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Persistence Changes Test for Heavy Tail Series in the Presence of Index Breaks

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Abstract

In this paper we consider the effect of persistence change test when the series exists an index change point at the moment. It is shown that under the null hypothesis that the circumstance of the series only existed an index change point, if the heavy tail index k change from large to small, the statistics is diverging at a rate of $T^{2/\kappa_2-\kappa_1}$, and the larger of the κ_2-κ_1 is, the faster the divergence is. If the index changes from small to large, the statistics converge to the bounded constant. The numerical simulation shows that no matter how the change of k will lead to the size distortions, and the size distortions shows more serious when $\kappa_1>\kappa_2$.

Key words: Persistence change point; Heavy tail series; Ratio statistics

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INTRODUCTION

Persistence change is one of the hotspots in the research of change points in recent years. Statisticians have found that many financial data are changed from stationary sequences to unit root sequences under the influence of certain factors such as the actual output of the United States and the European Union Data, US fiscal deficit data. This change from the smooth to the unit root, that is, the change of the persistence component, is called the persistence change point, and vice versa. A large number of scholars have studied the persistence change point, and Kim (2000) detection of change in persistence of a linear time. Kim (2002) Corrigendum to "detection of change in persistence of a linear time series". Leybourne and Kim (2003) tests for a change in persistence against the null of difference stationarity. Busetti and Taylor (2004) Tests of stationarity against a change in persistence. and so on. But found that in the past three decades, the study of persistent variable point is mostly based on the normal distribution of the new information process. However, Mandelbrot (1963) The variation of certain speculative prices. Article shows that many financial asset yields can not be characterized by a good distribution of normal, they are more spikes, heavy tail and other characteristics, so the heavy tail sequence of persistent change point of the study more practical significance.

Guillaume (1997) From the bird's eye to the microscope: a survey of new stylized facts of the intra-daily foreign exchange markets and Anderson (1997) Meerschaert, M. M. Periodic moving average of random variables with regularly varying tails found that many types of data in the financial and economic arena, all of which are tail variance-infinite sequences of normal distribution; Rechev (2000) Stable Paretian Models in Finance and Kokoszka (2001) Can one use the Durbin-Levinson algorithm to generate infinite variance fractional ARIMA time series studied linear models with heavy-tailed sequences; Horvath (2003) Bootstrap approximation to a unit root test statistics for heavy-tailed observations. Jin (2009) Subsampling tests for mean change point with heavy-tailed innovations studied the new the process of infinite variance of the heavy tail sequence, the various variable point of the test problem, and so on. On the basis of these predecessors, this paper will also be based on the heavy tail sequence of persistent change point to make an analysis, the specific content and chapter arrangements see the next section.

In this paper, the goal of the paper is to extend the persistent change point test with index change in heavy tail series. The following work on the limit distribution of statistics is obtained and analyzes the influence of the statistics for the break points. We can conclude that the persistent change point not only depends on the heavy tail index, but also depends on the magnitude of heavy tail index and any other.

The remainder of the paper is structured as follows. Section 1 introduces the model, assumption and test statistic. Main results in Section 2. Section 3 included some Monte Carlo experiments. The conclusions draw in Section 4. Finally, all proofs are given in the appendix.

1. MODEL, HYPOTHESES AND STATISTIC

In order to analyze the effect of Persistent change point test with index change in heavy tail series, given the following generation process:

$$y_t=\mu+z_t,$$

$$z_t=\begin{cases}\rho_1 z_{t-1}+\varepsilon_t, & t=1,\cdots,[\tau^*T],\\ \rho_2 z_{t-1}+\varepsilon_t, & t=[\tau^*T]+1,\cdots,T.\end{cases}$$

$$\varepsilon_t=\varepsilon_{t,1}1_{\{t\le\lambda^*T\}}+\varepsilon_{t,2}1_{\{t>\lambda^*T\}}.$$

Among them, $1_{\{\cdot\}}$ is the indicator function, $\varepsilon_{t,1}$ and $\varepsilon_{t,2}$ are heavy tail sequence, the heavy tail index is κ_1 and κ_2, λ^* and $\tau^*$are unknown change point moment, and indicate the heavy tail index change point and the persistence change point respectively. In order to be able to better show the effect of the heavy tail index change point on the persistence change point, consider the following hypothesis test problem: the null hypothesis H_0 is that there no persistent change point, but there presence a heavy tail index change point, ie

$$H_0:\rho_1=\rho_2=0,\kappa_1\neq\kappa_2,$$

against the alternative hypothesis that the series exist a persistent change point and a heavy tail index change point, expressed as follow:

$$H_1:\rho_1\neq\rho_2,|\rho_1|<1,\rho_2=1,\kappa_1\neq\kappa_2.$$

Since this paper is determine to the direction of the persistence change point from $I(0)$ to $I(1)$, the statistic will use Kim's ratio test statistic.

$$\Xi(\tau)=\frac{(T-[\tau T])^{-2}\sum_{t=[\tau T]+1}^{T}\left(\sum_{i=[\tau T]+1}^{t}\hat{z}_{1,i}\right)^2}{[\tau T]^{-2}\sum_{t=1}^{[\tau T]}\left(\sum_{i=1}^{t}\hat{z}_{0,i}\right)^2}.$$

Where,

$$\hat{z}_{0,i}=y_i-\frac{1}{[\tau T]}\sum_{i=1}^{[\tau T]}y_i,\ \ \hat{z}_{1,i}=y_i-\frac{1}{T-[\tau T]}\sum_{i=[\tau T]+1}^{T}y_i.$$

In the model given by Kim, the error process is a mixed Gaussian sequence, which satisfies the general central limit theorem, and in this paper is based on the persistence point of the heavy tail sequence. In order to derive the limit distribution of $E(\tau)$, we give the following assumptions and lemmas.

Assumption 1.1. is a symmetric independent distribution of the heavy tail sequence, $\kappa\in(1,2)$ in the stable absorption domain, and $E\varepsilon_t=0$.

Lemma 1.1. When assumption 2.1. is hold, there are

$$\left(a_T^{-1}\sum_{t=1}^{[\tau T]}\varepsilon_t,\ a_T^{-2}\sum_{t=1}^{[\tau T]}\varepsilon_t^2\right)\xrightarrow{d}(U_1(\tau),U_2(\tau)).$$

among them

$$a_T=\inf\{x:P(|\varepsilon_t|>x)\le T^{-1}\}.$$

And $U_1(\tau)$ and $U_2(\tau)$ are on the [0,1], and the heavy tail exponent is the Levy process of k and $k/2$ respectively. Kokoszka and Wolf proved the lemma and pointed out that a_T could be expressed as

$$a_T=T^{1/k}L(T).$$

Where $L(\cdot)$ is slow variable function.

2. MAIN RESULTS

Theorem 2.1. The sequence $\{\varepsilon_t\}$ satisfies assumption 2.1. Under the condition that the null hypothesis H_0 is satisfied, if $(\lambda^*,\ 1]\cap\Lambda\neq\emptyset$, then

$\Xi(\tau)=O_p(T^{2/\kappa_2-2/\kappa_1})$, when $\kappa_1>\kappa_2$;

$\Xi(\tau)=O_p(1)$, when $\kappa_1\le\kappa_2$.

Remark 2.1. It can be seen from the conclusion in theorem 3.1 that when the heavy tail exponent k is changed from large to small, the statistic tends to infinity when $T\to\infty$, but when the heavy tail exponent k is changed from small to large, the statistic converges to the bounded constant. the conclusion shows that when the heavy tail exponent has a change point, it will affect the test of the persistence change point, so that the level of the empirical value under the null hypothesis is distorted and the false rejects, that is, the heavy tail index change point is judged as the persistence change point. When k is large, the statistic diverges to infinity, which will produce very large horizontal distortions. When k is small, the statistic also converges to bounded constants, but in the derivation it is found that the statistic has changed, so it will also lead to distortions in the level of experience, but the specific level of distortion will be given in the numerical simulation of the next section.

Theorem 2.2. When the sequence $\{\varepsilon_t\}$ satisfies assumption 1.1, under the alternative assumptions that H_1 is established, when $\lambda^*\le\tau^*$, have $\Xi(\tau)=O_p\left(T^{2+2\left(\frac{1}{\kappa_2}-\frac{1}{\kappa_1}\right)}\right)$; when $\lambda^*>\tau^*$, if $\kappa_1>\kappa_2$, have $\Xi(\tau)=O_p\left(T^{2+2\left(\frac{1}{\kappa_2}-\frac{1}{\kappa_1}\right)}\right)$, if $\kappa_1\le\kappa_2$, have $\Xi(\tau)=O_p(T^2)$.

Remark 2.2. It can be seen from the conclusion in theorem 2.2 that the position of the two kinds of change point and the change magnitude and direction of the heavy tail index all affect the divergence velocity of the statistics. On one hand, considering the influence of the direction of the heavy tail index, when the exponential change point λ^* is before the persistence change τ^*, that is $\lambda^* \leq \tau^*$, then the statistics are divergent at the rate of $T^{2+2(\frac{1}{\kappa_2}-\frac{1}{\kappa_1})}$ and the divergence velocity is $T^{1+\frac{2}{\kappa_1}}$ when there is no exponential change point. It can be found that as long as $\kappa_1 > \kappa_2$, and $\kappa_1, \kappa_2 \in [1,2]$, have $1+\frac{2}{\kappa_2}-\frac{2}{\kappa_1} \geq 0$, that is, the heavy tail index change point at this time will increase the rejection rate of the persistence change test, and reject the null hypothesis; but when $\kappa_1 \leq \kappa_2$, that $1+\frac{2}{\kappa_2}-\frac{2}{\kappa_1} < 0$, in this case, the power function of the test will be a certain loss. reconsider when the exponential change point λ^* is after persistent change point τ^*, that is when $\lambda^* > \tau^*$, if $\kappa_1 > \kappa_2$, there are the same conclusions as $\lambda^* \leq \tau^*$ and $\kappa_1 > \kappa_2$ but when $\kappa_1 \leq \kappa_2$, the divergence velocity of the statistics is T^2, then $1-\frac{2}{\kappa_1} \leq 0$, so it will weaken the power function. On the other hand, considering the influence of two kinds of change position on the test statistic, we can see that when $\kappa_1 \leq \kappa_2$ the divergence velocity of $\lambda^* > \tau^*$ is T^2 is bigger than the divergence velocity of $\lambda^* \leq \tau^*$ is $T^{2+2(\frac{1}{\kappa_2}-\frac{1}{\kappa_1})}$. when $\kappa_1 > \kappa_2$, the size of λ^* and τ^* is independent, and the statistics are divergent at the same velocity. A detailed proof of theorem 2.2 will be given in the appendix.

3. MONTE CARLO STUDY

In this section we use Monte Carlo study to verify the theory of the previous section, and through Matlab to achieve. First consider the data generation process as follows:

$$e_t = \begin{cases} \mu + \varepsilon_{t,1}, & t = 1, \cdots, \lambda^* T, \\ \mu + \varepsilon_{t,2}, & t = \lambda^* T + 1, \cdots, T. \end{cases}$$

Then

$$y_t = \begin{cases} e_t, & t = 1, \cdots, \tau^* T, \\ y_{t-1} + e_t, & t = \tau^* T + 1, \cdots, T. \end{cases}$$

We have $\tau^*=1$ under the null hypothesis. Without loss of generality, make $\mu=0$.

$\varepsilon_{t,1}$ and $\varepsilon_{t,2}$ be heavy tail sequences with heavy tail index κ_1, κ_2, respectively. Heavy tailed data is generated by Prof. Nolan's Stable software. In the numerical simulation process, the selected sample is T=200, 300, 500, and the significance level of the test is 0.05, the critical value of the test is shown in Table 1.

Table 1
Statistic $E(\tau)$ Critical Value

κ	MC($\Xi(\tau)$)	κ	MC($\Xi(\tau)$)
1.1	1559.79	1.6	66.48
1.2	550.30	1.7	43.44
1.3	318.51	1.8	35.78
1.4	192.36	1.9	23.24
1.5	95.20	2.0	17.23

Table 2
The Level of Experience Under the Heavy Tail Index Change Point (a)

$\kappa_1 \to \kappa_2$	T	λ			$\kappa_1 \to \kappa_2$	T	λ		
		0.3	0.5	0.7			0.3	0.5	0.7
	200	0.858	0.852	0.752		200	0.055	0.190	0.292
2.0→1.1	300	0.909	0.894	0.833	1.1→2.0	300	0.064	0.182	0.301
	500	0.945	0.931	0.891		500	0.064	0.189	0.310
	200	0.743	0.748	0.664		200	0.046	0.146	0.243
2.0→1.2	300	0.810	0.793	0.712	1.2→2.0	300	0.048	0.158	0.255
	500	0.865	0.873	0.795		500	0.052	0.142	0.254
	200	0.646	0.648	0.534		200	0.045	0.136	0.211
2.0→1.3	300	0.708	0.685	0.598	1.3→2.0	300	0.047	0.129	0.232
	500	0.763	0.755	0.674		500	0.036	0.126	0.204
	200	0.547	0.528	0.449		200	0.034	0.104	0.188
2.0→1.4	300	0.586	0.584	0.475	1.4→2.0	300	0.037	0.093	0.193
	500	0.642	0.626	0.536		500	0.029	0.109	0.177
	200	0.414	0.397	0.331		200	0.031	0.082	0.149
2.0→1.5	300	0.468	0.421	0.351	1.5→2.0	300	0.022	0.082	0.133
	500	0.486	0.488	0.408		500	0.023	0.075	0.138
	200	0.328	0.313	0.234		200	0.029	0.073	0.119
2.0→1.6	300	0.334	0.326	0.282	1.6→2.0	300	0.032	0.062	0.115
	500	0.380	0.351	0.321		500	0.024	0.057	0.122
	200	0.218	0.207	0.157		200	0.025	0.053	0.078
2.0→1.7	300	0.227	0.224	0.200	1.7→2.0	300	0.023	0.053	0.093
	500	0.251	0.255	0.202		500	0.031	0.039	0.084
	200	0.158	0.146	0.113		200	0.033	0.047	0.071
2.0→1.8	300	0.155	0.173	0.116	1.8→2.0	300	0.029	0.050	0.072
	500	0.186	0.175	0.140		500	0.026	0.044	0.070
	200	0.089	0.087	0.061		200	0.044	0.047	0.065

To be continued

Continued

$\kappa_1 \to \kappa_2$	T		λ		$\kappa_1 \to \kappa_2$	T		λ	
2.0→1.9	300	0.091	0.088	0.076	1.9→2.0	300	0.039	0.049	0.061
	500	0.094	0.092	0.081		500	0.036	0.042	0.050
	200	0.032	0.036	0.040		200	0.037	0.037	0.036
2.0→2.0	300	0.043	0.043	0.038	2.0→2.0	300	0.048	0.039	0.033
	500	0.047	0.056	0.049		500	0.041	0.046	0.043

Table 3
The Level of Experience Under the Heavy Tail Index Change Point (b)

$\kappa_1 \to \kappa_2$	T		λ		$\kappa_1 \to \kappa_2$	T		λ	
		0.3	0.5	0.7			0.3	0.5	0.7
	200	0.463	0.412	0.311		200	0.045	0.109	0.165
1.6→1.1	300	0.542	0.501	0.402	1.1→1.6	300	0.039	0.091	0.155
	500	0.542	0.492	0.427		500	0.040	0.096	0.165
	200	0.337	0.288	0.226		200	0.043	0.082	0.131
1.6→1.2	300	0.371	0.339	0.274	1.2→1.6	300	0.037	0.077	0.118
	500	0.373	0.328	0.260		500	0.041	0.068	0.115
	200	0.220	0.208	0.147		200	0.035	0.075	0.090
1.6→1.3	300	0.243	0.235	0.177	1.3→1.6	300	0.033	0.071	0.089
	500	0.242	0.236	0.196		500	0.036	0.065	0.102
	200	0.148	0.119	0.109		200	0.052	0.060	0.083
1.6→1.4	300	0.158	0.143	0.123	1.4→1.6	300	0.037	0.053	0.071
	500	0.177	0.142	0.123		500	0.039	0.056	0.072
	200	0.101	0.088	0.079		200	0.053	0.050	0.052
1.6→1.5	300	0.079	0.086	0.077	1.5→1.6	300	0.053	0.054	0.060
	500	0.082	0.092	0.082		500	0.049	0.056	0.060
	200	0.050	0.048	0.063		200	0.060	0.053	0.054
1.6→1.6	300	0.052	0.051	0.052	1.6→1.6	300	0.057	0.051	0.060
	500	0.055	0.052	0.053		500	0.052	0.052	0.050
	200	0.150	0.133	0.104		200	0.046	0.062	0.084
1.3→1.1	300	0.165	0.125	0.108	1.1→1.3	300	0.037	0.060	0.071
	500	0.160	0.157	0.110		500	0.040	0.047	0.073
	200	0.098	0.085	0.062		200	0.051	0.047	0.060
1.3→1.2	300	0.073	0.077	0.077	1.2→1.3	300	0.040	0.053	0.057
	500	0.084	0.090	0.064		500	0.038	0.053	0.054
	200	0.053	0.058	0.052		200	0.049	0.055	0.044
1.3→1.3	300	0.048	0.052	0.050	1.3→1.3	300	0.052	0.059	0.043
	500	0.049	0.046	0.052		500	0.057	0.043	0.048

Table 4
For the Empirical Power Function of $\kappa_1 > \kappa_2$ ($T = 500$)

$\kappa_1 \to \kappa_2$	τ		λ		$\kappa_1 \to \kappa_2$	τ		λ	
		0.3	0.5	0.7			0.3	0.5	0.7
	0.3	1.000	1.000	1.000		0.3	1.000	1.000	1.000
2.0→1.1	0.5	1.000	1.000	1.000	1.9→1.1	0.5	1.000	1.000	1.000
	0.7	1.000	1.000	1.000		0.7	1.000	1.000	1.000
	0.3	1.000	1.000	1.000		0.3	1.000	1.000	1.000
2.0→1.6	0.5	1.000	1.000	1.000	1.9→1.5	0.5	1.000	1.000	1.000
	0.7	1.000	1.000	1.000		0.7	0.999	0.999	1.000
	0.3	1.000	1.000	1.000		0.3	1.000	1.000	1.000
2.0→1.9	0.5	1.000	1.000	1.000	1.9→1.8	0.5	1.000	1.000	1.000
	0.7	1.000	1.000	1.000		0.7	0.998	1.000	1.000
	0.3	1.000	0.999	1.000		0.3	1.000	1.000	0.999
1.8→1.1	0.5	1.000	1.000	1.000	1.7→1.1	0.5	1.000	1.000	1.000
	0.7	1.000	1.000	1.000		0.7	0.999	1.000	1.000
	0.3	1.000	0.999	0.999		0.3	1.000	1.000	0.999
1.8→1.4	0.5	0.999	1.000	1.000	1.7→1.3	0.5	0.999	1.000	0.998
	0.7	0.996	0.999	0.999		0.7	0.999	0.999	0.998
	0.3	0.998	0.999	0.997		0.3	0.999	0.998	0.998

To be continued

Continued

$\kappa_1 \to \kappa_2$	τ	λ			$\kappa_1 \to \kappa_2$	τ	λ		
1.8→1.7	0.5	1.000	0.999	0.998	1.7→1.6	0.5	0.998	0.998	0.997
	0.7	0.998	0.996	0.997		0.7	0.993	0.992	0.993
	0.3	1.000	0.998	0.999		0.3	0.999	0.997	0.994
1.6→1.1	0.5	1.000	1.000	0.998	1.5→1.1	0.5	0.996	0.999	0.995
	0.7	0.997	0.998	1.000		0.7	0.992	0.997	0.992
	0.3	0.997	0.998	0.996		0.3	0.994	0.995	0.989
1.6→1.3	0.5	1.000	0.997	0.997	1.5→1.3	0.5	0.994	0.994	0.993
	0.7	0.993	0.994	0.994		0.7	0.985	0.987	0.982
	0.3	0.997	0.995	0.993		0.3	0.991	0.990	0.991
1.6→1.5	0.5	0.994	0.993	0.995	1.5→1.4	0.5	0.988	0.990	0.980
	0.7	0.988	0.986	0.991		0.7	0.973	0.978	0.968
	0.3	0.991	0.986	0.983		0.3	0.981	0.985	0.979
1.4→1.1	0.5	0.989	0.991	0.984	1.4→1.3	0.5	0.982	0.985	0.961
	0.7	0.972	0.967	0.975		0.7	0.945	0.934	0.952

Table 5
For the Empirical Power Function of $\kappa_1 \le \kappa_2$ ($T = 500$)

$\kappa_1 \to \kappa_2$	τ	λ			$\kappa_1 \to \kappa_2$	τ	λ		
		0.3	0.5	0.7			0.3	0.5	0.7
	0.3	0.878	0.947	0.976		0.3	0.995	0.999	0.996
1.1→2.0	0.5	0.866	0.867	0.969	1.6→2.0	0.5	0.995	0.994	0.995
	0.7	0.804	0.808	0.821		0.7	0.987	0.988	0.990
	0.3	0.931	0.973	0.984		0.3	0.996	0.999	0.997
1.2→2.0	0.5	0.926	0.930	0.977	1.7→2.0	0.5	0.997	0.999	0.999
	0.7	0.885	0.873	0.885		0.7	0.996	0.993	0.996
	0.3	0.965	0.977	0.991		0.3	0.998	0.999	1.000
1.3→2.0	0.5	0.956	0.961	0.983	1.8→2.0	0.5	0.997	0.999	1.000
	0.7	0.938	0.925	0.937		0.7	0.999	0.999	0.997
	0.3	0.975	0.987	0.994		0.3	1.000	1.000	1.000
1.4→2.0	0.5	0.975	0.971	0.992	1.9→2.0	0.5	1.000	0.999	1.000
	0.7	0.967	0.960	0.959		0.7	0.997	1.000	0.999
	0.3	0.992	0.993	0.995		0.3	1.000	1.000	1.000
1.5→2.0	0.5	0.994	0.990	0.995	2.0→2.0	0.5	1.000	1.000	1.000
	0.7	0.979	0.981	0.980		0.7	1.000	1.000	1.000
	0.3	0.855	0.935	0.968		0.3	0.989	0.988	0.992
1.1→1.8	0.5	0.849	0.854	0.954	1.5→1.8	0.5	0.988	0.984	0.991
	0.7	0.784	0.773	0.784		0.7	0.972	0.976	0.979
	0.3	0.917	0.956	0.972		0.3	0.990	0.995	0.997
1.2→1.8	0.5	0.910	0.908	0.966	1.6→1.8	0.5	0.995	0.994	0.995
	0.7	0.865	0.856	0.869		0.7	0.990	0.980	0.981
	0.3	0.954	0.975	0.986		0.3	0.999	0.997	0.997
1.3→1.8	0.5	0.955	0.956	0.972	1.7→1.8	0.5	0.996	0.998	0.997
	0.7	0.916	0.912	0.905		0.7	0.990	0.992	0.993
	0.3	0.972	0.984	0.992		0.3	0.999	0.999	0.997
1.4→1.8	0.5	0.969	0.977	0.989	1.8→1.8	0.5	0.999	0.997	1.000
	0.7	0.942	0.944	0.942		0.7	0.998	0.995	0.992
	0.3	0.868	0.929	0.966		0.3	0.977	0.981	0.986
1.1→1.6	0.5	0.869	0.864	0.937	1.4→1.6	0.5	0.979	0.975	0.982
	0.7	0.775	0.772	0.766		0.7	0.952	0.951	0.942
	0.3	0.922	0.948	0.974		0.3	0.987	0.992	0.990
1.2→1.6	0.5	0.911	0.911	0.958	1.5→1.6	0.5	0.987	0.985	0.987
	0.7	0.872	0.850	0.872		0.7	0.970	0.972	0.963
	0.3	0.951	0.973	0.979		0.3	0.995	0.997	0.993
1.3→1.6	0.5	0.955	0.948	0.967	1.6→1.6	0.5	0.993	0.991	0.994
	0.7	0.908	0.912	0.895		0.7	0.985	0.985	0.988
	0.3	0.851	0.899	0.933		0.3	0.950	0.958	0.966
1.1→1.4	0.5	0.833	0.851	0.896	1.3→1.4	0.5	0.943	0.939	0.957
	0.7	0.761	0.748	0.737		0.7	0.896	0.878	0.889
	0.3	0.918	0.932	0.949		0.3	0.972	0.974	0.979
1.2→1.4	0.5	0.907	0.911	0.940	1.4→1.4	0.5	0.972	0.968	0.965
	0.7	0.844	0.821	0.826		0.7	0.922	0.925	0.934

Table 2 shows that the heavy tail index changes from κ_1=2.0 to the smaller heavy tail index κ_2, or the small tail index κ_1 changes to κ_2=2.0 in the absence of a persistent change point in the null hypothesis. It can be seen from the data in the table that: a) When $\kappa_1>\kappa_2$, that is, on the left side of Table 2, the empirical level is already far greater than the significance level, ie, the level of experience is seriously distorted, Mismatched as a permanent variable point, and the level of distortion increases with the sample increases, such as the heavy tail index from 2.0 to 1.1, λ take 0.5, then T=200, 300, 500 experience level values were 0.852, 0.894, 0.931.

In addition, the severity of the horizontal twist is also affected by the magnitude of the heavy-tail index. The greater the change of the heavy tail index, the more severe the horizontal twist. If T=300, λ=0.5, κ_1 changes from 2.0 to 1.1, 1.5 and 1.9 The value of the experience level is 0.894, 0.421, 0.088 respectively. Finally, the position of the heavy tail index change point also affects the size of the experience level, the more the position of the change point, the smaller the experience level, such as the heavy tail index from 2.0 to 1.1, and T=300, λ=0.3, 0.5, 0.7 the empirical level is 0.858, 0.852, 0.752 respectively; b) When $\kappa_1\leqslant\kappa_2$, from the theorem 2.1, we can see that when $\kappa_1\leqslant\kappa_2$, The statistic of convergence is constant, the right side of Table 3.3 can be seen, there are still horizontal distortion, but compared to $\kappa_1>\kappa_2$, the level of distortion has been much smaller, and the level of distortion has not been affected by the sample size. The same is true for $\kappa_1>\kappa_2$, the greater the magnitude of the heavy tail index, the more severe the horizontal twist, but the effect of the variable position on the empirical level is opposite to that of $\kappa_1>\kappa_2$. The higher the level, such as κ_1=1.1, κ_2=2.0, T=300, λ=0.3, 0.5, 0.7, the empirical level of 0.064, 0.182, 0.301, respectively.

The level of experience given in Table 2 is the change between the Gaussian sequence and the heavy tail sequence of the different indices. To make the conclusion more general, the changes between the different heavy tail indices are given in Table 3. Comparing the data in Table 2, the data in Table 3 have the same conclusion.

Table 4 shows the empirical power function values for numerical simulations when $\kappa_1>\kappa_2$ under the alternative assumptions. It can be concluded that:

a) First, from the whole, in the case of $\kappa_1>\kappa_2$, the value of the power function is mostly close to 1, which is a reinforcement of the power, but found that when the heavy tail index from 1.4 to 1.3, the power function value only 0.945, which is due to the heavy tail index itself in the smaller when the power is low, so this does not affect the overall law;

b) The value of the power function decreases with the decrease of the change magnitude. As the statistic given in theorem 2.2 is divergent at the rate of $T^{2+2(\frac{1}{\kappa_2}-\frac{1}{\kappa_1})}$ when $\kappa_1>\kappa_2$, so the greater the $\kappa_1-\kappa_2$, the faster the divergence rate, the higher the rejection rate will be. For example, let $\tau=\lambda$=0.7, then the heavy tail index from 1.6 to 1.1, 1.3, 1.5 the rejection rate was 0.997, 0.993, 0.988, respectively.

In Table 5, given the empirical power value of $\kappa_1\leq\kappa_2$ under the alternative hypothesis. Since the data is too big, just to give the numerical results for the sample at 500. From the results in the table can be drawn the following rules:

First consider the relationship between the position of the change point and the power function.

a) The larger the value of the heavy tail index change point λ, change point position is more backward. The greater the value of the empirical power function. For example, κ_1 to κ_2 are 1.1 to 2.0 and τ=0.3, then the power function values of λ=0.3, 0.5, 0.7 are 0.878, 0.947 and 0.976 respectively, which is consistent with the conclusion of $\kappa_1\leq\kappa_2$ in Table 1.

b) The smaller of the persistence change point τ, change point position is more forward. the greater the value of the empirical power function. For example κ_1 to κ_2 is 1.1 to 2.0, λ=0.3, the power function values of τ=0.3, 0.5, 0.7 are 0.878, 0.866 and 0.804 respectively.

c) The value of the power function at $\lambda\geqslant\tau$ is greater than the power function value at $\lambda<\tau$. For example, when κ_1 to κ_2 is taken from 1.1 to 2.0, the empirical power function of τ=0.3, 0.5 is 0.947 and 0.867, and the power function value at τ=0.7 is 0.808, which is obviously less than 0.3 and 0.5 The rejection rate. This is consistent with the conclusion drawn from Theorem 2.2.

Then, consider the influence of the heavy tail index on the power function value.

a) The smaller of the heavy tail index κ is, the smaller the empirical power function is. If we take τ=0.7, λ=0.3, κ is 2.0 to 2.0, 1.8 to 1.8, 1.6 to 1.6, 1.4 to 1.4, respectively, the rejection rate is 1.00, 0.998, 0.985, 0.922 respectively.

b) The value of the power function increases with the decrease of the magnitude of the heavy tail index, that is, the larger the change magnitude, the lower the rejection rate. For example, let $\tau=\lambda$=0.3, the heavy tail index from 1.1 to 2.0, 1.5 to 2.0, 1.9 to 2.0, respectively, the magnitude of 0.9, 0.4, 0.1 respectively. Their corresponding power function values were 0.878, 0.992, 1.000 respectively. This is from theorem 2.2 we conclude that $\kappa_1\leq\kappa_2$, when statistic $E(\tau)$ divergence rate is $T^{2+2(\frac{1}{\kappa_2}-\frac{1}{\kappa_1})}$of the decision, by numerical simulation. Also verify the correctness of the theorem.

c) Finally, consider the effect of sample size on power function values. As with the data in other tables, on the alternative assumptions the same rejection rate increases as the sample increases, but given that the amount of data is too large, this paper does not give the alternative hypothesis that the sample is an empirical power value of 200, 300. The above numerical simulation can be a good description of the correctness of the theoretical method in this chapter, and in the null hypothesis and alternative hypothesis, the detection of change points have a good effect.

CONCLUSION

In this paper, change points are consider in the heavy tail indexes and the persistence change. We obtained conclusions as follows: under the null hypothesis that the circumstance of the series only existed an index change point, if the heavy tail index κ change from large to small, the statistics is diverging at a speed of $T^{2/\kappa_2-2/\kappa_1}$, and the larger of the $\kappa_2-\kappa_1$ is, the faster the divergence is. If the index changes from small to large, the statistics converges to the bounded constant. But the numerical simulation shows that no matter how the change of κ will lead to the size distortions, and the size distortions shows more serious when $\kappa_1>\kappa_2$. Under the alternative hypothesis, the series have persistence change point and index change point, in the case of $\kappa_1>\kappa_2$, the power is increased which makes it easier to reject the null hypothesis. But the power will loss when $\kappa_1\le\kappa_2$.

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APPENDIX

Proof of Theorem 2.1. Under the condition of the null hypothesis H_0 is established, First, discussing the $\kappa_1>\kappa_2$.

The first case is when $\lambda^*>\tau$, for the denominator

$$a_{T,1}^{-1}\sum_{i=1}^{t}\hat{z}_{0,i}\xrightarrow{d}U_{1,1}(t),$$

$$[\tau T]^{-2}\sum_{t=1}^{[\tau T]}\left(\sum_{i=1}^{t}\hat{z}_{0,i}\right)^2=O_p\left(\frac{a_{T,1}^2}{[\tau T]}\right).$$

Where $a_{T,1} = O_p(T^{1/\kappa_1})$, $U_{1,1}$ is a stable Levy process of the κ_1.

For the numerator, let $t=[rT]$, $r\in[0,1]$, when $\tau<r<\lambda^*$ we have

$$\sum_{i=[\tau T]+1}^{[rT]} \hat{z}_{1,i} = \sum_{i=[\tau T]+1}^{[rT]} \left(y_i - \frac{1}{T-[\tau T]} \sum_{j=[\tau T]+1}^{T} y_j \right)$$

$$= \sum_{i=[\tau T]+1}^{[rT]} (\mu + \varepsilon_{i,1}) - \sum_{i=[\tau T]+1}^{[rT]} \left\{ \frac{1}{T-[\tau T]} \left[\sum_{j=[\tau T]+1}^{[\lambda^* T]} (\mu + \varepsilon_{j,1}) + \sum_{j=[\lambda^* T]+1}^{T} (\mu + \varepsilon_{j,2}) \right.\right.$$

$$= \sum_{i=[\tau T]+1}^{[rT]} \varepsilon_{i,1} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau T]+1}^{[\lambda^* T]} \varepsilon_{i,1} - \frac{r-\tau}{1-\tau} \sum_{i=[\lambda^* T]+1}^{T} \varepsilon_{i,2}.$$

Then, by Lemma 1.1 we have

$$a_{T,2}^{-1} \sum_{i=[\tau T]+1}^{[rT]} \hat{z}_{1,i} \xrightarrow{d} o_p(1) - \frac{r-\tau}{1-\tau} U_{1,2}(1-\lambda^*). \qquad \text{(A.1)}$$

Where $a_{T,2} = O_p(T^{1/\kappa_2})$, $U_{1,2}$ is a stable Levy process of the κ_2.

Available from type A.1

$$\sum_{i=[\tau T]+1}^{t} \hat{z}_{1,i} = O_p(a_{T,2}) = O_p\left(T^{\frac{1}{\kappa_2}}\right). \qquad \text{(A.2)}$$

when $\lambda^*<r<1$ we have

$$\sum_{i=[\tau T]+1}^{[rT]} \hat{z}_{1,i} = \sum_{i=[\tau T]+1}^{[\lambda^* T]} (\mu + \varepsilon_{i,1}) + \sum_{i=[\lambda^* T]+1}^{[rT]} (\mu + \varepsilon_{i,2}) - \sum_{i=[\tau T]+1}^{[rT]} \left\{ \frac{1}{T-[\tau T]} \left[\sum_{j=[\tau T]+1}^{[\lambda^* T]} (\mu + \varepsilon_{j,1}) + \sum_{j=[\lambda^* T]+1}^{T} (\mu + \varepsilon_{j,2}) \right] \right\}$$

$$= \sum_{i=[\tau T]+1}^{[\lambda^* T]} \varepsilon_{i,1} + \sum_{i=[\lambda^* T]+1}^{[rT]} \varepsilon_{i,2} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau T]+1}^{[\lambda^* T]} \varepsilon_{i,1} - \frac{r-\tau}{1-\tau} \sum_{i=[\lambda^* T]+1}^{T} \varepsilon_{i,2}.$$

Then

$$a_{T,2}^{-1} \sum_{i=[\tau T]+1}^{[rT]} \hat{z}_{1,i} \xrightarrow{d} o_p(1) + U_{1,2}(r-\lambda^*) - \frac{r-\tau}{1-\tau} U_{1,2}(1-\lambda^*). \qquad \text{(A.3)}$$

Available from type A.3

$$\sum_{i=[\tau T]+1}^{t} \hat{z}_{1,i} = O_p(a_{T,2}) = O_p\left(T^{\frac{1}{\kappa_2}}\right). \qquad \text{(A.4)}$$

Comprehensive formula A.2 and formula A.4 the numerator can be simplify as follow,

$$(T-[\tau T])^{-2} \sum_{t=[\tau T]+1}^{T} \left(\sum_{i=[\tau T]+1}^{t} \hat{z}_{1,i} \right)^2 = O_p\left(\frac{a_{T,2}^2}{T-[\tau T]} \right).$$

Thus

$$\Xi(\tau) = O_p\left(\frac{a_{T,2}^2}{a_{T,1}^2}\right) = O_p\left(T^{\frac{1}{\kappa_2}-\frac{1}{\kappa_1}}\right). \quad (A.5)$$

The second case is when $\lambda^* < \tau$, the numerator can be simplify as follow:

$$a_{T,2}^{-1}\sum\nolimits_{i=[\tau T]+1}^{t}\hat{z}_{1,i} \xrightarrow{d} U_{1,2}(t-\tau).$$

$$(T-[\tau T])^{-2}\sum_{t=[\tau T]+1}^{T}\left(\sum\nolimits_{i=[\tau T]+1}^{t}\hat{z}_{1,i}\right)^2 = O_p\left(\frac{a_{T,2}^2}{T-[\tau T]}\right).$$

For denominator, in the case of $0<r<\lambda^*$,

$$\sum_{i=1}^{[rT]}\hat{z}_{0,i} = \sum_{i=1}^{[rT]}\left(y_i - \frac{1}{[\tau T]}\sum_{j=1}^{[\tau T]}y_j\right)$$

$$= \sum_{i=1}^{[rT]}(\mu+\varepsilon_{i,1}) - \sum_{i=1}^{[rT]}\left[\frac{1}{[\tau T]}\left([\tau T]\mu + \sum_{i=1}^{[\lambda^* T]}\varepsilon_{i,1} + \sum_{i=[\lambda^* T]+1}^{[\tau T]}\varepsilon_{i,2}\right)\right]$$
$$= \sum_{i=1}^{[rT]}\varepsilon_{i,1} - \frac{r}{\tau}\sum_{i=1}^{[\lambda^* T]}\varepsilon_{i,1} - \frac{r}{\tau}\sum_{i=[\lambda^* T]+1}^{[\tau T]}\varepsilon_{i,2}.$$

Then

$$a_{T,2}^{-1}\sum_{i=1}^{[rT]}\hat{z}_{0,i} \xrightarrow{d} o_p(1) - \frac{r}{\tau}U_{1,2}(\tau-\lambda^*). \quad (A.6)$$

Available from type A.6

$$[\tau T]^{-2}\sum_{t=1}^{[\tau T]}\left(\sum\nolimits_{i=1}^{t}\hat{z}_{0,i}\right)^2 = O_p\left(\frac{a_{T,1}^2}{[\tau T]}\right).$$

In the case of $\lambda^* < r < \tau$,

$$\sum_{i=1}^{[rT]}\hat{z}_{0,i} = \sum_{i=1}^{[\lambda^* T]}(\mu+\varepsilon_{i,1}) + \sum_{i=[\lambda^* T]+1}^{[rT]}(\mu+\varepsilon_{i,2})$$
$$-\sum_{i=1}^{[rT]}\left[\frac{1}{[\tau T]}\left([\tau T]\mu + \sum_{i=1}^{[\lambda^* T]}\varepsilon_{i,1} + \sum_{i=[\lambda^* T]+1}^{[\tau T]}\varepsilon_{i,2}\right)\right]$$
$$= \sum_{i=1}^{[\lambda^* T]}\varepsilon_{i,1} + \sum_{i=[\lambda^* T]+1}^{[rT]}\varepsilon_{i,2} - \frac{r}{\tau}\sum_{i=1}^{[\lambda^* T]}\varepsilon_{i,1} - \frac{r}{\tau}\sum_{i=[\lambda^* T]+1}^{[\tau T]}\varepsilon_{i,2}.$$

Then

$$a_{T,2}^{-1}\sum_{i=1}^{[rT]}\hat{z}_{0,i} \xrightarrow{d} o_p(1) + U_{1,2}(r-\lambda^*) - \frac{r}{\tau}U_{1,2}(\tau-\lambda^*). \quad (A.7)$$

Available from type A.7

$$[\tau T]^{-2}\sum_{t=1}^{[\tau T]}\left(\sum\nolimits_{i=1}^{t}\hat{z}_{0,i}\right)^2 = O_p\left(\frac{a_{T,1}^2}{[\tau T]}\right).$$

Then

$$\Xi(\tau) = O_p\left(\frac{a_{T,2}^2}{a_{T,1}^2}\right) = O_p\left(T^{\frac{1}{\kappa_2}-\frac{1}{\kappa_1}}\right). \quad (A.8)$$

Comprehensive formula A.2 and formula A.4, we can get when $\kappa_1 > \kappa_2$, then

$$\Xi(\tau) = O_p\left(T^{\frac{1}{\kappa_2}-\frac{1}{\kappa_1}}\right). \Xi(\tau) = O_p\left(T^{\frac{1}{\kappa_2}-\frac{1}{\kappa_1}}\right).$$

Similarly, when $\kappa_1 \leq \kappa_2$, if $\lambda^* > \tau$, the denominations is

$$[\tau T]^{-2}\sum_{t=1}^{[\tau T]}\left(\sum\nolimits_{i=1}^{t}\hat{z}_{0,i}\right)^2 = O_p\left(\frac{a_{T,1}^2}{[\tau T]}\right). \quad \text{(A.9)}$$

While the numerator is similar to the distribution of $\kappa_1 > \kappa_2$

$$\sum\nolimits_{i=[\tau T]+1}^{t}\hat{z}_{1,i} = O_p(a_{T,1}).$$

Then

$$(T-[\tau T])^{-2}\sum_{t=[\tau T]+1}^{T}\left(\sum\nolimits_{i=[\tau T]+1}^{t}\hat{z}_{1,i}\right)^2 = O_p\left(\frac{a_{T,1}^2}{T-[\tau T]}\right). \quad \text{(A.10)}$$

Comprehensive formula A.9 and formula A.10

$$\Xi(\tau) = O_p(1). \quad \text{(A.11)}$$

If $\lambda^* \leq \tau$, the numerator is

$$(T-[\tau T])^{-2}\sum_{t=[\tau T]+1}^{T}\left(\sum\nolimits_{i=[\tau T]+1}^{t}\hat{z}_{1,i}\right)^2 = O_p\left(\frac{a_{T,2}^2}{T-[\tau T]}\right). \quad \text{(A.12)}$$

While the denominator is similar to the distribution of $\kappa_1 > \kappa_2$

$$[\tau T]^{-2}\sum_{t=1}^{[\tau T]}\left(\sum\nolimits_{i=1}^{t}\hat{z}_{0,i}\right)^2 = O_p\left(\frac{a_{T,1}^2}{[\tau T]}\right). \quad \text{(A.13)}$$

Comprehensive formula A.12 and formula A.13

$$\Xi(\tau) = O_p\left(T^{\frac{1}{\kappa_2}-\frac{1}{\kappa_1}}\right). \quad \text{(A.14)}$$

This proves the theorem 2.1.

Proof of Theorem 2.2.

First of all, the simplification of the statistics, the following points to discuss, first consider the $\lambda^* \leq \tau^*$, can be divided into three cases.

For the first case: When $\tau \leq \lambda^* \leq \tau^*$, then the denominator can be simplified

$$[\tau T]^{-2}\sum_{t=1}^{[\tau T]}\left(\sum\nolimits_{i=1}^{t}\hat{z}_{0,i}\right)^2 = O_p\left(\frac{a_{T,1}^2}{[\tau T]}\right).$$

While for numerator

$$\begin{aligned}\hat{z}_{1,i} &= y_i - \frac{1}{T-[\tau T]}\sum_{i=[\tau T]+1}^{T} y_i \\ &= y_i - \frac{1}{T-[\tau T]}\left(\sum_{i=[\tau T]+1}^{[\lambda^* T]}(\mu+\varepsilon_{i,1}) + \sum_{i=[\lambda^* T]+1}^{[\tau^* T]}(\mu+\varepsilon_{i,2}) \right.\\ &\qquad\left. + \sum_{i=[\tau^* T]+1}^{T}(\mu + z_{i-1} + \varepsilon_{i,2})\right) \\ &= y_i - \frac{1}{T-[\tau T]}\left\{(T-[\tau T])\mu + \sum_{i=[\tau T]+1}^{[\lambda^* T]}\varepsilon_{i,1} + \sum_{i=[\lambda^* T]+1}^{T}\varepsilon_{i,2} + \sum_{i=[\tau^* T]+1}^{T} z_{i-1}\right\}.\end{aligned}$$

Let $t=[rT]$, $r \in (0,1)$. If $\tau < r < \lambda^*$, then

$$\begin{aligned}\sum_{i=[\tau T]+1}^{[rT]} \hat{z}_{1,i} &= \sum_{i=[\tau T]+1}^{[rT]} \left(\mu + \varepsilon_{i,1}\right) \\ &\quad - \sum_{i=[\tau T]+1}^{[rT]} \frac{1}{T-[\tau T]} \sum_{i=[\tau T]+1}^{[\lambda^* T]} \varepsilon_{i,1} \\ &\quad - \sum_{i=[\tau T]+1}^{[rT]} \frac{1}{T-[\tau T]} \sum_{i=[\lambda^* T]+1}^{T} \varepsilon_{i,2} - \sum_{i=[\tau T]+1}^{[rT]} \frac{1}{T-[\tau T]} \sum_{i=[\tau^* T]+1}^{T} z_{i-1} \\ &= \sum_{i=[\tau T]+1}^{[rT]} \varepsilon_{i,1} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau T]+1}^{[\lambda^* T]} \varepsilon_{i,1} - \frac{r-\tau}{1-\tau} \sum_{i=[\lambda^* T]+1}^{T} \varepsilon_{i,2} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau^* T]+1}^{T} z_{i-1} \\ &= O_p\left(a_{T,1} - \frac{r-\tau}{1-\tau} a_{T,1} - \frac{r-\tau}{1-\tau} a_{T,2} - \frac{r-\tau}{1-\tau} T a_{T,2}\right) \\ &= O_p\left(\frac{r-\tau}{1-\tau} T a_{T,2}\right).\end{aligned}$$

If $\lambda^* < r < \tau^*$, then

$$\begin{aligned}\sum_{i=[\tau T]+1}^{[rT]} \hat{z}_{1,i} &= \sum_{i=[\tau T]+1}^{[\lambda^* T]} \varepsilon_{i,1} + \sum_{i=[\lambda^* T]+1}^{[rT]} \varepsilon_{i,2} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau T]+1}^{[\lambda^* T]} \varepsilon_{i,1} - \frac{r-\tau}{1-\tau} \sum_{i=[\lambda^* T]+1}^{T} \varepsilon_{i,2} \\ &\quad - \frac{r-\tau}{1-\tau} \sum_{i=[\tau^* T]+1}^{T} z_{i-1} \\ &= O_p\left(\frac{r-\tau}{1-\tau} T a_{T,2}\right).\end{aligned}$$

If $\tau^* < r < 1$, then

$$\begin{aligned}\sum_{i=[\tau T]+1}^{[rT]} \hat{z}_{1,i} &= \sum_{i=[\tau T]+1}^{[\lambda^* T]} \varepsilon_{i,1} + \sum_{i=[\lambda^* T]+1}^{[rT]} \varepsilon_{i,2} + \sum_{i=[\tau^* T]+1}^{[rT]} z_{i-1} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau T]+1}^{[\lambda^* T]} \varepsilon_{i,1} \\ &\quad - \frac{r-\tau}{1-\tau} \sum_{i=[\lambda^* T]+1}^{T} \varepsilon_{i,2} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau^* T]+1}^{T} z_{i-1} \\ &= O_p\left(\frac{1-r}{1-\tau} T a_{T,2}\right).\end{aligned}$$

Then there is when $\tau \le \lambda^* \le \tau^*$

$$(T-[\tau T])^{-2} \sum_{t=[\tau T]+1}^{T} \left(\sum\nolimits_{i=[\tau T]+1}^{t} \hat{z}_{1,i}\right)^2 = O_p\left(T a_{T,2}^2\right).$$

So there is

$$\Xi(\tau) = O_p\left(T^2 \frac{a_{T,2}^2}{a_{T,1}^2}\right) = O_p\left(T^{2+2\left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right)}\right).$$

For the second case: when $\lambda^* \le \tau \le \tau^*$, then the denominator can be simplified

$$\begin{aligned}&[\tau T]^{-2} \sum_{t=1}^{[\tau T]} \left(\sum\nolimits_{i=1}^{t} \hat{z}_{0,i}\right)^2 \\ &\quad = [\tau T]^{-2} \left\{ \sum_{t=1}^{[\lambda^* T]} \left(\sum\nolimits_{i=1}^{t} \hat{z}_{0,i}\right)^2 + \sum_{t=[\lambda^* T]+1}^{[\tau T]} \left(\sum\nolimits_{i=[\lambda^* T]+1}^{t} \hat{z}_{0,i}\right)^2 \right\} \\ &\quad = O_p\left(\frac{a_{T,1}^2}{[\tau T]}\right) + O_p\left(\frac{a_{T,2}^2}{[\tau T]}\right).\end{aligned}$$

If $\tau < r < \tau^*$, then the numerator

$$\sum_{i=[\tau T]+1}^{[rT]} \hat{z}_{1,i} = \sum_{i=[\tau T]+1}^{[rT]} \varepsilon_{i,2} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau T]+1}^{T} \varepsilon_{i,2} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau^* T]+1}^{T} z_{i-1}$$
$$= O_p\left(\frac{r-\tau}{1-\tau} T a_{T,2}\right).$$

If $\tau^* < r < 1$, then the numerator

$$\sum_{i=[\tau T]+1}^{[rT]} \hat{z}_{1,i} = \sum_{i=[\tau T]+1}^{[\tau^* T]} \varepsilon_{i,2} + \sum_{i=[\tau^* T]+1}^{[rT]} z_{i-1} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau T]+1}^{T} \varepsilon_{i,2} - \frac{r-\tau}{1-\tau} \sum_{i=[\tau^* T]+1}^{T} z_{i-1}$$
$$= O_p\left(\frac{1-r}{1-\tau} T a_{T,2}\right).$$

Then we have

$$(T-[\tau T])^{-2} \sum_{t=[\tau T]+1}^{T} \left(\sum\nolimits_{i=[\tau T]+1}^{t} \hat{z}_{1,i}\right)^2 = O_p\left(T a_{T,2}^2\right).$$

Thus, when $\lambda^* \leq \tau \leq \tau^*$

$$\Xi(\tau) = O_p\left(T^2 \frac{a_{T,2}^2}{a_T^2}\right).$$

Where $a_T = \max\{a_{T,1}, a_{T,2}\}$. When $\kappa_1 > \kappa_2$, that $\Xi(\tau)=O_p(T^2)$; when $\kappa_1 \leq \kappa_2$, then $\Xi(\tau) = O_p\left(T^{2+2\left(\frac{1}{\kappa_2}-\frac{1}{\kappa_1}\right)}\right)$.

For the third case: when $\lambda^* \leq \tau^* \leq \tau$, Then the numerator can be simplified

$$(T-[\tau T])^{-2} \sum_{t=[\tau T]+1}^{T} \left(\sum\nolimits_{i=[\tau T]+1}^{t} \hat{z}_{1,i}\right)^2 = O_p\left(T a_{T,2}^2\right).$$

And denominator, if $0 < r < \lambda^*$

$$\sum_{i=1}^{[rT]} \hat{z}_{0,i} = \sum_{i=1}^{[rT]} \varepsilon_{i,1} - \frac{r}{\tau} \sum_{i=1}^{[\lambda^* T]} \varepsilon_{i,1} - \frac{r}{\tau} \sum_{i=[\lambda^* T]+1}^{[\tau T]} \varepsilon_{i,2} - \frac{r}{\tau} \sum_{i=[\tau^* T]+1}^{[\tau T]} z_{i-1}$$
$$= O_p\left(\frac{r}{\tau} T a_{T,2}\right).$$

If $\lambda^* < r < \tau^*$

$$\sum_{i=1}^{[rT]} \hat{z}_{0,i} = \sum_{i=1}^{[\lambda^* T]} \varepsilon_{i,1} + \sum_{i=[\lambda^* T]+1}^{[rT]} \varepsilon_{i,2} - \frac{r}{\tau} \sum_{i=1}^{[\lambda^* T]} \varepsilon_{i,1} - \frac{r}{\tau} \sum_{i=[\lambda^* T]+1}^{[\tau T]} \varepsilon_{i,2} - \frac{r}{\tau} \sum_{i=[\tau^* T]+1}^{[\tau T]} z_{i-1}$$
$$= O_p\left(\frac{r}{\tau} T a_{T,2}\right).$$

If $\tau^* < r < \tau$,

$$\sum_{i=1}^{[rT]} \hat{z}_{0,i} = \sum_{i=1}^{[\lambda^* T]} \varepsilon_{i,1} + \sum_{i=[\lambda^* T]+1}^{[rT]} \varepsilon_{i,2} + \sum_{i=[\tau^* T]+1}^{[rT]} z_{i-1} - \frac{r}{\tau} \sum_{i=1}^{[\lambda^* T]} \varepsilon_{i,1}$$
$$- \frac{r}{\tau} \sum_{i=[\lambda^* T]+1}^{[\tau T]} \varepsilon_{i,2} - \frac{r}{\tau} \sum_{i=[\tau^* T]+1}^{[\tau T]} z_{i-1}$$
$$= O_p\left(\frac{\tau-r}{\tau} T a_{T,2}\right).$$

So have

$$[\tau T]^{-2} \sum_{t=1}^{[\tau T]} \left(\sum\nolimits_{i=1}^{t} \hat{z}_{0,i}\right)^2 = O_p\left(T a_{T,2}^2\right).$$

Thus, when $\lambda^* \le \tau^* \le \tau$, $\Xi(\tau)=O_p(1)$.

Based on the above three cases, when $\lambda^* \le \tau^*$, $\kappa_1>\kappa_2$, have $\Xi(\tau) = O_p\left(T^{2+2(\frac{1}{\kappa_2}-\frac{1}{\kappa_1})}\right)$, when $\kappa_1 \le \kappa_2$, have $\Xi(\tau) = O_p\left(T^{2+2(\frac{1}{\kappa_2}-\frac{1}{\kappa_1})}\right)$.

Similarly, when $\lambda^* > \tau^*$, we can get:

If $\tau<\tau^* < \lambda^*$, then $\Xi(\tau) = O_p\left(T^2 \frac{a_T^2}{a_{T,1}^2}\right)$; if $\tau^* < \tau < \lambda^*$, then $\Xi(\tau) = O_p\left(\frac{a_T^2}{a_{T,1}^2}\right)$; if $\tau^* < \lambda^* < \tau$, then $\Xi(\tau) = O_p\left(\frac{a_{T,2}^2}{a_T^2}\right)$.

Comprehensive these three cases, available

When $\kappa_1 > \kappa_2$, $\Xi(\tau) = O_p\left(T^{2+2\left(\frac{1}{\kappa_2}-\frac{1}{\kappa_1}\right)}\right)$; when $\kappa_1 \le \kappa_2$, then $\Xi(\tau)=O_p(T^2)$.

The proof of theorem 2.1 is then completed.

