CSCanada
Advances in Natural Science
Vol. 4, No. 2, 2011, pp. 18-21
DOI:10.3968/j.ans.1715787020110402.251

ISSN 1715-7862 [PRINT]
ISSN 1715-7870 [ONLINE]
www.cscanada.net
www.cscanada.org

The Error Caused by Relativity in GPS Positioning System

ZHAO Min[1,*]; WANG Yunzhi[1]; MU Xiaoyun[1]

[1]Basic Courses Department of Beijing Union University, Beijing, 100101, China
[*]Corresponding author.

Received 23 September 2011; accepted 19 November 2011

Abstract

In GPS Positioning System, distance measurement between the receiver and the satellite is affected by all kinds of factors. So it has errors. Some of these errors, which are caused by relativity can't be neglected when it is being accurate positioned. This essay analyses and estimates these errors.

Key words: GPS; Positioning; Measurement; Error Special; Theory of Relativity; General; Theory of Relativity

ZHAO Min, WANG Yunzhi, MU Xiaoyun (2011). The Error Caused by Relativity in GPS Positioning System. *Advances in Natural Science, 4*(2), 18-21. Available from: URL: http://www.cscanada.net/index.php/css/article/view/j.ans.1715787020110402.251 DOI: http://dx.doi.org/10.3968/j.ans.1715787020110402.251.

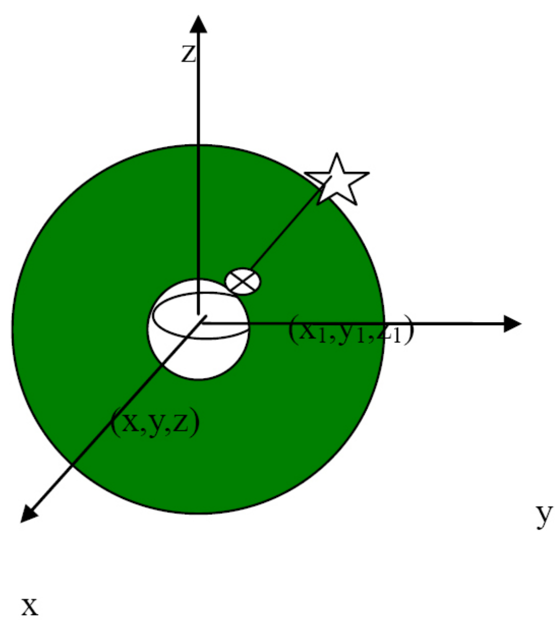
1. THE GPS POSITIONING SYSTEM

GPS is The Navigation Satellite Timing Ranging Global Positioning System--- A Positioning System using the Navigation Satellite to measure the time and distance .Now the GPS has already been used in military, agriculture, exploration and navigation all of the world[1-4].

Its principle is very simple. The satellite transmits the electromagnetic carrier wave. The receiver on the earth receives the radio signals and calculates the relative distance between the satellite and the receiver according to the wave's information to fix the receiver's place. This measuring method is called false distance measuring[5-9], since the measuring has errors caused by various factors. The measured distance needs to be revised.

2. THE MEASURING EQUATION

As chart 1, supposing the satellite is at the earth coordinate (x_i, y_i, z_i), it transmits a radio signal at the time t_i. The receiver is at the earth coordinate (x, y, z), it receives this signal at the time t. The distance between the satellite and the receiver is:



$$\rho = \left[(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2\right]^{\frac{1}{2}} = c\Delta t_i = c(t - t_i) \qquad (1)$$

The radio transmitted by the satellite contains the information about the satellite's place and time. In theory, if the errors are disregarded, we can ascertain the receiver's place through solving the equation (1). But the equation (1) needs to be revised considering all kinds of errors which are caused by various factors.

3. THE ERROR CAUSED BY THE SPECIAL RELATIVITY

In eqution (1), t is the time at the receiver's clock when receiver receives the radio. (x_i, y_i, z_i) can be obtained from the information of the radio. But t_i is the time at the receiver's clock when the satellite transmits the radio. The time t_i', which we obtained from the information of the radio, is the time at the satellite. The time t_i is quantitatively different from the time t_i'. This quantitative difference is caused by all kinds of factors. The difference caused by special relativity should not be neglected.

Supposing the receiver is near the equator of the earth. The satellite moves at v_i in the earth coordinate. According to the special Relativity:

$$\Delta t_i = \frac{\Delta t_i^{'}}{\sqrt{1-\frac{v_i^2}{c^2}}} \tag{2}$$

$\Delta t_i'$ is the time interval at the satellite clock, Δt_i is the time interval at the earth clock. If when the satellite transmits the radio, the satellite clock and the earth clock have already been calibrated, that means $t_{i0}=t'_{i0}$, so the time at the earth clock when the radio reaches the receiver is:

$$t_i = \frac{t_i^{'}}{\sqrt{1-\frac{v_i^2}{c^2}}} \tag{3}$$

According to equation (1),we obtain

$$\begin{aligned}\rho &= \left[(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2\right]^{\frac{1}{2}} \\ &= c(t - t_i) \\ &= c(t - t_i^{'}) - ct_i^{'}\left(\frac{1-\sqrt{1-\beta^2}}{\sqrt{1-\beta^2}}\right)\end{aligned} \tag{4}$$

In (4), $\beta = \frac{v_i}{c}$, the equation (4) becomes:

$$\rho = \left[(x_i - x)^2 + (y_i - y)^2 + (z_i - z\ )^2\right]^{\frac{1}{2}} + ct_i^{'}\left(\frac{1-\sqrt{1-\beta^2}}{\sqrt{1-\beta^2}}\right) = c(t - t_i^{'}) \tag{5}$$

We can calculate the place of the receiver according to equation (5), the error caused by the special Relativity is: $(\frac{1-\sqrt{1-\beta^2}}{\sqrt{1-\beta^2}})$. It is approximately calculated in Taile progression: $\beta^2/2$, because β^2 is infinitely small.

The velocity of the satellite around the earth is about 4000m/s, the velocity of light is 3×10^8 *m/s*, so that this relative error is:

$$\left(\frac{1-\sqrt{1-\beta^2}}{\sqrt{1-\beta^2}}\right) \approx \frac{\beta^2}{2} \approx 9\times10^{-11}$$

As a high precision positioning System, its error is limited in scope. The most advanced military GPS's error is limited in centimeter .Its relative error is limited in 10^{-13}. This error can not be neglected.

4. ERROR CAUSED BY THE GENERAL RELATIVITY

We can simply only consider the force exerted by the earth, according to the General Relativity, the time and space around the earth becomes winding. The winding degree is decided by the metric. Because the metrics of the places in the satellite and the receiver are not the same, the error exists in the measured distance between the satellite and the receiver.

4.1 The Error Caused by the Time Effect of the General Relativity---the Gravity Frequency Shift Error

According to the general Relativity, the gravity field around the earth is Schwarzschild field, the Schwarzschild metric in ball coordinates is:

$$ds^2 = -c^2\left(1-\frac{2Gm}{c^2 r}\right)dt^2 + \left(1-\frac{2Gm}{c^2 r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \tag{6}$$

The relation between the inherent time and the coordinate time of the receiver who is still in the earth coordinate is:

$$d\tau = \sqrt{-g_{00}}\,dt \tag{7}$$

$d\tau$ is the inherent time, dt is the Schwarzschild metric time. Meanwhile the inherent time $d\tau$ is significant in measurement.

The receiver's inherent time in space 1 is $d\tau_1$, in space 2 is $d\tau_2$. Their relation is:

$$d\tau_2 = \frac{(\sqrt{-g_{00}})_{(2)}}{(\sqrt{-g_{00}})_{(1)}}d\tau_1 \tag{8}$$

$g_{00} = -c^2\left(1-\frac{2Gm}{c^2 r}\right)$ is the Schwarzschild metric time. If the satellite is in space 1, the receiver is in space 2, the time interval at the satellite clock is $\Delta\tau'$, the time interval at the receiver is $\Delta\tau$, their relation is:

$$\Delta\tau = \frac{(\sqrt{-g_{00}})_{(接)}}{(\sqrt{-g_{00}})_{(卫)}}\Delta\tau^{'} \tag{9}$$

Meanwhile G is gravitational constant, m is the mass of the earth, c is the velocity of light, r is the distance between the Geocentric and the object. Because the satellite moves in relative velocity v in the earth coordinate, according to the special Relativity, the time at the satellite converts into the time at the earth. There is:

$$\Delta\tau^{'} = \Delta\tau^{''} / \sqrt{1-\frac{v^2}{c^2}} \tag{10}$$

Equation (9) becomes:

$$\Delta\tau = \frac{\sqrt{1-\frac{2Gm}{c^2 r_2}}}{\sqrt{1-\frac{2Gm}{c^2 r_1}}}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Delta\tau^{''} \tag{11}$$

Where r_1 and r_2 is the position of the satellite and the receiver. If the receiver is situated on the surface of the earth, h is the height of the satellite from the surface of the earth, we have: $r_2=R$, $r_1=R+h$.

Equation (11) can be expressed:

$$\nu = \frac{\sqrt{1-\frac{2Gm}{c^2 r_1}}}{\sqrt{1-\frac{2Gm}{c^2 r_2}}}\sqrt{1-\frac{v^2}{c^2}}\nu^{*} \quad (12)$$

In Equation (12), v is the frequency received by the receiver, v'' is the radiation frequency from the satellite. Equation (12) shows v and v'' is not the same. There is quantitative difference between these two. This error is called the gravity frequency shift error. So the quantity of gravity frequency shifts:

$$\Delta\nu = |\nu - \nu^{*}| = \left(\sqrt{\frac{1-\frac{2Gm}{c^2 r_1}}{1-\frac{2Gm}{c^2 r_2}}}\sqrt{1-\frac{v^2}{c^2}}-1\right)\nu^{*} \quad (13)$$

The relative quantity is:

$$E = \frac{\Delta\nu}{\nu^{*}} = \left(\sqrt{\frac{1-\frac{2Gm}{c^2 r_1}}{1-\frac{2Gm}{c^2 r_2}}}\sqrt{1-\frac{v^2}{c^2}}-1\right) \quad (14)$$

Because $\frac{2Gm}{c^2 r}, \frac{v^2}{c^2}$ are all infinitely small, Equation (11) is expanded by Taylor series and the Second-Order infinitesimals is neglected, we have:

$$\begin{aligned} E &\approx \left(\frac{1-\frac{Gm}{c^2 r_1}}{1-\frac{Gm}{c^2 r_2}}\right)\left(1-\frac{v^2}{2c^2}\right)-1 \\ &\approx \left(1-\frac{Gm}{c^2 r_1}\right)\left(1+\frac{Gm}{c^2 r_2}\right)\left(1-\frac{v^2}{2c^2}\right)-1 \\ &\approx \frac{Gm}{c^2 r_2}-\frac{Gm}{c^2 r_1}-\frac{v^2}{2c^2} \end{aligned} \quad (15)$$

The height of the satellite from the surface of the earth is about 20200 km, the mass of the earth m=5.976×10^{24} kg, the average radius of the earth is 4371 km, the velocity of the satellite around the earth is 4000m/s, the light velocity v is 3×10^8 m/s, the gravitational constant G is $6.67\text{x}10^{-11} Nm^2/kg^2$, we have:

$$E \approx 4.5\times10^{-10} \quad (15)$$

So this error can not be neglected.

4.2 The Error Caused by the Space Effect of the General Relativity

According to equation (4), the radial inherent distance is:

$$d\rho_r = \sqrt{g_{11}}dr = \left(1-\frac{2Gm}{c^2 r}\right)^{-\frac{1}{2}}dr \quad (16)$$

Meanwhile $g_{11} = (1-\frac{2Gm}{c^2 r})^{-1}$ is the Schwarzschild radial space metric.

Equation (16) is expanded by Taylor series and the Second-Order infinitesimals is neglected, we have :

$$d\rho_r = \left(1+\frac{Gm}{c^2 r}\right)dr \quad (17)$$

If the receiver is located at the equator of the earth, Equation (17) is integrated, we get:

$$\begin{aligned} \rho_r &= \int_r^{r_1}\left(1+\frac{Gm}{c^2 r}\right)dr \\ &= (r_1-r)+\frac{Gm}{c^2}\ln\frac{r_1}{r} \\ &= \sqrt{(x_1-x)^2+(y_1-y)^2+(z_1-z)^2}+\frac{Gm}{c^2}\ln\frac{R+h}{R} \end{aligned} \quad (18)$$

This is the inherent distance between the receiver and the satellite. Compared with equation (1), this distance is longer, $\frac{Gm}{c^2}\ln\frac{R+h}{R}$ is just the error caused by the space effect of the general Relativity. Because the gravitational constant G, the mass of the earth m, the average radius of the earth R, the light velocity v and The height of the satellite h are all known, so we get:

$$\Delta\rho_r = \frac{Gm}{c^2}\ln\frac{R+h}{R} \approx 6.32\times10^{-3}m \quad (19)$$

This error can not be neglected when accurate Positioning.

5. THE END WORDS

Except these errors, GPS accurate positioning needs to consider the ionosphere's chromatic dispersion and the troposphere's refraction. We also need to think over the error caused by the rotation of the earth and the tides.

REFERENCES

[1] Tennessean, P. J. G. (1999). An Optimality Property of the Integer Least-Squares Estimator. *Journal of Geodesy*, *73*(11), 587-593.

[2] Hofmann Wellenhof, B., Lichtenberger, H., & Collins, J. (1993). *Global Positioning System*. *Theory and practice* (p. 347). Springer, Wien: Austria.

[3] Randell, C. D. Cliff & Muller, H. L. (2003). Personal position measurement using dead reck-oning. In *Proceedings of the Seventh International Symposium on Wearable Computers,* C. Narayanaswami, Ed. IEEE Computer Society, October 2003, pp. 166–173.

[4] Čapkun, S., Hamdi, M., & Hubaux, Jean-Pierre (2002). GPS-Free Positioning in Mobile Ad Hoc Networks. *Cluster Computing*, *5*(2), 157-167.

[5] ZHAO Zheng (2004). *The Basic of the General Relativity* (pp. 101-103). Beijing, China: Beijing Normal University Publishing House.

[6] BIAN Lijun. Principle and Application of GPS Positioning System (2010). *Telecommunication Engineering Technology*

and Standardization, 1, 25.

[7] Ashby, Neil (2002). Relativity and the Global Positioning System. *Physics Today*, (5), 11.

[8] XU Shaoquan (2010). *GPS Measurement Principle and Application* (p. 113). Hubei: Wuhan University Publishing House.

[9] Alison K. Brown et al. (1993). Vehicle Tracking System Employing Global Positioning System (GPS) Satellites. United States Patent number: 5225842, Filing date: 9 May 1991, Issue date: 6 Jul 1993.

