a phenomenon, however, the economic and financial time series exist structural change points both in variance and mean. In this paper we consider the detection problem of variance change in presence of breaks in mean. The limiting distribution of CUSSQ tests are derive under the null and alternative hypothesis. Then, we found breaks in mean can be easily confused with detecting for variances changes, which makes the size and power distorted. The size can gains from the mean breaks, and the power will loss depending on the location of the mean breaks. Finally, theoretical analysis is verified via the numerical simulation.

The paper is organized at follow. Section 1 introduces the detail of data generating process (DGP), the hypothesis and CUSSQ tests. Section 2 analyzes the asymptotic properties of the CUSSQ tests under the null and alternative. Section 3 contains a Monte Carlo study showing the size and power. Finally, presents brief concluding. All proofs are given in the Appendix.

1. MODEL, HYPOTHESES AND TEST STATISTICS

In order to analyze the effect of mean shifts on testing for variance changes, we suppose the simple DGP is given as follows:

$$y_t = \begin{cases} \mu_t + \sigma_1 \varepsilon_t, & 1 \le t \le [T\tau] \\ \mu_t + \sigma_2 \varepsilon_t, & [T\tau]+1 \le t \le T \end{cases}, \quad (1.1)$$

where

$$\mu_t = \mu_1 1_{[t \le \lambda T]} + \mu_2 1_{[t > \lambda T]} . \quad (1.2)$$

For convenient writing, we signify $z_t = \sigma_t \varepsilon_t$. The notation $[\cdot]$ denotes the largest integer less than or equal to its argument and $1_{\{\cdot\}}$ is the indicator function. In this model, this series $\{y_t\}_{t=1}^{T}$ has a variance change at $\tau \in [0,1]$. Specially, the standard deviation increases (decreases) from σ_1 to σ_2.

Unlike previous literature assuming constant of mean, we consider parameter μ_1 also having a change point at $\lambda \in [0,1]$. The pre-break mean of $\{y_t\}_{t=1}^{T}$ is μ_1 and post-break mean is μ_2: The difference $\mu_2 - \mu_1$ represents the magnitude of break, which can be either random or nonrandom, and is assumed to be independent of error process $\{\varepsilon_t\}_{t=1}^{T}$. In addition, $E(\mu_2 - \mu_1)^2 \le M$. The location of change points τ and λ is unknown. Of course, we do not force these two locations to happen at the same time. In order to derive the asymptotic distribution, the series $\{\varepsilon_t\}_{t=1}^{T}$ is a stationary variable satisfying the following regularity conditions:

Assumption 1.1. The process $\{\varepsilon_t\}_{t=1}^{T}$ is such that

- $E(\varepsilon_t)=0$, $E(\varepsilon_t^2)$;
- $E|\varepsilon_t|^{4+\pi} < \infty$ for some $\pi > 4$;
- $\{\varepsilon_t\}_{t=1}^{T}$ is α -mixing with mixing coefficients α_n such that $\sum_{n=1}^{\infty} \alpha_n^{1-2/\beta} < \infty$ for some $\beta > 2$;
- The long-run variance $\lim_{T\to\infty} D\left(T^{1/2} \sum_{t=1}^{T} \varepsilon_t\right) = \sigma_\varepsilon^2$ exists.

The above conditions allow for a broad class of weakly dependent time series and have been used by (Phillips, 1987; Phillips & Solo, 1992; Kim, 2000) among others, to derive limiting behavior of a stochastic process.

Under the circumstance of $\{y_t\}_{t=1}^{T}$ have mean shift, we consider the following hypotheses. The null hypothesis H_0 is that there is no variance change throughout the sample period

H_0: $\{y_t\}_{t=1}^{T}$ is a sample with $\sigma_1 = \sigma_2$, $\mu_1 \ne \mu_2$,

against the alternative hypothesis that the series exist a break in variance,

H_1: $\{y_t\}_{t=1}^{T}$ is a sample involves $\sigma_1 \ne \sigma_2$, $\mu_1 \ne \mu_2$.

The CUSSQ statistic based on $\hat{z}_t$ is defined as follows,

$$\hat{K}_T = \max_{1 \le k \le T} \frac{\sqrt{T}}{\hat{\gamma}} \left| \frac{\sum_{t=1}^{k} \hat{z}_t^2}{\sum_{t=1}^{T} \hat{z}_t^2} - \frac{k}{T} \right|$$

where

$$\hat{z}_t = y_t - \hat{\mu}, \quad \hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} y_t, \quad \hat{\gamma}^2 = \frac{1}{T}\sum_{t=1}^{T} \hat{z}_t^4 - \left(\frac{1}{T}\sum_{t=1}^{T} z_t^2\right)^2 .$$

The estimator $\hat{\gamma}^2$ is estimate of the $\gamma^2 = EZ_t^4 - (EZ_t^2)^2$.

2. MAIN RESULTS

In this section we show the asymptotic properties of the CUSSQ test both under the null and the alternative hypothesis, assuming the series involving mean shifts.

Theorem 2.1. Suppose Assumption 1 hold. Under the null hypothesis H_0, then

$$\hat{K}_T \xrightarrow{d} \sup_{0 \le r \le 1} \left| \frac{B(r) + \Psi_2}{\sigma_1^2 + \Psi_1} \right|, \quad (2.1)$$

where

$$\Psi_1 = \lambda(1-\lambda)(\mu_1 - \mu_2)^2,$$

$$\Psi_2 = \frac{\sqrt{T}}{\hat{\gamma}} \bar{r} \bar{\lambda} (1-2\lambda)(\mu_1 - \mu_2)^2,$$

$\bar{r} = \max\{r, (1-r)\}$, $\bar{\lambda} = \max\{\lambda, (1-\lambda)\}$, and $r = \frac{k}{T}$, $B(r)$ is a standard Brownian bridge: $B(r) = W(r) - rW(1)$, $W(\cdot)$ is a Wiener process, the symbol "$\xrightarrow{d}$" signifies convergence in distribution.

Remark 2.1. The result shows that the statistics $\hat{}$ diverge at the rate of $\sqrt{T}$ and hence can cause the size distortion. If there is no shift in mean, i.e. $\mu_1 = \mu_2$, $\hat{K}_T$ converge on the radio of a standard Brownian bridge to the variance of z_t. The critical values for the test can obtain by (Ploberger, Krämer, & Kontrus, 1989). It is interesting when the mean shifts occur in the middle of the sample, viz., $\lambda = 0.5$, $\hat{K}_T$ will converge on $\sup_{0 \le r \le 1} \left| \frac{B(r)}{\sigma_1^2 + \Psi_1} \right|$ and is less than the standard distribution $\sup_{0 \le r \le 1} \left| \frac{B(r)}{\sigma_1^2} \right|$. In other words, empirical size will be slightly conservative.

Theorem 2.2. Suppose that Assumption 1 is true under the hypothesis H_1. Then

$$T^{-1/2} \hat{K}_T(r) \xrightarrow{d} \Phi,$$

where

$$\Phi = \frac{\left[\bar{\tau} r (\sigma_1^2 - \sigma_2^2) + \bar{r} \bar{\lambda} (1-2\lambda)(\mu_1 - \mu_2)^2\right]}{\left[\tau \sigma_1^2 + (1-\tau)\sigma_2^2 + \lambda(1-\lambda)(\mu_1 - \mu_2)^2\right] \hat{\gamma}},$$

$$\bar{\tau} = \max\{\tau, (1-\tau)\}.$$

Remark 2.2. Theorem 2.2 implies that the statistics of $\hat{K}_T$ still diverge to infinite at rate of $\sqrt{T}$. We can receive an entertaining conclusion via analyzing Φ. When $\sigma_1^2 > \sigma_2^2$, the first part of the numerator is positive and the second part also is positive if $\lambda < 0.5$, this case will gain in power from the mean shifts. However, if $\lambda > 0.5$ the second part is minus, then it will cause a loss in power. Another case $\sigma_1^2 < \sigma_2^2$ has reverse conclusions for the case $\sigma_1^2 > \sigma_2^2$.

3. SIMULATIONS

In this section we use Monte Carlo simulation methods to investigate the finite sample size and power properties of the tests. Let ε_t is an independent identical standard normal distributed. For each scenario, we simulate the replications 2000 times and report empirical rejection frequencies of the tests with sample size T=100,200,300 for tests run at 5% critical value in various combinations.

To analyze the effects for detecting variance changes caused by break in mean, we consider the magnitude of mean shifts $\triangle = \mu_2 - \mu_1$ varying among $\{0, 0.5, 1, 2, 3\}$, and the ratio of the square standard deviation σ_2^2/σ_1^2 varying among $\{1/4, 1/3, 1/2, 2, 3, 4\}$. Without loss of generality, the location of change points belong to $\{0.3, 0.5, 0.7\}$, so that both early and late breaks are considered.

Table 1
Size of the CUSSQ Test under the Null Hypothesis

		***T*=100**			***T*=200**			***T*=300**	
		λ			λ			λ	
△	0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
0	0.044	0.037	0.043	0.040	0.047	0.048	0.051	0.050	0.056
0.5	0.041	0.034	0.043	0.060	0.046	0.056	0.078	0.048	0.061
1	0.092	0.017	0.088	0.207	0.022	0.196	0.316	0.022	0.307
2	0.650	0.001	0.665	0.978	0.001	0.977	1.000	0.002	0.999
3	0.998	0.000	0.997	1.000	0.000	1.000	1.000	0.000	1.000

We found that, in Table 1, the size percentage of the CUSSQ test varies according to the sample size and the magnitude of mean shifts. The size increases with the increase of sample size, the rejection rate are 8.8% and 30.7% for T=100,300 when △=1, λ=0.7 The larger the magnitude of mean shifts is, the more severe the size distortion is. The size all lie close the asymptotic 5% level when △=0, the affect can be slightly when △=0.5 but the rejection rate are almost 100% at △=3 What is surprising is that if λ=0.5 the size tend to 0 as the magnitude of mean shifts △ increase. This phenomenon is consistent with the conclusions of Theorem 2.1.

Table 3 shows the power of the test under H_1, to save space, we just only focus on the magnitude of mean change △=1 In order to facilitate analysis of the power in Table 3, we also provide the power for only existing variance change point in Table 2. First, Table 2 and Table 3 all indicate,the power increase as the sample size tends to infinite. Let $\tau=\lambda=0.5$, σ_2^2/σ_1^2=1/2, the power respective is 0.26, 0.66 and 0.87 when T=100,200,300. Furthermore, the test is more powerful when the magnitude of variance changes becomes large. The power respective is 1.00, 0.98 and 0.66 as σ_2^2/σ_1^2=1/4,1/3,1/2.

Table 2
Power for Only Existing Variance Change Point

		***T*=100**			***T*=200**			***T*=300**	
σ_2^2/σ_1^2		λ			λ			λ	
	0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
1/4	0.98	0.99	0.89	1.00	1.00	1.00	1.00	1.00	1.00
1/3	0.88	0.93	0.69	1.00	1.00	0.98	1.00	1.00	1.00
1/2	0.52	0.54	0.28	0.81	0.88	0.67	0.94	0.97	0.88
2	0.29	0.57	0.52	0.67	0.88	0.81	0.88	0.98	0.93
3	0.66	0.91	0.89	0.98	1.00	1.00	1.00	1.00	1.00
4	0.90	0.99	0.98	1.00	1.00	1.00	1.00	1.00	1.00

Table 3
Power of the CUSSQ Test under the H_1

			***T*=100**			***T*=200**			***T*=300**	
$\frac{\sigma_2^2}{\sigma_1^2}$			τ			τ			τ	
	λ	0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
	0.3	1.00	1.00	0.91	1.00	1.00	1.00	1.00	1.00	1.00
1/4	0.5	0.77	0.85	0.48	0.98	1.00	0.93	1.00	1.00	1.00
	0.7	0.53	0.43	0.09	0.86	0.80	0.19	0.98	0.96	0.34
	0.3	0.99	0.98	0.79	1.00	1.00	0.99	1.00	1.00	1.00
1/3	0.5	0.58	0.64	0.28	0.93	0.98	0.79	0.99	1.00	0.96
	0.7	0.32	0.26	0.05	0.66	0.57	0.10	0.84	0.77	0.17

To be continued

Continued

		T=100			T=200			T=300		
$\frac{\sigma_2^2}{\sigma_1^2}$	λ	τ 0.3	0.5	0.7	τ 0.3	0.5	0.7	τ 0.3	0.5	0.7
	0.3	0.81	0.78	0.51	0.99	0.99	0.89	1.00	1.00	0.98
1/2	0.5	0.24	0.26	0.12	0.53	0.66	0.37	0.76	0.87	0.64
	0.7	0.13	0.08	0.03	0.25	0.17	0.03	0.41	0.26	0.05
	0.3	0.08	0.23	0.24	0.17	0.49	0.49	0.28	0.69	0.68
2	0.5	0.19	0.39	0.36	0.52	0.76	0.68	0.75	0.94	0.87
	0.7	0.39	0.68	0.68	0.79	0.95	0.96	0.95	1.00	1.00
	0.3	0.34	0.73	0.70	0.72	0.97	0.96	0.92	1.00	1.00
3	0.5	0.52	0.87	0.80	0.94	0.99	0.98	0.99	1.00	1.00
	0.7	0.71	0.95	0.94	0.99	1.00	1.00	1.00	1.00	1.00
	0.3	0.60	0.94	0.93	0.96	1.00	1.00	1.00	1.00	1.00
4	0.5	0.80	0.98	0.95	1.00	1.00	1.00	1.00	1.00	1.00
	0.7	0.89	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Moreover, the power depends heavily on the location of the structure breaks. If the standard deviation decreases, i.e., $\sigma_2^2/\sigma_1^2<1$, the extent of loss power is larger when the variance change occurs in the later of the sample. For example, when T=100, σ_2^2/σ_1^2=1/4 and λ=0.5, the rejection rate is 0.77, 0.85 and 0.48 as τ=0.3, 0.5, 0.7, respectively. Note that as if the time of mean shift moves backwards, the power-distortion becomes larger, which is consistent with the results explained in Theorem 2.2. Such as, let T=200 and σ_2^2/σ_1^2=1/2, τ=0.5, the rejection rate respective is 0.99, 0.66 and 0.17 as λ increases from 0.3 to 0.5 and 0.7. If the standard deviation increases, i.e., $\sigma_2^2/\sigma_1^2>1$, the reverse conclusions is obtained for the case $\sigma_2^2/\sigma_1^2<1$.

CONCLUSION

In this paper we use CUSSQ tests for the variance changes presence of breaks in mean. We derive the asymptotic distribution of the CUSSQ tests under the null and the alternative hypothesis, and find that the size suffer distortion. But the location of mean breaks at 0.5, the size is conservative. The power depends heavily on the location of the structure breaks. If the standard deviation decreases, this case will gain in power from the mean shifts as λ<0.5, and loss in power when λ>0.5. Similarly to the case $\sigma_2^2/\sigma_1^2<1$, the reverse conclusions is obtained if the standard deviation increases. Finally, the theoretical analysis is verified via the numerical simulation.

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APPENDIX

Proof of Theorem 2.1. Observe that

$$\hat{K}_T(r) = \max_{0\le r\le 1} \frac{\sqrt{T}}{\hat{\gamma}} \left| \frac{\sum_{t=1}^{[rT]} \hat{z}_t^2 - \frac{[rT]}{T} \sum_{t=1}^{T} \hat{z}_t^2}{\sum_{t=1}^{T} \hat{z}_t^2} \right| = \max_{0\le r\le 1} \left| \frac{\frac{1}{\hat{\gamma}\sqrt{T}} \sum_{t=1}^{[rT]} \hat{z}_t^2 - \frac{1}{\hat{\gamma}\sqrt{T}} \frac{[rT]}{T} \sum_{t=1}^{T} \hat{z}_t^2}{T^{-1} \sum_{t=1}^{T} \hat{z}_t^2} \right|$$

then,

$$\hat{z}_t^2 = z_t^2 + (\hat{\mu} - \mu_t)^2 + 2(\hat{\mu} - \mu_t) z_t .$$

because of $\mu_t = \mu_1 1_{[t\le \lambda T]} + \mu_2 1_{[t>\lambda T]}$, then

$$\sum_{t=1}^{T} (\hat{\mu} - \mu_t)^2 = \sum_{t=1}^{T} \left(\lambda\mu_1 + (1-\lambda)\mu_2 + \frac{1}{T}\sum_{t=1}^{T} z_t - \mu_t \right)^2$$

$$= \sum_{t=1}^{[\lambda T]} \left((1-\lambda)(\mu_2 - \mu_1) + \frac{1}{T}\sum_{t=1}^{T} z_t \right)^2 + \sum_{t=[\lambda T]+1}^{T} \left(\lambda(\mu_1 - \mu_2) + \frac{1}{T}\sum_{t=1}^{T} z_t \right)^2$$

$$= \lambda T (1-\lambda)(\mu_1 - \mu_2)^2 .$$

and

$$\sum_{t=1}^{T} (\mu_t - \hat{\mu}) z_t = \sum_{t=1}^{T} \left(\mu_t - \lambda\mu_1 - (1-\lambda)\mu_2 - \frac{1}{T}\sum_{i=1}^{T} z_i \right) z_t$$

$$= \sum_{t=1}^{[\lambda T]} \left((1-\lambda)(\mu_1 - \mu_2) - \frac{1}{T}\sum_{i=1}^{T} z_i \right) z_t + \sum_{t=[\lambda T]+1}^{T} \left(\lambda(\mu_2 - \mu_1) - \frac{1}{T}\sum_{i=1}^{T} z_i \right) z_t$$

$$= (\mu_1 - \mu_2) \left[\sum_{t=1}^{[\lambda T]} z_t - \lambda \sum_{t=1}^{T} z_t \right] - \frac{1}{T} \left(\sum_{t=1}^{T} z_t \right)^2 = \mathrm{I}_1 + \mathrm{I}_2 .$$

Then, by a functional central limit theorem as in Chan and Wei [18]. We have

$$\frac{(\mu_1 - \mu_2)}{\sqrt{T}} \mathrm{I}_1 = \frac{1}{\sqrt{T}} \left[\sum_{t=1}^{[\lambda T]} z_t - \lambda \sum_{t=1}^{T} z_t \right] \xrightarrow{w} \sigma\left(W(\lambda) - \lambda W(1) \right),$$

$$\mathrm{I}_2 = \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} z_t \right)^2 \xrightarrow{w} \sigma^2 W^2(1),$$

The results imply that $\mathrm{I}_1 = O_p(T^{1/2}), \mathrm{I}_2 = O_p(1).$

Hence the denominator of the $\hat{K}_T(r)$ can be expressed as

$$T^{-1} \sum_{t=1}^{T} \hat{z}_t^2 = \sigma_1^2 + \lambda(1-\lambda)(\mu_1 - \mu_2)^2 + o_p(1)$$

$$= \sigma_1^2 + \lambda(1-\lambda)(\mu_1 - \mu_2)^2 .$$

Considering the numerator of the $\hat{\mathrm{K}}_T(r)$,

$$\sum_{t=1}^{[rT]}\hat{z}_t^2=\sum_{t=1}^{[rT]}z_t^2+\sum_{t=1}^{[rT]}(\hat{\mu}-\mu_t)^2+2\sum_{t=1}^{[rT]}(\mu_t-\hat{\mu})z_t=\sum_{t=1}^{[rT]}z_t^2+\Lambda_1+\Lambda_2.$$

If $\lambda\le r$,

$$\Lambda_1=\sum_{t=1}^{[\lambda T]}\left((1-\lambda)(\mu_2-\mu_1)+\frac{1}{T}\sum_{i=1}^{T}z_i\right)^2+\sum_{t=[\lambda T]+1}^{[rT]}\left(\lambda(\mu_1-\mu_2)+\frac{1}{T}\sum_{i=1}^{T}z_i\right)^2$$

$$=\lambda T\left(1-(2-r)\lambda\right)(\mu_1-\mu_2)^2, \tag{A.1}$$

$$\Lambda_2=2(\mu_1-\mu_2)\left[\sum_{t=1}^{[\lambda T]}z_t-\lambda\sum_{t=1}^{[rT]}z_t\right]-\frac{2}{T}\sum_{i=1}^{T}z_i\sum_{t=1}^{[rT]}z_t=O_p(T^{1/2}). \tag{A.2}$$

then, the numerator of the $\hat{\mathrm{K}}_T(r)$ can be simplify as follow,

$$\frac{1}{\hat{\gamma}\sqrt{T}}\sum\nolimits_{t=1}^{[rT]}\hat{z}_t^2-\frac{1}{\hat{\gamma}\sqrt{T}}\frac{[rT]}{T}\sum\nolimits_{t=1}^{T}\hat{z}_t^2$$

$$=\frac{1}{\hat{\gamma}\sqrt{T}}\left[\sum_{t=1}^{[rT]}z_t^2-r\sum_{t=1}^{T}z_t^2+(1-r)(1-2\lambda)\lambda T(\mu_1-\mu_2)^2\right]+O_p(1)$$

$$=\frac{1}{\hat{\gamma}\sqrt{T}}\left[\sum_{t=1}^{[rT]}(z_t^2-Ez_t^2)-r\sum_{t=1}^{T}(z_t^2-Ez_t^2)\right]+\frac{(1-r)(1-2\lambda)\lambda T(\mu_1-\mu_2)^2}{\hat{\gamma}\sqrt{T}}+O_p(1)$$

$$\xrightarrow{w}W(r)-rW(1)+\frac{\sqrt{T}}{\hat{\gamma}}\lambda(1-r)(1-2\lambda)(\mu_1-\mu_2)^2.$$

Similarly, if $\lambda>r$, we can obtain that

$$\Lambda_1=rT(1-\lambda)^2(\mu_1-\mu_2)^2,$$

$$\Lambda_2=2(1-\lambda)(\mu_1-\mu_2)\sum_{t=1}^{[rT]}z_t-\frac{2}{T}\sum_{i=1}^{T}z_i\sum_{t=1}^{[rT]}z_t=O_p(T^{1/2}).$$

And

$$\frac{1}{\hat{\gamma}\sqrt{T}}\sum\nolimits_{t=1}^{[rT]}\hat{z}_t^2-\frac{1}{\hat{\gamma}\sqrt{T}}\frac{[rT]}{T}\sum\nolimits_{t=1}^{T}\hat{z}_t^2\xrightarrow{w}W(r)-rW(1)+\frac{\sqrt{T}}{\hat{\gamma}}r(1-\lambda)(1-2\lambda)(\mu_1-\mu_2)^2$$

Combining the result for the nominator and the denominator, we can obtain (2.1).

To proof the Theorem 2.1, we need explain that the estimator $\hat{\gamma}$ converge on a constant, i.e. $\hat{\gamma}^2 - \gamma^2 = O_p(1)$. Note that

$$\frac{1}{T}\sum_{t=1}^{T}(\hat{z}_t^2 - z_t^2) = \frac{1}{T}\sum_{t=1}^{T}(\hat{\mu} - \mu_t)^2 + \frac{2}{T}\sum_{t=1}^{T}(\hat{\mu} - \mu_t)z_t$$

$$= \lambda\left(1-\lambda\right)\left(\mu_1 - \mu_2\right)^2 + o_p(1). \qquad \text{(A.3)}$$

This is proved

$$P\left(\frac{1}{T}\sum_{t=1}^{T}\hat{z}_t^2 - Ez_t^2 > \lambda\left(1-\lambda\right)\left(\mu_1 - \mu_2\right)^2\right) = 0$$

Analogous

$$\frac{1}{T}\sum_{t=1}^{T}(\hat{z}_t^2 - z_t^2)^2 = \frac{1}{T}\sum_{t=1}^{T}(\hat{\mu} - \mu_t)^4 + \frac{4}{T}\sum_{t=1}^{T}(\hat{\mu} - \mu_t)^2 z_t^2 + \frac{4}{T}\sum_{t=1}^{T}(\hat{\mu} - \mu_t)^3 z_t$$

$$= \lambda(1-\lambda)(1+3\lambda^2 - 3\lambda)(\mu_1 - \mu_2)^4 + 4\lambda(1-\lambda)(\mu_1 - \mu_2)^2\sigma^2 + o_p(1).$$

and

$$\frac{1}{T}\sum_{t=1}^{T}(\hat{z}_t^2 + z_t^2)^2 = \frac{1}{T}\sum_{t=1}^{T}(\hat{z}_t^2 - z_t^2)^2 + \frac{4}{T}\sum_{t=1}^{T}\hat{z}_t^2 z_t^2$$

$$\le \frac{2}{T}\sum_{t=1}^{T}(\hat{z}_t^2 - z_t^2)^2 + \frac{2}{T}\sum_{t=1}^{T}z_t^4$$

$$= 2\lambda(1-\lambda)(1+3\lambda^2 - 3\lambda)(\mu_1 - \mu_2)^4 + 8\lambda(1-\lambda)(\mu_1 - \mu_2)^2\sigma^2 + O_p(1).$$

Therefore

$$\left|\frac{1}{T}\sum_{t=1}^{T}\hat{z}_t^4 - \frac{1}{T}\sum_{t=1}^{T}z_t^4\right| \le \left(\frac{1}{T}\sum_{t=1}^{T}(\hat{z}_t^2 - z_t^2)^2\right)^{1/2}\left(\frac{1}{T}\sum_{t=1}^{T}(\hat{z}_t^2 + z_t^2)^2\right)^{1/2}$$

$$= \sqrt{2}\lambda(1-\lambda)(1+3\lambda^2 - 3\lambda)(\mu_1 - \mu_2)^4 + 4\sqrt{2}\lambda(1-\lambda)(\mu_1 - \mu_2)^2\sigma^2. \qquad \text{(A.4)}$$

Combing (A.3) and (A.4), we can obtain

$$\hat{\gamma}^2 - \gamma^2 = O_p(1).$$

The proof of theorem 2.1 is then completed.

Proof of Theorem 2.2. Through the proof of the Theorem 2.1, we can obtain the denominator of the $\hat{\mathrm{K}}_T\left(r\right)$ is

$$T^{-1}\sum_{t=1}^{T}\hat{z}_t^2 = \tau\sigma_1^2+(1-\tau)\sigma_2^2+\lambda(1-\lambda)(\mu_1-\mu_2)^2.$$

Then, we simplify the numerator of the $\hat{\mathrm{K}}_T(r)$. If $\lambda \le r \le \tau$, together with (A.1) and (A.2), yields that

$$\frac{1}{\hat{\gamma}\sqrt{T}}\left[\sum_{t=1}^{[rT]}\hat{z}_t^2-\frac{[rT]}{T}\sum_{t=1}^{T}\hat{z}_t^2\right]$$

$$=\frac{1}{\hat{\gamma}\sqrt{T}}\left\{\sum_{t=1}^{[rT]}\sigma_1^2\varepsilon_t^2-r\left[\sum_{t=1}^{\tau T}\sigma_1^2\varepsilon_t^2+\sum_{t=\tau T+1}^{T}\sigma_2^2\varepsilon_t^2\right]+(1-r)(1-2\lambda)\lambda T(\mu_1-\mu_2)^2\right\}+O_p(1)$$

$$=\frac{T}{\hat{\gamma}\sqrt{T}}\left[r\sigma_1^2-r\tau\sigma_1^2-r(1-\tau)\sigma_2^2+(1-r)(1-2\lambda)\lambda(\mu_1-\mu_2)^2\right]$$

$$=\frac{\sqrt{T}}{\hat{\gamma}}\left[r(1-\tau)(\sigma_1^2-\sigma_2^2)+(1-r)(1-2\lambda)\lambda(\mu_1-\mu_2)^2\right].$$

Analogous, if $\lambda \le r$ and $\tau \le r$,

$$\frac{1}{\hat{\gamma}\sqrt{T}}\left[\sum_{t=1}^{[rT]}\hat{z}_t^2-\frac{[rT]}{T}\sum_{t=1}^{T}\hat{z}_t^2\right]=\frac{\sqrt{T}}{\hat{\gamma}}\left[\tau(1-r)(\sigma_1^2-\sigma_2^2)+(1-r)(1-2\lambda)\lambda(\mu_1-\mu_2)^2\right].$$

If $\lambda > r$ and $\tau > r$,

$$\frac{1}{\hat{\gamma}\sqrt{T}}\left[\sum_{t=1}^{[rT]}\hat{z}_t^2-\frac{[rT]}{T}\sum_{t=1}^{T}\hat{z}_t^2\right]=\frac{\sqrt{T}}{\hat{\gamma}}\left[r(1-\tau)(\sigma_1^2-\sigma_2^2)+r(1-\lambda)(1-2\lambda)(\mu_1-\mu_2)^2\right]$$

.

If $\lambda > r > \tau$,

$$\frac{1}{\hat{\gamma}\sqrt{T}}\left[\sum_{t=1}^{[rT]}\hat{z}_t^2-\frac{[rT]}{T}\sum_{t=1}^{T}\hat{z}_t^2\right]=\frac{\sqrt{T}}{\hat{\gamma}}\left[\tau(1-r)(\sigma_1^2-\sigma_2^2)+r(1-\lambda)(1-2\lambda)(\mu_1-\mu_2)^2\right]$$

.

In conclusion

$$\frac{1}{\hat{\gamma}\sqrt{T}}\left[\sum_{t=1}^{[rT]}\hat{z}_t^2-\frac{[rT]}{T}\sum_{t=1}^{T}\hat{z}_t^2\right]=\frac{\sqrt{T}}{\hat{\gamma}}\left[\overline{\tau r}(\sigma_1^2-\sigma_2^2)+\overline{r\lambda}(1-2\lambda)(\mu_1-\mu_2)^2\right].$$

Hence,

$$\hat{\mathrm{K}}_T(r)\xrightarrow{d}\frac{\frac{\sqrt{T}}{\hat{\gamma}}\left[\overline{\tau r}(\sigma_1^2-\sigma_2^2)+\overline{r\lambda}(1-2\lambda)(\mu_1-\mu_2)^2\right]}{\tau\sigma_1^2+(1-\tau)\sigma_2^2+\lambda(1-\lambda)(\mu_1-\mu_2)^2}.$$

This proves the theorem.

