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A Resolvent Algorithm for System of General Mixed Variational Inequalities

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Abstract

In this paper, we suggest and analyze a new resolvent algorithm for finding the common solutions for a generalized system of relaxed cocoercive mixed variational inequality problems and fixed point of a nonexpansive mapping in Hilbert spaces. We also prove the convergence analysis of the proposed algorithm under some suitable mild conditions. In this respect, our results present a refinement and improvement of the previously known results.

Key words: Generalized system mixed variational inequality problem; Fixed point problem; Relaxed cocoercive mapping; Resolvent operator.

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INTRODUCTION

Variational inequality has become a rich of inspiration in pure and applied mathematics. In recent years, classical variational inequality problems have been extended and generalized to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences. The projection and contraction method and its invariant forms represent an important tool for finding the approximation solution of various types of variational inequalities and complementarity problems.

In recent years variational inequalities have been extended in various directions using novel and innovative techniques. A useful and important generation of variational inequalities is the general mixed variational inequality containing a nonlinear term φ. Due to the presence of the nonlinear term the projection method and its variant forms can not be applied to suggest iterative algorithms for solving mixed variational inequalities. To overcome these drawbacks, some iterative methods have been suggested for a special cases of the general mixed variational inequalities. For example, if the nonlinear term is a proper, convex and lower semicontinuous function, then using the resolvent operator technique, one usually establishes the equivalence between the general mixed variational inequalities and the resolvent equations. It turned out that the resolvent equations are more general and flexible. This approach has played an important part in developing various efficient resolvent-type methods for solving general mixed variational inequalities. The convergence of these methods requires that the operator is both strongly monotone and Lipschitz continuous. Secondly, it is very difficult to evaluate the resolvent of the operator except for very simple cases. This fact has motivated many authors to develop the auxiliary principle technique, Lions and Stampacchia[7], Glowinski et al.[14] used this technique to study the existence of solution for the mixed variational inequalities. Noor[9–11] has extended the auxiliary principle technique to investigate the existence of a solution of various classes of variational inequalities and to suggest and analysis several algorithms for mixed (quasi) variational inequalities. In recent years, some new and interesting problems, which are called the system of variational inclusions were introduced and studied. Chang et al.[4], Huang and Noor[6], Noor and Noor[1], Noor[8], Verma[15–17], He and Gu[5], Petro and

Yang et al.[19] introduced studied a system of variational inclusions involving four, three, two different nonlinear operators.

Inspired and motivated by research going on in this area, we introduce and consider a system of general mixed variational inequalities involving nonlinear operators in Hilbert space. We establish the equivalence between this system of general variational inequalities and the fixed point problem and then by this equivalent formulation, we suggest and analyze a new iterative algorithm for finding a solution of the aforementioned system by using the resolvent operator technique. We also prove the convergence analysis of the proposed algorithm under some suitable mild conditions. Since this class of systems includes the system of variational inequalities involving three, two operators and the classical variational inequalities as special cases, results obtained in this paper continue to hold for these problems. It is expected that these results may inspire and motivate others to find novel and innovative applications in various branches of pure and applied sciences.

1. PRELIMINARIES

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, let I be the identity mapping on H, and $\partial\varphi$ denotes the subdifferential of function φ, where $\varphi : H \to R\cup\{+\infty\}$ is a proper convex lower semi continuous function on H. It is well known that the subdifferential $\partial\varphi$ is a maximal monotone operator. For given nonlinear operators $T_i : H\times H \longrightarrow H$ and $g_i : H \longrightarrow H (i=1,2)$, we consider the system of general mixed variational inequalities of finding $(x^*,y^*)\in H\times H$ such that

$$\begin{cases} \langle \rho T_1(y^*,x^*)+g_1(x^*)-g_1(y^*),g_1(x)-g_1(x^*)\rangle \\ +\varphi(g_1(x))-\varphi(g_1(x^*))\geq 0;\forall x\in H \text{ and } \rho>0, \\ \langle \eta T_2(x^*,y^*)+g_2(y^*)-g_2(x^*),g_2(x)-g_2(y^*)\rangle \\ +\varphi(g_2(x))-\varphi(g_2(y^*))\geq 0;\forall x\in H \text{ and } \eta>0. \end{cases} \tag{1.1}$$

In this paper, we denote the solution set of the problem (1.1) by Ω^*. We now discuss several special cases of the problem (1.1).

If $T_1=T_2=T$ and $g_1=g_2=I$, then the problem (1.1) reduces of finding $(x^*,y^*)\in H\times H$ such that

$$\begin{cases} \langle \rho T(y^*,x^*)+x^*-y^*,x-x^*\rangle+\varphi(x)-\varphi(x^*)\geq 0; \\ \forall x\in H \text{ and } \rho>0, \\ \langle \eta T(x^*,y^*)+y^*-x^*,x-y^*\rangle+\varphi(x)-\varphi(y^*)\geq 0; \\ \forall x\in H \text{ and } \eta>0. \end{cases} \tag{1.2}$$

which has been considered and studied by He and Gu[5], and Petrot[12].

If K is closed convex set in H and $\varphi(v)\equiv I_K(v)$ for all $v\in H$, where I_K is the indicator function of K defined by

$$I_K(v)=\begin{cases} 0, & \text{if } v\in K; \\ +\infty, & \text{otherwise} \end{cases}$$

and $T_i(x,y)=T_i(x), g_1=g_2=g$, then the problem (1.1) reduces to finding $(x^*,y^*)\in K\times K$ such that

$$\begin{cases} \langle \rho T_1(y^*)+g(x^*)-g(y^*),g(x)-g(x^*)\rangle\geq 0; \\ \forall x\in H, g(x)\in K \text{ and } \rho>0, \\ \langle \eta T_2(x^*)+g(y^*)-g(x^*),g(x)-g(y^*)\rangle\geq 0; \\ \forall x\in H, g(x)\in K \text{ and } \eta>0, \end{cases} \tag{1.3}$$

which has been introduced and studied by Yang et al.[19].
If K is closed convex set in H and $\varphi(v)\equiv I_K(v)$ for all $v\in H$, and $g_1=g_2=I$, then the problem (1.1) reduces of finding $(x^*,y^*)\in K\times K$ such that

$$\begin{cases} \langle \rho T_1(y^*,x^*)+x^*-y^*,x-x^*\rangle\geq 0;\forall x\in K \text{ and } \rho>0, \\ \langle \eta T_2(x^*,y^*)+y^*-x^*,x-y^*\rangle\geq 0;\forall x\in K \text{ and } \eta>0, \end{cases} \tag{1.4}$$

which has been considered and studied by Huang and Noor[6].
If $T_1=T_2=T$, then the problem (1.4) reduces of finding $(x^*,y^*)\in K\times K$ such that

$$\begin{cases} \langle \rho T(y^*,x^*)+x^*-y^*,x-x^*\rangle\geq 0;\forall x\in K \text{ and } \rho>0, \\ \langle \eta T(x^*,y^*)+y^*-x^*,x-y^*\rangle\geq 0;\forall x\in K \text{ and } \eta>0, \end{cases} \tag{1.5}$$

The system (1.5) has been studied and investigated by Chang et al.[4] and Verma[16].
If $T_1=T_2=T$, and $g=I$, then the problem (1.3) reduces of finding $(x^*,y^*)\in K\times K$ such that

$$\begin{cases} \langle \rho T(y^*)+x^*-y^*,x-x^*\rangle\geq 0;\forall x\in K \text{ and } \rho>0, \\ \langle \eta T(x^*)+y^*-x^*,x-y^*\rangle\geq 0;\forall x\in K \text{ and } \eta>0, \end{cases} \tag{1.6}$$

which has been introduced and studied by Verma[15,17].
If $x^*=y^*$, then the problem (1.3) collapses to finding $x^*\in K$. such that

$$\langle T(x^*),x-x^*\rangle\geq 0,\ \forall x\in K. \tag{1.7}$$

Inequality of type (1.7) is called variational inequality, which was introduced and studied by Stampacchia[13] in 1964. The system of general mixed variational inequalities (1.1) includes several classes of variational inequalities and related optimization problems as special cases.

We now recall some well-known results and concepts, which are needed.

Definition 1.1 (See [3]) For any maximal operator T, the resolvent operator associated with T, for any $\nu>0$, is defined as

$$J_T^{\nu}(u)=(I+\nu T)^{-1}(u),\ \forall u\in H. \tag{1.8}$$

Lemma 1.1 (See [3]) For a given $w\in H$ and $\nu>0$, the inequality

$$\langle w-z,z-v\rangle+\nu\varphi(v)-\nu\varphi(z)\geq 0,\ \forall v\in H.$$

holds if and only if $z=J_{\varphi}^{\nu}(w)$, where $J_{\varphi}^{\nu}=(I+\nu\partial\varphi)^{-1}$ is the resolvent operator.
It follows from Lemma 1.1 that

$$\langle w-J_{\varphi}^{\nu}(w),J_{\varphi}^{\nu}(w)-v\rangle+\nu\varphi(v)-\nu\varphi(J_{\varphi}^{\nu}(w))\geq 0,\ \forall v,w\in H. \tag{1.9}$$

It is well-known that J_φ^ν is nonexpansive i.e.,

$$\|J_\varphi^\nu(u) - J_\varphi^\nu(v)\| \le \|u - v\|, \qquad \forall u, v \in H.$$

Definition 1.2 A mapping $h : H \longrightarrow H$ is called

(a) r-strongly monotone if, there exists a constant $r > 0$ such that

$$\langle h(x) - h(y), x - y\rangle \ge r\|x - y\|^2, \forall x, y \in H;$$

(b) (ξ, ς)-relaxed cocoercive if, there exist constants $\xi, \varsigma > 0$ such that

$$\langle h(x) - h(y), x - y\rangle \ge -\xi\|h(x) - h(y)\|^2 + \varsigma\|x - y\|^2; \quad \forall x, y \in H.$$

(c) γ-Lipschitz continuous if, there exists a constant $\gamma > 0$ such that

$$\|h(x) - h(y)\| \le \gamma\|x - y\|, \forall x, y \in H.$$

Definition 1.3 Let $T : H \times H \longrightarrow H$. Then T is called

(a) r-strongly monotone in the first variable if there exists a constant $r > 0$ such that for each $x_1, x_2 \in H$,

$$\langle T(x_1, y_1) - T(x_2, y_2), x_1 - x_2\rangle \ge r\|x_1 - x_2\|^2 \quad \forall y_1, y_2 \in H;$$

(b) relaxed (ξ, ς)-cocoercive in the first variable if there exist constants $\xi, \varsigma > 0$ such that for each $x_1, x_2 \in H$,

$$\langle T(x_1, y_1) - T(x_2, y_2), x_1 - x_2\rangle \ge -\xi\|T(x_1, y_1) - T(x_2, y_2)\|^2 + \varsigma\|x_1 - x_2\|^2, \quad \forall y_1, y_2 \in H;$$

(c) γ-Lipschitz continuous in the first variable if there exists a constant $\gamma > 0$ such that for each $x_1, x_2 \in H$,

$$\|T(x_1, y_1) - T(x_2, y_2)\| \le \gamma\|x_1 - x_2\|, \forall y_1, y_2 \in H.$$

Definition 1.4 A mapping $S : H \longrightarrow H$ is said to be nonexpansive if

$$\|S(x) - S(y)\| \le \|x - y\|, \forall x, y \in H.$$

We will denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in H : S(x) = x\}$.

Lemma 1.2 (see[18] Lemma 4, p. 729) Suppose $\{\delta_n\}_{n=0}^\infty$ is a non-negative sequence satisfying the following inequality:

$$\delta_{n+1} \le (1 - \lambda_n)\delta_n + \sigma_n \quad \text{for all} \quad n \ge 0,$$

with $\lambda_n \in [0, 1], \Sigma_{n=0}^\infty \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \delta_n = 0$.

2. THE PROPOSED METHOD

In this section, we first verify the equivalence between the variational inequality system (1.1) and the fixed point problems. Then by using the obtained fixed point formulation, we construct a new iterative algorithm for solving the system (1.1).

Lemma 2.1 Let H be a real Hilbert space, for given nonlinear operators $T_i : H \times H \longrightarrow H$, $g_i : H \longrightarrow H (i = 1, 2)$ and $\rho, \eta > 0$. Then the point $(x^*, y^*) \in H \times H$ is a solution of the variational inequality system (1.1) if and only if

$$\begin{cases} g_1(x^*) = J_\varphi[g_1(y^*) - \rho T_1(y^*, x^*)] \\ g_2(y^*) = J_\varphi[g_2(x^*) - \eta T_2(x^*, y^*)]. \end{cases} \tag{2.1}$$

Proof. The first variational inequality of (1.1) can be written as follows:

$$\langle g_1(y^*) - \rho T_1(y^*, x^*) - g_1(x^*), g_1(x^*) - g_1(x)\rangle + \varphi(g_1(x)) - \varphi(g_1(x^*)) \ge 0; \forall x \in H \text{ and } \rho > 0,$$

We can deduce from Lemma 1.1 with $\nu = 1$ that the above inequality is equivalent to

$$g_1(x^*) = J_\varphi[g_1(y^*) - \rho T_1(y^*, x^*)],$$

where $J_\varphi = (I + \partial\varphi)^{-1}$ is the resolvent operator.
Similar, the second variational inequality of (1.1) is equivalent to

$$g_2(y^*) = J_\varphi[g_2(x^*) - \eta T_2(x^*, y^*)]. \qquad \square$$

By rewriting (2.1), we have

$$\begin{cases} x^* = x^* - g_1(x^*) + J_\varphi[g_1(y^*) - \rho T_1(y^*, x^*)] \\ y^* = y^* - g_2(y^*) + J_\varphi[g_2(x^*) - \eta T_2(x^*, y^*)]. \end{cases} \tag{2.2}$$

(2.2) enables us to construct the following algorithm:

Algorithm 2.1 Let $T_i : H \times H \longrightarrow H$, $g_i : H \longrightarrow H (i = 1, 2)$ be nonlinear operators, $S : H \longrightarrow H$ be nonexpansive mapping and ρ, $\eta > 0$ are two constants. For arbitrary chosen initial points $x^0, y^0 \in H$, compute the iterative sequences $\{x^k\}$ and $\{y^k\}$ by using

$$\begin{cases} y^k = (1 - \beta_k)x^k + \beta_k S\{y^k - g_2(y^k) \\ \quad + J_\varphi[g_2(x^k) - \eta T_2(x^k, y^k)]\} \\ x^{k+1} = (1 - \alpha_k)x^k + \alpha_k S\{x^k - g_1(x^k) \\ \quad + J_\varphi[g_1(y^k) - \rho T_1(y^k, x^k)]\}, \end{cases} \tag{2.3}$$

where $J_\varphi = (I + \partial\varphi)^{-1}$ is the resolvent operator and $\{\alpha_k\}, \{\beta_k\}$ are sequences in $[0, 1]$.

Remark 2.1

- If $T_1 = T_2 = T$ and $g_1 = g_2 = I$, then Algorithm 2.1 reduces to Algorithm I[12] and Algorithm 2.1 in[5](with $S = I$).
- If $T_1(x, y) = T_2(x, y) = T(x)$, $g_1 = g_2 = I$, and φ is an indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto K and consequently Algorithm 2.1 collapses to Algorithm 2.1 in[4,17].

- If K is closed convex set in H and $\varphi(v) \equiv I_K(v)$ for all $v \in H$, where I_K is the indicator function of K, $g_1 = g_2 = I$ and $\beta_k = 1$, then Algorithm 2.1 reduces to Algorithm 2.1 in[6].

We now establish the strongly convergence of the sequences generated by Algorithm 2.1.

Theorem 2.1 Let H be a real Hilbert space. Let $T_i : H \times H \longrightarrow H$ be a relaxed (κ_i, θ_i)-cocoercive and γ_i-Lipschitz continuous in the first variable and $g_i : H \longrightarrow H$ be a (ξ_i, ν_i)-relaxed cocoercive and δ_i-Lipschitz continuous. Let $S : H \longrightarrow H$ be nonexpansive mapping. If the following conditions are satisfied:

(i) $\{\alpha_k\} \subset [0,1]$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$;

(ii) $\{\beta_k\} \subset [0,1]$ and $\lim_{k\to} \beta_k = 1$;

(iii) $\Lambda_2 + \Lambda_4 < 1$ and $\frac{\Lambda_3+\Lambda_1}{1-\beta_k\Lambda_4} < 1 - \Lambda_3$. where

$$\Lambda_1 := \sqrt{1 - 2\rho\theta_1 + (2\rho\kappa_1 + \rho^2)\gamma_1^2}$$
$$\Lambda_2 := \sqrt{1 - 2\eta\theta_2 + (2\eta\kappa_2 + \eta^2)\gamma_2^2};$$

and

$$\Lambda_3 := \sqrt{(1-2\nu_1) + (1+2\xi_1)\delta_1^2}$$
$$\Lambda_4 := \sqrt{(1-2\nu_2) + (1+2\xi_2)\delta_2^2}.$$

If $\Omega^* \bigcap F(S) \neq \emptyset$ then x^k and y^k obtained from Algorithm 2.1 converge strongly to x^* and y^*, respectively, such that $(x^*, y^*) \in \Omega^*$, and $\{x^*, y^*\} \subset F(S)$.

Proof. It follows from (2.3) that

$$\begin{aligned}
&\|x^{k+1} - x^*\| \\
=&\|(1-\alpha_k)x^k + \alpha_k S\{x^k - g_1(x^k) + J_\varphi[g_1(y^k) \\
&- \rho T_1(y^k, x^k)]\} - (1-\alpha_k)x^* \\
&+ \alpha_k S\{x^* - g_1(x^*) + J_\varphi[g_1(y^*) - \rho T_1(y^*, x^*)]\}\| \\
\le&(1-\alpha_k)\|x^k - x^*\| + \alpha_k\|x^k - g_1(x^k) \\
&+ J_\varphi[g_1(y^k) - \rho T_1(y^k, x^k)] \\
&- \{x^* - g_1(x^*) + J_\varphi[g_1(y^*) - \rho T_1(y^*, x^*)]\}\| \\
\le&(1-\alpha_k)\|x^k - x^*\| + \alpha_k\|x^k - x^* - [g_1(x^k) - g_1(x^*)]\| \\
&+ \alpha_k\|J_\varphi[g_1(y^k) - \rho T_1(y^k, x^k)] \\
&- J_\varphi[g_1(y^*) - \rho T_1(y^*, x^*)]\| \\
\le&(1-\alpha_k)\|x^k - x^*\| + \alpha_k\|x^k - x^* - [g_1(x^k) - g_1(x^*)]\| \\
&+ \alpha_k\|y^k - y^* - [g_1(y^k) - g_1(y^*)]\| \\
&+ \alpha_k\|y^k - y^* - \rho(T_1(y^k, x^k) - T_1(y^*, x^*))\|.
\end{aligned} \tag{2.4}$$

Since T_1 is relaxed (κ_1, θ_1)-cocoercive and γ_1-Lipschitz continuous in the first variable, we can conclude that

$$\begin{aligned}
&\|y^k - y^* - \rho(T_1(y^k, x^k) - T_1(y^*, x^*))\|^2 \\
=&\|y^k - y^*\|^2 - 2\rho\langle T_1(y^k, x^k) - T_1(y^*, x^*), y^k - y^*\rangle \\
&+ \rho^2\|T_1(y^k, x^k) - T_1(y^*, x^*)\|^2 \\
\le&\|y^k - y^*\|^2 + 2\rho\kappa_1\|T_1(y^k, x^k) - T_1(y^*, x^*)\|^2 \\
&- 2\rho\theta_1\|y^k - y^*\|^2 + \rho^2\|T_1(y^k, x^k) - T_1(y^*, x^*)\|^2 \\
\le&\left(1 - 2\rho\theta_1 + (2\rho\kappa_1 + \rho^2)\gamma_1^2\right)\|y^k - y^*\|^2.
\end{aligned} \tag{2.5}$$

From (ξ_1, ν_1)-relaxed cocoercive and δ_1-Lipschitz continuous, we have

$$\begin{aligned}
&\|y^k - y^* - (g_1(y^k) - g_1(y^*))\|^2 \\
\le&\|y^k - y^*\|^2 - 2\langle g_1(y^k) - g_1(y^*), y^k - y^*\rangle \\
&+ \|g_1(y^k) - g_1(y^*)\|^2 \\
\le&\|y^k - y^*\|^2 + 2\xi_1\|g_1(y^k) - g_1(y^*)\|^2 - 2\nu_1\|y^k - y^*\|^2 \\
&+ \|g_1(y^k) - g_1(y^*)\|^2 \\
\le&\left((1-2\nu_1) + (1+2\xi_1)\delta_1^2\right)\|y^k - y^*\|^2.
\end{aligned} \tag{2.6}$$

In a similar way, one can show that

$$\|x^k - x^* - (g_1(x^k) - g_1(x^*))\| \le \Lambda_3\|x^k - x^*\|, \tag{2.7}$$

where $\Lambda_3 := \sqrt{(1-2\nu_1) + (1+2\xi_1)\delta_1^2}$.
Substituting (2.5), (2.6) and (2.7) in (2.4), we get

$$\begin{aligned}
&\|x^{k+1} - x^*\| \\
\le& (1 - \alpha_k(1-\Lambda_3))\|x^k - x^*\|| + \alpha_k(\Lambda_3 + \Lambda_1)\|y^k - y^*\|.
\end{aligned} \tag{2.8}$$

It follows from (2.3) that

$$\begin{aligned}
&\|y^k - y^*\| \\
=&\|(1-\beta_k)x^k + \beta_k S\{y^k - g_2(y^k) + J_\varphi[g_2(x^k) \\
&- \eta T_2(x^k, y^k)]\} - (1-\beta_k)y^* - \beta_k S\{y^* - g_2(y^*) \\
&+ J_\varphi[g_2(x^*) - \eta T_2(x^*, y^*)]\}\| \\
\le&(1-\beta_k)\|x^k - y^*\| + \beta_k\|y^k - g_2(y^k) + J_\varphi[g_2(x^k) \\
&- \eta T_2(x^k, y^k)] - \{y^* - g_2(y^*) \\
&+ J_\varphi[g_2(x^*) - \eta T_2(x^*, y^*)]\}\| \\
\le&(1-\beta_k)\|x^k - y^*\| + \beta_k\|y^k - y^* - [g_2(y^k) - g_2(y^*)]\| \\
&+ \beta_k\|x^k - x^* - [g_2(x^k) - g_2(x^*)]\| \\
&+ \beta_k\|x^k - x^* - \eta(T_2(x^k, y^k) - T_2(x^*, y^*))\|.
\end{aligned} \tag{2.9}$$

Since T_2 is relaxed (κ_2,θ_2)-cocoercive and γ_2-Lipschitz continuous in the first variable, we can conclude that

$$\begin{aligned}
&\|x^k-x^*-\eta(T_2(x^k,y^k)-T_2(x^*,y^*))\|^2\\
=&\|x^k-x^*\|^2-2\eta\langle T_2(x^k,y^k)-T_2(x^*,y^*),x^k-x^*\rangle\\
&+\eta^2\|T_2(x^k,y^k)-T_2(x^*,y^*)\|^2\\
\le&\|x^k-x^*\|^2+2\eta\kappa_2\|T_2(x^k,y^k)-T_2(x^*,y^*)\|^2\\
&-2\eta\theta_2\|x^k-x^*\|^2+\eta^2\|T_2(x^k,y^k)-T_2(x^*,y^*)\|^2\\
\le&\left(1-2\eta\theta_2+(2\eta\kappa_2+\eta^2)\gamma_2^2\right)\|x^k-x^*\|^2.
\end{aligned}\tag{2.10}$$

Like in the proof (2.6), we can prove that

$$\|y^k-y^*-(g_2(y^k)-g_2(y^*))\|\le\Lambda_4\|y^k-y^*\|\tag{2.11}$$

and

$$\|x^k-x^*-(g_2(x^k)-g_2(x^*))\|\le\Lambda_4\|x^k-x^*\|,\tag{2.12}$$

where $\Lambda_4:=\sqrt{(1-2\nu_2)+(1+2\xi_2)\delta_2^2}$.
Substituting (2.10), (2.11) and (2.12) in (2.9), we get

$$\begin{aligned}
\|y^k-y^*\|\le&(1-\beta_k)\|x^k-y^*\|\\
&+\beta_k(\Lambda_2+\Lambda_4)\|x^k-x^*\|+\beta_k\Lambda_4\|y^k-y^*\|.
\end{aligned}$$

Since $\Lambda_4<1$, it is easy to prove that $\beta_k\Lambda_4<1$. By simple manipulation, we obtain

$$\begin{aligned}
\|y^k-y^*\|\le&\frac{(1-\beta_k(1-(\Lambda_2+\Lambda_4)))}{1-\beta_k\Lambda_4}\|x^k-x^*\|+\\
&\frac{1-\beta_k}{1-\beta_k\Lambda_4}\|x^*-y^*\|.
\end{aligned}\tag{2.13}$$

It follows from (2.8) and (2.13) that

$$\begin{aligned}
&\|x^{k+1}-x^*\|\\
\le&(1-\alpha_k(1-\Lambda_3))\|x^k-x^*\|\\
&+\frac{\alpha_k(\Lambda_3+\Lambda_1)(1-\beta_k(1-(\Lambda_2+\Lambda_4)))}{1-\beta_k\Lambda_4}\|x^k-x^*\|\\
&+\frac{\alpha_k(1-\beta_k)(\Lambda_3+\Lambda_1)}{1-\beta_k\Lambda_4}\|x^*-y^*\|
\end{aligned}\tag{2.14}$$

and

$$\begin{aligned}
&\|x^{k+1}-x^*\|\\
\le&\left[1-\alpha_k\left(1-\Lambda_3-\frac{(\Lambda_3+\Lambda_1)(1-\beta_k(1-(\Lambda_2+\Lambda_4)))}{1-\beta_k\Lambda_4}\right)\right]\\
&\cdot\|x^k-x^*\|+\frac{\alpha_k(1-\beta_k)(\Lambda_3+\Lambda_1)}{1-\beta_k\Lambda_4}\|x^*-y^*\|.
\end{aligned}\tag{2.15}$$

Put

$$\delta_k=\|x^k-x^*\|,$$
$$\lambda_k=\alpha_k\left(1-\Lambda_3-\frac{(\Lambda_3+\Lambda_1)(1-\beta_k(1-(\Lambda_2+\Lambda_4)))}{1-\beta_k\Lambda_4}\right)$$

and

$$\sigma_k=\frac{\alpha_k(1-\beta_k)(\Lambda_3+\Lambda_1)}{1-\beta_k\Lambda_4}\|x^*-y^*\|.$$

From (iii) we have $\lambda_k\in(0,1)$ for all $k\in N$. Meanwhile, the condition (ii) implies that $\sigma_k=o(\lambda_k)$; moreover, by using the condition (iii), it is easy to see that $\lambda_k>\alpha_k\left(1-\Lambda_3-\frac{\Lambda_3+\Lambda_1}{1-\beta_k\Lambda_4}\right)$ for all $k\in N$ and so, from the condition (iii), we obtain $\Sigma_{k=0}^{\infty}\lambda_k=\infty$. Hence all the conditions in Lemma 1.2 are satisfied and so $\lim_{k\to\infty}\|x^k-x^*\|=0$, i.e., $\lim_{k\to\infty}x^k=x^*$. Consequently, by the condition (ii) and (2.13), we obtain $\lim_{k\to\infty}y^k=y^*$. This completes the proof. □

If K is closed convex set in H and $\varphi(v)\equiv I_K(v)$ for all $v\in H$, where I_K is the indicator function of K, then $J_\varphi\equiv P_K$ the projection of H onto K. Consequently, the following result can be obtain from Theorem 2.1 immediately.

Theorem 2.2 Let H be a real Hilbert space, K be a nonempty closed convex subset of H. Let $T_i:K\times K\longrightarrow K$ be a relaxed (κ_i,θ_i)-cocoercive and γ_i-Lipschitz continuous in the first variable and $g_i:K\longrightarrow K$ be a (ξ_i,ν_i)-relaxed cocoercive and δ_i-Lipschitz continuous. Let $S:K\longrightarrow K$ be nonexpansive mapping. For arbitrary chosen initial points $x^0,y^0\in K$, compute the iterative sequences $\{x^k\}$ and $\{y^k\}$ by using

$$\begin{cases}
y^k=(1-\beta_k)x^k+\beta_kS\{y^k-g_2(y^k)\\
+P_K[g_2(x^k)-\eta T_2(x^k,y^k)]\}\\
x^{k+1}=(1-\alpha_k)x^k+\alpha_kS\{x^k-g_1(x^k)\\
+P_K[g_1(y^k)-\eta T_1(y^k,x^k)]\},
\end{cases}\tag{2.16}$$

If the following conditions are satisfied:

(i) $\{\alpha_k\}\subset[0,1]$ and $\sum_{k=0}^{\infty}\alpha_k=\infty$;

(ii) $\{\beta_k\}\subset[0,1]$ and $\lim_{k\to}\beta_k=1$;

(iii) $\Lambda_2+\Lambda_4<1$ and $\frac{\Lambda_3+\Lambda_1}{1-\beta_k\Lambda_4}<1-\Lambda_3$ where

$$\Lambda_1:=\sqrt{1-2\rho\theta_1+(2\rho\kappa_1+\rho^2)\gamma_1^2}$$
$$\Lambda_2:=\sqrt{1-2\eta\theta_2+(2\eta\kappa_2+\eta^2)\gamma_2^2},$$

and

$$\Lambda_3:=\sqrt{(1-2\nu_1)+(1+2\xi_1)\delta_1^2}$$
$$\Lambda_4:=\sqrt{(1-2\nu_2)+(1+2\xi_2)\delta_2^2}.$$

If $\Omega^*\bigcap F(S)\ne\emptyset$ then x^k and y^k obtained from Algorithm 2.1 converge strongly to x^* and y^*, respectively, such that $(x^*,y^*)\in\Omega^*$, and $\{x^*,y^*\}\subset F(S)$.

Remark 2.2 Theorem 2.2 extends and improves the main result of [4].

Remark 2.3 It is clear from Definition 1.3 that ς-strongly monotone mappings are relaxed (ξ,ς)-cocoercive, but the converse is not true. This shows that relaxed cocoercivity is a weaker condition than strongly monotonicity. The underlying operator $Ti(i=1,2)$ in our paper needs to be relaxed (ξ,ς)-cocoercive while the underlying operator T in [17] needs to be ς-strongly monotone. Hence, Theorem 2.1 extends and improves the main results of Theorem 3.1 of [17].

CONCLUSIONS

In this paper, we suggest and analyze a new method for a system of general mixed variational inequalities involving nonlinear operators in Hilbert space, which can be viewed as a refinement and improvement of some existing resolvent methods and projection descent methods. It is easy to verify that Algorithm 2.1 include some existing methods (e.g. [2,4–6,12,16,19]) as special cases. Therefore, the new algorithm is expected to be widely applicable.

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