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Comparative Loss Aversion: Some New Behavioral Implications

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Abstract
This paper proposes a new loss aversion coefficient for decision makers. Considering this loss aversion coefficient and Shalev's (2002) perceptive utility function, we define a new version of rank-dependent expected utility theory. Our main results extend the restrictions of comparative loss aversion, and reveal the behavioral implications of comparative loss aversion. The new loss aversion coefficient is well defined as a measure of degree of loss aversion. Our main findings of comparative loss aversion can also be applied to welfare, health, insurance and other topical economic problems.

Key words: Loss aversion coefficient; Comparative loss aversion; Rank-dependent expected utility; Decision making under risk

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INTRODUCTION

Loss aversion is one of the most significant concepts in behavioral finance. It has received wide attention in decision making under risk. Kahneman and Tversky (1979) pioneer the idea of loss aversion. Loss aversion implies decision makers are more sensitive to losses than to gains. Loss aversion is a vague concept, it has various measure methods (see e.g. Kahneman & Tversky, 1979; Wakker & Tversky, 1993; Bowman *et al.*, 1999; Breiter *et al.*, 2001; Neilson, 2002; Bleichrodt & Miyamoto, 2003). However, these measure skills don't make loss aversion be separated from utility curvature. This paper proposes a new separated loss aversion coefficient as a measure of loss aversion of decision makers; we hope that this loss aversion coefficient will play an important role for the comparison of loss aversion of decision makers.

Many utility theories characterize the preference relations of decision makers, such as rank-dependent expected utility theory (Quiggin, 1981), prospect theory (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992). Although these utility theories have tractability and psychological behavioral evidences (see e.g. Laibson & Zeckhauser, 1998), they do not well reflect the perceptive utility and loss aversion of decision maker. Therefore, this paper proposes a new version of rank-dependent expected utility theory to make comparative loss aversion will be more unambiguous.

The comparison of loss aversion between decision makers has been dissussed in the literature (see e.g. Neilson, 2002; Köbberling & Wakker, 2005; Blavatskyy, 2011). This paper extends the restrictions of comparative loss aversion, and reveals the behavioral implications of comparative loss aversion.

Many empirical results imply loss aversion (see e.g. Samuelson & Zeckhauser, 1988; Benartzi & Thaler, 1995; Camerer *et al.*, 1997; Genesove & Mayer, 2001; Chen *et al.*, 2006). Our main results of comparative loss aversion can be applied to the comparison of consumptions of families with different income levels. Their different loss averse degrees of less income lead to different consumptions. Our main findings can also be applied to welfare, health, insurance and other topical economic problems.

This paper is constructed as follows. Section 2 defines a new version of rank-dependent expected utility theory. Section 3 presents main results of comparison of loss aversion between decision makers. Section 4 provides an example as an illustration of main results of comparative loss aversion. Section 5 concludes. Proofs are presented in the Appendix.

1. RANK-DEPENDENT EXPECTED UTILITY THEORY

Let $X = \{x_1, x_2, ..., x_n\}$ be a finite set of riskless monetary outcomes, containing at least three elements, the outcomes in X can be ordered from best to worst, i.e., $x_1 \geq ... \geq x_{k-1} \geq x_k > x_{k+1} \geq ... \geq x_n$ for some $1 \leq k \leq n$, where $x_k = r \in X$ denoted by a riskless reference outcome that a decision maker perceives. Let $X_+ \subseteq X$ be nonempty set of gain x_+ that above the reference outcome, i.e., $X_+ = \{x_1, x_2, ..., x_k\}$. Let $X_- = X \setminus X_+$ be nonempty set of loss x_- that below the reference outcome, i.e., $X_- = \{x_{k+1}, x_{k+2}, ..., x_n\}$. Lottery $l: X \to [0,1]$ is a probability distribution on X, i.e., $l(x) \in [0,1]$ for all $x \in X$ and $\sum_{x \in X} l(x) = 1$. A degenerate lottery that yields one outcome $x \in X$ with probability 1 is denoted by x. Let L denote the set of all lotteries over X. Lottery $l_+: X \to [0,1]$ is a loss-free lottery, i.e., $l_+(x_+) > 0$ for all $x_+ \in X_+$, $\sum_{x_+ \in X_+} l_+(x_+) = 1$ $l_+(x_-) = 0$ for all $x_- \in X_-$. Let $L_+ \subseteq L$ denote the set of all such loss-free lotteries over X. Similarly, lottery $l_-: X \to [0,1]$ is a gain-free lottery, i.e., $l_-(x_-) > 0$ for all $x_- \in X_-$, $\sum_{x_- \in X_-} l_-(x_-) = 1$ $l_-(x_+) = 0$ for all $x_+ \in X_+$. Let $L_- \subseteq L$ denote the set of all such gain-free lotteries over X. A decision maker has a unique preference relation $\succsim^r$ on L if the fixed reference outcome is r. We write $\succsim^r$ as $\succsim$ in simplified form if all decision makers have the same reference outcome r, $\succ$ is denoted as the asymmetric component of $\succsim$, $\sim$ is denoted as the symmetric component of $\succsim$.

We next define a new version of rank-dependent expected utility theory proposed by Quiggin (1981). We begin with a definition of perceptive utility function in loss aversion model proposed by Shalev (2002). Let the increasingly differentiable function denoted by $u: X \to R$ be the basic utility function of a decision maker, let $r \in X$ be a riskless reference outcome, let real number $\lambda \geq 0$ be loss aversion coefficient that is constant for different outcomes and reference outcomes. Loss aversion is considered as a new risk attitude, the perceptive utility function of a decision maker is defined by

$$U(x) = \begin{cases} u(x), & x \in X_+ \\ u(x) - \lambda[u(r) - u(x)], & x \in X_- \end{cases} \quad (1)$$

Behavioral implications behind definition (1) are very intuitive. It shows the perceptive utility of a decision maker equals to his basic utility for the gain above reference outcome, and it equals to his basic utility by subtracting the loss multiplied by loss aversion coefficient for the loss below reference outcome. This perceptive utility function implies the core idea of loss aversion is that perceptive utility function is kinked at reference point. Köbberling and Wakker (2005) take the riskless reference outcome r as zero, and take the ratio of the left derivative and the right derivative of the utility function at reference point as index of loss aversion. According to this implication of loss aversion, we can define a plausible loss aversion coefficient at the reference point in the following.

Definition 2.1. The loss aversion coefficient of a decision maker is defined as

$$\lambda = \frac{U'_-(r) - U'_+(r)}{U'_+(r)} \quad (2)$$

Formally, we propose a new rank-dependent expected utility theory using perceptive utility function (1). A decision maker with basic utility function u and reference outcome r uses perceptive utility function U to compute his rank-dependent expected utility of risky lottery. So the rank-dependent expected utility of any lottery l with respect to reference outconme r is denoted by

$$EU(l) = \sum_{x \in X} U(x)[w(\sum_{\substack{y \in X \\ U(y) \geq U(x)}} l(y)) - w(\sum_{\substack{y \in X \\ U(y) > U(x)}} l(y))] \quad (3)$$

Theorem 1 in Peters (2012) shows a decision maker's preference over lotteries can be respresented by a unique utility function (up to a positive affine transformation) in expected utility theory. In this paper, we assume decision makers are rank-dependent expected utility maximizers, and their preferences over lotteries exist the new rank-dependent expected utility representations. Namely, there exists a unique rank-dependent utility function (up to a positive affine transformation) $u: X \to R$, a real number $\lambda \geq 0$ (i.e. a unique perceptive utility function U) and a unique continuous and strictly increasing perceptive probability weighting function $w: [0,1] \to [0,1]$ with $w(0) = 0$ and $w(1) = 1$, we have

$$l \succsim h \Leftrightarrow EU(l) \geq EU(h) \quad (4)$$

for $\forall\ r \in X, \forall\ l, h \in L$.

2. MAIN RESULTS

Blavatskyy (2011) gives a benchmark of comparison of loss aversion between decision maker 1 and decision maker 2 whose preference relations are characterized by $\succsim_1$ and $\succsim_2$ respectively in the following.

Lemma 3.1. Decision maker 1 is more loss aversion than decision maker 2 if and only if

(i) $l_+ \succ_2 l \Rightarrow l_+ \succ_1 l$ for $\forall\ l_+ \in L_+,\ l \in L$;

(ii) $l_+ \sim_2 l \Rightarrow l_+ \succsim_1 l$ for $\forall\ l_+ \in L_+,\ l \in L$;

(iii) $\exists\ l_+ \in L_+,\ l \in L$, such that $l_+ \sim_2 l,\ l_+ \succ_1 l$.

Behavioral implications behind Lemma 3.1 are very intuitive. A less loss averse decision maker strictly prefers a certain loss-free lottery to a risky lottery, and then a more loss averse decision maker should more do so. And a less loss averse decision maker perceives indifferent between a certain loss-free lottery and a risky lottery, then a more loss averse decision maker weakly prefers a loss-free lottery to a risky lottery. Blavatskyy (2011) points out the advantage of Lemma 3.1 is that we can compare loss aversion between decision makers according to their observable preferences rather than depending on some specific decision theory that represents their preferences.

As the core idea of this paper, we next present main results of comparative loss aversion between different decision makers in the new rank-dependent expected utility theory. In this paper, decision maker 1, 2 are characterized by basic utility function $u_1, u_2 : X \to R$ and loss aversion coefficient $\lambda_1, \lambda_2 \geq 0$ i.e., perceptive utility function U_1, U_2 respectively. We can obtain several propositions of comparative loss aversion between different decision makers in the following.

Proposition 3.1. A decision maker 1 with perceptive utility U_1 and perceptive probability weighting w_1 is more loss aversion than a decision maker 2 with perceptive utility U_2 and perceptive probability weighting w_2 if and only if $\exists\ a \in R_+,\ b \in R$ such that

(i) $w_1(p) = w_2(p)$, for $\forall\ p \in [0,1]$;

(ii) $\Delta U_{1,2}(x_+) = 0$, for $\forall\ x_+ \in X_+$;

(iii) $\Delta U_{1,2}(x_-) \geq 0$, for $\forall\ x_- \in X_-$;

(iv) $\exists\ x_- \in X_-$, such that $\Delta U_{1,2}(x_-) > 0$;

Where, $\Delta U_{1,2}(x) = NU_2(x) - U_1(x)$,

$NU_2(x) = aU_2(x) + b,\ \forall\ x \in X$.

Proof. See Appendix.

Behavioral implications behind Proposition 3.1 are very intuitive. It shows the comparison of loss aversion between two decision makers can be characterized by perceptive utility difference; it doesn't depend on the shape of perceptive probability weighting function. In other words, decision maker 1 is more loss aversion than decision maker 2 if the two decision makers have the same perceptive probability weighting, there is indifference between decision maker 1's perceptive utility U_1 and decision maker 2's normalized perceptive utility NU_2 (i.e. U_1 and U_2 are strategically identical) in the domain of gains, and difference between the two decision makers is nonnegative (and it is strictly positive for at least one loss) in the domain of losses, or vice versa.

We can obtain an intuitive idea according to Proposition 3.1. It is that two decision makers can be ranked by loss attitudes only if they have the same perceptive probability weighting, the same perceptive utility in the domain of gains, and the perceptive utility of decision maker 1 is lower than the one of decision maker 2 in the domain of losses. So we immediately obtain some strong restrictions that characterize comparative loss aversion between different decision makers in the following propositions.

Proposition 3.2. A decision maker 1 with perceptive utility U_1 and perceptive probability weighting w_1 is more loss aversion than a decision maker 2 with perceptive utility U_2 and perceptive probability weighting w_2 if $\exists\ \mu \in R_+,\ a \in R_+,\ b \in R$ such that

(i) $w_1(p) = w_2(p)$, for $\forall\ p \in [0,1]$;

(ii) $U_1'(x_+) \equiv NU_2'(x_+)$, for $\forall\ x_+ \in X_+$;

(iii) $U_1'(x_-) \geq \mu \geq NU_2'(y_-)$, for $\forall\ x_-, y_- \in X_-$;

where, $NU_2(x) = aU_2(x) + b$, for $\forall\ x \in X$.

Proof. See Appendix.

The behavioral implications of proposition 3.2 are intuitive. It shows comparative loss aversion between two decision makers can be characterized by perceptive marginal utility, it doesn't depend on the shape of perceptive probability weighting function. Namely, decision maker 1 is more loss aversion than decision maker 2 if the two decision makers have the same perceptive probability weighting, the same perceptive marginal utility in the domain of gains, and the perceptive marginal utility of decision maker 1 is greater than the one of decision maker 2 in the domain of losses. The parameter μ can be interpreted as a boundary value between perceptive marginal utilities of the two decision makers.

Proposition 3.3. A decision maker 1 with perceptive utility U_1 and perceptive probability weighting w_1 is more loss aversion than a decision maker 2 with perceptive utility U_2 and perceptive probability

(i) $w_1(p) = w_2(p)$, for $\forall\ p \in [0,1]$;

(ii) $U_1(x_+) = NU_2(x_+)$, for $\forall\ x_+ \in X_+$;

(iii) $\dfrac{U_1(r) - U_1(x_-)}{r - x_-} \geq \eta \geq \dfrac{NU_2(r) - NU_2(y_-)}{r - y_-}$, for

$\forall\ x_-, y_- \in X_-$;

where, $NU_2(x) = aU_2(x) + b,\ \forall\ x \in X$.

Proof. See Appendix.

Proposition 3.3 contains some deeply behavioral implications. It reflects comparative loss aversion between two decision makers can be characterized by average perceptive utility; it doesn't depend on the shape of perceptive probability weighting function. Namely, decision maker 1 is more loss aversion than decision maker 2 if the two decision makers have the same perceptive probability weighting, the same perceptive utility in the domain of gains, and the average perceptive utility of decision maker 1 is greater than the one of decision maker 2 in the domain of losses. The parameter η can be seen as a boundary value between two decision makers' average perceptive utilities.

Proposition 3.4. A decision maker 1 with basic utility function u_1, loss aversion coefficient λ_1 and perceptive probability weighting w_1 is more loss aversion than a decision maker 2 with basic utility function u_2, loss aversion coefficient λ_2 and perceptive probability weighting w_2

if and only if $\exists\ a \in R_+$, $b \in R$ such that

(i) $w_1(p) = w_2(p)$, for $\forall\ p \in [0,1]$;

(ii) $u_1(x) = nu_2(x)$, for $\forall\ x \in X$;

(iii) $\lambda_1 \geq \lambda_2$;

where, $nu_2(x) = au_2(x) + b$, $\forall\ x \in X$.

Proof. See Appendix.

Proposition 3.4 reveals the fundamental nature of comparative loss aversion between decision makers. It shows the comparison of loss aversion between two decision makers can be characterized by loss aversion coefficient. Loss aversion coefficient can be used as a measure of degree of loss aversion of decision makers. So decision maker 1 is more loss aversion than decision maker 2 if the two decision makers have the same perceptive probability weighting, there is indifference between decision maker 1's basic utility and decision maker 2's normalized basic utility nu_2 (i.e. u_1 and u_2 are strategically identical) in the domain of all outcomes, and the loss aversion coefficient of decision maker 1 is greater than the one of decision maker 2.

In this paper, comparative loss aversion is characterized by observed preferences over lotteries, but Köbberling and Wakker (2005) characterize comparative loss aversion by Yaari's acceptance sets. Proposition 3.4 makes these two formulations of comparative loss aversion coincide in the new rank-dependent expected utility theory.

3. EXAMPLE

In this section, we provide an example as an illustration of Proposition 3.4 that implies the essence of comparative loss aversion between decision makers. The basic utility function of decision maker 1 is $u_1(x) = \sqrt{x} + 1$, $x \in R_+$. Without loss of generality, we take $a = 1$, $b = 0$, then the normalized basic utility function of decision maker 2 is $nu_2(x) = u_2(x) = \sqrt{x} + 1, x \in R_+$. We take $w_1(p) = w_2(p) = p$, $p \in [0,1]$, reference outcome $r = 4$ and $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$, so $\lambda_1 > \lambda_2$. Now we demonstrate decision maker 1 is more loss aversion than decision maker 2 if the conditions (i)-(iii) of Proposition 3.4 hold. According to the definition of perceptive utility, we have $U_1(x) = NU_2(x) = u_1(x) = \sqrt{x} + 1$, when $x \geq r = 4$, so $\Delta U_{1,2}(x) = NU_2(x) - U_1(x) = 0$, when $x \geq r = 4$. We also obtain that

$$\Delta U_{1,2}(x) = U_1(x) - NU_2(x) = (\lambda_1 - \lambda_2)[u_1(r) - u_1(x)]$$
$$= \frac{\sqrt{r} - \sqrt{x}}{4} > 0$$

when $x < r = 4$.

According to Proposition 3.1, we demonstrate that decision maker 1 is more loss aversion than decision maker 2. We can also demonstrate the inverse conclusion of Proposition 3.4. We give an illustration of behavioral implications of comparative loss aversion of Proposition 3.4 in Figure 1.

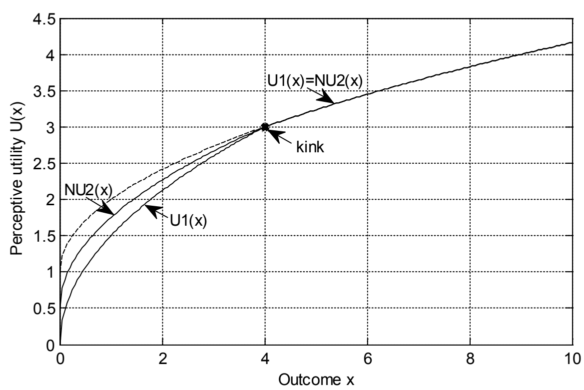

Figure 1
Decision Maker 1 Is More Loss Aversion Than Decision Maker 2

In Figure 1, we see that decision maker 1 is more loss aversion than decision maker 2, then they have the same perceptive utility in the domain of gains, and the perceptive utility of decision maker 1 is lower than the one of decision maker 2 in the domain of losses, namely, decision maker 1 is more sensitive to losses. Conversely, the loss aversion coefficient of decision maker 1 is greater than the one of decision maker 2, of course, they must have the same perceptive probability weighting and the same basic utility on the set of all outcomes, we have $U'_{1-}(r) \geq NU'_{2-}(r)$ in terms of Definition 2.1 and $U'_{1+}(r) = NU'_{2+}(r)$. Therefore, the perceptive utility function of decision maker 1 exhibits a greater kink at reference point $r = 4$, so that the curve graph of decision maker 1 falls below the one of decision maker 2. In this sense, $U'_{1-}(r) \geq NU'_{2-}(r)$ can be used as a measure of comparative loss aversion between the two decision makers. According to Proposition 3.1, we conclude that decision maker 1 is more loss aversion than decision maker 2. Based on the above analysis, we note that loss aversion coefficient in Definition 2.1 is well defined as a measure of degree of loss aversion of decision makers in the new rank-dependent expected utility theory.

CONCLUDING REMARKS

This paper proposes a new loss aversion coefficient for the perceptive utility function of decision maker. This loss aversion coefficient will result in an unambiguous decomposition of risk attitude into three distinct components: perceptive utility, perceptive probability weighting and loss aversion. Considering this loss aversion coefficient, we define a new version of rank-dependent expected utility theory. Our main results of comparison of loss aversion are obtained by characterizing some restrictions of perceptive utility functions. These restrictions show some very meaningful behavioral implications of comparison of loss aversion. In fact, comparative loss aversion between decision makers is implied by the comparison of perceptive utility difference, perceptive marginal utility, average perceptive utility and loss aversion coefficient. Our main results extend the restrictions of comparative loss aversion, reveal the essence of comparative loss aversion, and also make comparative loss aversion proposed by us coincide with the one proposed by Köbberling and Wakker (2005) in the new rank-dependent expected utility theory. The new loss aversion coefficient is well defined as a measure of degree of loss aversion of decision makers. Our main findings of comparative loss aversion can also be applied to welfare, health, insurance and other topical economic problems.

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APPENDIX. PROOFS

1. Proof of Proposition 3.1

According to Proposition 4 in Blavatskyy (2011), we use perceptive utility function (1) to replace its basic utility function, and consider the preference representation (4) in new rank-dependent expected utility theory. Considering Lemma 3.1, we have decision maker 1 is more loss aversion than decision maker 2 if and only if $\exists\ a \in R_+$, $b \in R$ such that $w_1(p) = w_2(p)$, for $\forall\ p \in [0,1]$; $U_1(x_+) = aU_2(x_+) + b$, for $\forall\ x_+ \in X_+$;

$U_1(x_-) \le aU_2(x_-) + b$, for $\forall\ x_- \in X_-$;

$\exists\ x_- \in X_-$, such that $U_1(x_-) < aU_2(x_-) + b$ Let $\Delta U_{1,2}(x)$ $= NU_2(x) - U_1(x)$, $NU_2(x) = aU_2(x) + b$, $\forall\ x \in X$, we obtain the parts (i)-(iv) of definition

2. Proof of Proposition 3.2

According to Proposition 3.1, we only need to proof part (ii) and (iii) of Proposition 3.2 implying part (ii) and (iii) of Proposition 3.1 respectively. Part (ii) of Proposition 3.2 implies $U_1(x_+) \equiv NU_2(x_+) + c$, for $\forall x_+ \in X_+$, c is constant, because $NU_2(x_+) = aU_2(x_+) + b$, for $\forall x_+ \in X_+$, so $U_1(x_+) = NU_2(x_+)$, for $\forall x_+ \in X_+$, i.e., $\Delta U_{1,2}(x_+) = 0$, for $\forall x_+ \in X_+$, which is the part (ii) of Proposition 3.1. Let $x_- = y_- \in X$, part (iii) of Proposition 3.2 implies $U'_1(x_+) \ge NU'_2(x_-)$, for $\forall x_- \in X_-$, i.e., $NU'_2(x_-) - U'_1(x_-) \le 0$, for $\forall x_- \in X_-$, that is to say, $NU_2(x_-) - U_1(x_-)$ is nondecreasing on X_-. We have $NU_2(x_-) - U_1(x_-) \ge NU_2(r) - U_1(r) = 0$, when $x_- < r$. So we have $\Delta U_{1,2}(x_-) \ge 0$, for $\forall x_- \in X_-$, which is the part (iii) of Proposition 3.1. We complete the proof of Proposition 3.3.

3. Proof of Proposition 3.3

According to Proposition 3.1, we only need to proof the parts (ii) , (iii) of Proposition 3.3 imply the parts (ii) , (iii) of Proposition 3.1 respectively. Part (ii) of Proposition 3.3 implies $\Delta U_{1,2}(x_+) = NU_2(x_+) - U_1(x_+) = 0$, for $\forall x_+ \in X_+$, which is the part (ii) of Proposition 3.1. Part (iii) of Proposition 3.3 implies $\dfrac{U_1(r) - U_1(x_-)}{r - x_-} \ge \dfrac{NU_2(r) - NU_2(x_-)}{r - x_-}$, for $\forall x_- \in X_-$ when $x_- = y_- \in X_-$. Because $U_1(r) = NU_2(r)$, we have $U_1(x_-) \le NU_2(x_-)$, for $\forall x_- \in X_-$, i.e., $\Delta U_{1,2}(x_-) \ge 0$, for $\forall x_- \in X_-$, which is the part (iii) of Proposition 3.1. We complete the proof of Proposition 3.3.

4. Proof of Proposition 3.4

According to Proposition 3.1, we only need to proof the parts (ii) , (iii) of Proposition 3.4 imply the parts (ii) , (iii) of Proposition 3.1, or vice versa. According to the definition of perceptive utility function, part (ii) of Proposition 3.4 implies $U_1(x_+) = u_1(x_+) = au_2(x_+) + b = aU_2(x_+) + b = NU_2(x_+)$, for $\forall x_+ \in X_+$, i.e., $\Delta U_{1,2}(x_+) = 0$, for $\forall x_+ \in X_+$, which is the part (ii) of Proposition 3.1. Because $U_i(x_-) = u_i(x_-) - \lambda_i[u_i(r) - u_i(x_-)]$ (i = 1, 2), for $\forall x_- \in X_-$, $u_1(x) = au_2(x) + b$, for $\forall x \in X$, part (ii) of Proposition 3.6, i.e., $\lambda_1 \ge \lambda_2$, and $u_1(r) > u_1(x_-)$ when $x_- \in X_-$, we have $\Delta U_{1,2}(x_-) = (aU_2(x_-) + b) - U_1(x_-)$, for $\forall x_- \in X_-$, i.e., $\Delta U_{1,2}(x_-) = (\lambda_1 - \lambda_2)[u_1(r) - u_1(x_-)] \ge 0$, for $\forall x_- \in X_-$, which is the part (iii) of Proposition 3.1. We complete the proof of Proposition 3.4.