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Reducible Property of a Finitely Generated Module[1]

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Abstract: In the article, G-invariant element, $Inv_H(V)$, $Hom_R(U,V)$ and other concepts were introduced, Several lemmas were proved to use these concepts, Finally, it had been proved that let *F* be a field whose characteristic dose not divide |*G*|. Then every finitely generated *F*[*G*] module is completely reducible.
Key words: finitely generated *R*[*G*] module; *B*-projective; *R*-homomorphism; *R*-homomorphism

Let *A* be a ring and let *V*, *W* be *A* modules. $Hom_A(V, W)$ is the additive abelian group of all homomorphisms from *V* to *W*, The same notation will be used in case *V*, *W* are left *A* modules. Let V_A, $_AV$ denote the fact that *V* is an *A* module, left *A* module respectively. $V={}_AV_B$ is a two sided (*A*, *B*) module. Let *A*, *B* be rings, Let $a \in A,\ b \in B\ f \in Hom_A(V,W)$ and let $v \in V,\ \ w \in W$ then the following hold.

Lemma 1: $Hom_A({}_BV_A, W_A)$ is a *B* module with $(fb)v=f(bv)$.

Lemma 2: $Hom_A(V_A, {}_BW_A)$ is a left *B* module with $(bf)v=b(fv)$.

In the following *A* is a ring which satisfies A.C.C

All the results in this article will be stated in terms of finitely generated modules (Jacobson, 1956). This is all that will be required in the sequel. However it is well-known that analogous results hold for arbitrary modules even if *A* does not satisfy any chain condition.

A finitely generated *A* module *P* is projective if every exact sequence

$$O \to W \to V \to P \to O$$

with *V*, *W* finitely generated *A* modules is a split exact sequence.

Lemma 3: Let *B* be a subring of A with $1 \in B$ such that *B* satisfies A.C.C., A_B is a finitely generated free *B* module and $_BA$ is a finitely generated free left *B* module. Let *P* be a finitely generated *A* module. The following are equivalent.

(i) *P* is *B*-projective.

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(ii) $P| P_B \otimes_{BB} A_A$.

(iii) $P| W \otimes_{BB} A_A$ for some finitely generated B module W.

The detail of proof of this lemma above can be found in Reference (Jacobson, 1956).

Lemma 4: Let e be an idempotent in A and let V be an A module. Define f: $Ve \rightarrow Hom_A(eA,V)$ by $f(v)ea=vea$. Then f is a group isomorphism. If A is an R algebra then Ve and $Hom_A(eA,V)$ are R modules and f is an R-isomorphism. If $V=eA$ then f: $eAe \rightarrow E_A(eA)$ is a ring isomorphism. Thus in particular $E_A(A_A) \approx A \approx E_A({}_A A)$.

Proof: By Lemma 1 and Lemma 2 Ve and $Hom(eA, V)$ are R modules if A is an R-algebra. In this case f is clearly an R-homomorphism. In any case f is a group homomorphism and if $V=eA$, f is a ring homomorphism. If $h \in Hom_A(eA, V)$ then $f(h(e))=h$. Thus f is an epimorphism (Robinson, 2003; Osima; LI & Skiba, 2008; GUO & Skiba, 2006).

The last statement follows by setting $e=1$. The lemma is proved. □

Let H be a subgroup of G and let V be an $R[G]$ module then $V_H=V_{R[H]}$ denotes the restriction of V to $R[H]$. If W is an $R[H]$ module then

$$W^G = W \underset{R[H]}{\otimes} {}_{R[H]}R[G]_{R[G]}$$

The $R[G]$ module W^G is said to be induced by W.

Let $\{x_i\}$ be a cross section (Alperin & Rowen, 1997; LI et al., 2003) of H in G. Then $R[G]$ is a free left $R[H]$ module with basis $\{x_i\}$. Thus if V is an $R[H]$ module then $V^G=\oplus_i V \otimes x_i$ where this is a direct sum of R modules and if $v \in V$, $x \in G$ then

$$(v \otimes x_i)x = v \otimes x_i x = v \otimes yx_j = vy \otimes x_j$$

Where $x_i x = yx_j$ with $y \in H$.

Suppose that σ is an automorphism of $R[G]$ such that $R^{\sigma}=R$ and $G^{\sigma}=G$. Thus σ defines an automorphism of R and one of G. If H is a subgroup of G and V is an $R[H]$ module define the $R[H]^{\sigma}$ module V^{σ} as follows. $V^{\sigma}=\{v\sigma \mid v \in V\}$ where

$v\sigma + w\sigma = (v+w)\sigma$ for $v, w \in V$.

$v\sigma a^{\sigma} = (va)\sigma$ for $v \in V$, $a \in R[H]$

If σ is an automorphism of R let $V^{\sigma}=V^{\sigma_1}$ where σ_1 is the automorphism of $R[G]$ defined by $(\sum_{x\in G} r_x x)^{\sigma_1} = \sum_{x \in G} r_x^{\sigma} x$.

If σ is an automorphism of G let $V^{\sigma}=V^{\sigma_1}$ where σ_1 is the automorphism of $R[G]$ defined by $(\sum_{x\in G} r_x x)^{\sigma_1} = \sum_{x \in G} r_x x^{\sigma}$.

For $x \in G$ define the $R[H^x]$-module

$$V^x = V \otimes x = \{v \otimes x | v \in V\}$$

Where $(v \otimes x)y^x = vy \otimes x$ for $y \in H$. clearly $V^x \approx V^{\sigma}$ where $z^{\sigma} = x^{-1}zx$ for all $z \in H$.

Lemma 5: Let H be a subgroup of G. Let V be an $R[G]$ module and let W be an $R[H]$ module. Then $V\otimes W^G \approx (V_H \otimes W)^G$.

Proof: Let $\{x_i\}$ be a cross section of H in G. Define f: $V\otimes W^G \rightarrow (V_H \otimes W)^G$ by

$$f\{v\otimes (w\otimes x_i)\}=(vx_i^{-1}\otimes w)\otimes x_i$$

for $v\in V$, $w\in W$. Clearly f is an R-isomorphism. Let $x\in G$. Suppose that $x_i x=yx_j$ with $y\in H$. Then

$$f\{v\otimes (w\otimes x_i)\}x=(vx_i^{-1}\otimes w)y\otimes x_j$$

$$=(vx_i^{-1}y\otimes wy)\otimes x_j=(vxx_j^{-1}\otimes wy)\otimes x_j$$

and

$$f[\{v\otimes (w\otimes x_i)\}x]=f\{vx\otimes (wy\otimes x_j)\}=(vxx_j^{-1}\otimes wy)\otimes x_j$$

Thus f is an $R[G]$ homomorphism (REN & Shum, 2004) as required. The lemma is proved. □

Let V be an $R[G]$ module. An element $v\in V$ is a G-invariant element (HUANG & GUO, 2007) or simply an invariant element if $vx=v$ for all $x\in G$. Let $Inv_G(V)$denote the set of G-invariant elements in V. clearly $Inv_G(V)$ is an R module.

Let V, W be $R[G]$ modules. For $f\in Hom_R(V, W)$ and $x\in G$ define fx by $(v)(fx)=\{(vx^{-1})f\}x$ for $v\in V$. clearly $fx\in Hom_R(V, W)$ and $(fx)y=f(xy)$. In this way $Hom_R(V, W)$ becomes an $R[G]$ module.

If V, W are finitely generated R-free $R[G]$ modules of rank m and n respectively then $Hom_R(V, V)\approx R_m$, $Hom_R(W, W)\approx R_n$ and $Hom_R(V, W)$ consists of all $m\times n$ matrices with entries in R. For $x\in G$ let a_x ,b_x respectively be the map sending v to vx, w to wx respecticely for $v\in V$, $w\in W$. Then $a_x\in Hom_R(W, W)$ and if $f\in Hom_R(V, W)$ then $fx=a_x^{-1}fb_x$. In case $W=R$ with $rx=r$ for all $r\in R$ and $x\in G$, $Hom_R(V, R)=\hat{V}$ as an R module. If $f\in \hat{V}$ and $v\in V$, $(v)(fx)=(vx^{-1})f=v(x^{-1}f)$ for $x\in G$. $Hom_R(V, R)$ made into an $R[G]$ module in this way will be dentoed by $V^{*(R)}$ or simply V^* if R is determined by context.

Lemma 6: Let V, W be $R[G]$ modules. Then $Inv_G(Hom_R(V, W))= Hom_{R[G]}(V, W)$.

Proof: By definition $f\in Inv_G(Hom_R(V, W))$ if and only if $f=fx$ for all $x\in G$. This is the case if and only if $(v)f=(v)(fx)=\{vx^{-1}\}f\}x$ for all $v\in V$, $x\in G$. This last condition is equivalent to the fact that $f\in Hom_{R[G]}(V, W)$. The lemma is proved. □

Lemma 7: Let H be a subgroup of G. Let V be an $R[G]$ module and let W be an $R[H]$ module.Then

(i) $\{Hom_R(W, V_H)\}^G \approx Hom_R(W^G, V)$.

(ii) $\{Hom_R(V_H, W)\}^G \approx Hom_R(V, W^G)$.

Proof. Let $\{x_i\}$ be a cross section of H in G.

(i) If f$\in \{Hom_R(W, V_H)\}^G$ then $f=\sum f_i\otimes x_i$ for $f_i\in Hom_R(W, V_H)$. Define $\hat{f}\in Hom_R(W^G, V)$ by $(\sum w_i\otimes x_i)\hat{f}=\sum\{(w_i)f_i\}x_i$. Let g be the map sending f to $\hat{f}$. Clearly g is an R-homomorphism. If $\hat{f}=0$ then $\sum\{(w_i)f_i\}x_i=0$ for all $w_i\in W$ and so $f_i=0$ for all i.Thus $f=0$. Therefore g is a monomorphism. If

$h\in Hom_R(W^G, V)$ then $(\sum w_i\otimes x_i)h=\sum\{(w_i)h_i\}x_i$ for some $h_i\in Hom_R(W, V_H)$. Hence $h=\hat{f}$ with $f=\sum h_i\otimes x_i$ and so g is an epimorphism.

Let $x\in G$. For each i, $x_ix=y_ix_i$ where $y_i\in H$ and $x_i\rightarrow x_i$ is a permutation of $\{x_i\}$. Thus if $f=\sum f_i\otimes x_i$ then $fx=\sum f_iy_i\otimes x_i$. Furthermore

$$\left(\sum w_i\otimes x_i\right)(\hat{f}x)=\left\{\left(\sum w_i\otimes x_ix^{-1}\right)\hat{f}\right\}x$$

$$=\left\{\left(\sum w_iy_i^{-1}\otimes x_i\right)\hat{f}\right\}x=\sum\{(w_iy_i^{-1})f_i\}x_ix$$

and

$$\left(\sum w_i\otimes x_i\right)(\widehat{fx})=\left(\sum w_i\otimes x_i\right)\left(\sum f_iy_i\otimes x_i\right)$$

$$=\sum\{(w_i)(f_iy_i)\}x_i=\sum\{(w_iy_i^{-1})f_i\}y_ix_i$$

Since $y_ix_i=x_ix$ this implies that g is an $R[G]$ isomorphism.

(ii)If $f\in\{Hom_R(V_H, W)\}^G$ then $f=\sum f_i\otimes x_i$ where $f_i\in Hom_R(V_H, W)$. Define $\hat{f}\in Hom_R(V, W^G)$ by $(v)\hat{f}=\sum(vx_i^{-1})f_i\otimes x_i$. Let g be the map sending f to $\hat{f}$. It is easily seen that g is an R-isomorphism.

Let $x\in G$, Let $x_ix=y_ix_i$ with $y_i\in H$ for each i. Then if $f=\sum f_i\otimes x_i$

$$(v)(\hat{f}x)=\{(vx^{-1})\hat{f}\}x=\sum(vx^{-1}x_i^{-1})f_i\otimes x_ix=\sum\{(vx^{-1}x_i^{-1})f_i\}y_i\otimes x_i$$

and

$$(v)(\widehat{fx})=(v)\left(\sum f_iy_i\otimes x_i\right)=\sum\{(vx_i^{-1}y_i^{-1})f_i\}y_i\otimes x_i$$

Since $x_i^{-1}y_i^{-1}=x^{-1}x_i^{-1}$ this implies that g is an $R[G]$ isomorphism. The lemma is proved. □

Lemma 8: Assume that R satisfies A.C.C. Let H be a subgroup of G. Let U, V be finitely generated $R[G]$ module The following hold.

(i)If V is R-free and U is a projective or free $R[G]$ module then each of $U\otimes V$, U^*, $Hom_R(U, V)$ and $Hom_R(V, U)$ is a projective or free $R[G]$ module respectively.

(ii) If U is $R[H]$-projective then $U\otimes V$, U^*, $Hom_R(U, V)$ and $Hom_R(V, U)$ are all $R[H]$-projective.

Proof: (i) Let $V_{<1>}=nRn$. By Lemma 5 $V\otimes mR_R^G\approx mnR_R^G$ is free. By Lemma 4 and Lemma 7

$$Hom_R(mR_R^G, V)\approx\{Hom_R(mR_R, nR_R)\}^G\approx mn\{Hom_R(R_R, R_R)\}^G\approx mnR_R^G$$

is free and

$$Hom_R(V, mR_R^G)\approx\{Hom_R(nR_R, mR_R)\}^G\approx mnR_R^G$$

is free The rest of (i) follows from (ii) with H=<1>.

(ii) By Lemma 3 $U\mid(U_H)^G$. Hence by Lemma 5

$$U \otimes V \mid (U_H)^G \otimes V \approx (U_H \otimes V_H)^G = \{(U \otimes V)_H\}^G$$

Thus by Lemma 3 $U \otimes V$ is $R[H]$-projective. By Lemma 7

$$Hom_R(U, V) \mid Hom_R((U_H)^G, V) \approx \{Hom_R(U, V)_H\}^G$$

$$Hom_R(V, U) \mid Hom_R(V, (U_H)^G) \approx \{Hom_R(V, U)_H\}^G$$

Thus $Hom_R(U, V)$ and $Hom_R(V, U)$ are $R[H]$-projective by Lemma 3. The lemma is proved. □

Lemma 9: Let H be a subgroup of G and let W be an module. Then $v \in Inv_G(W^G)$ if and only if $v = Tr_H^G(w \otimes 1)$ for some $w \in Inv_H(W)$. Furthermore $H^0(G, H, W^G) = (0)$.

Proof: If $v \in Inv_H(W)$ then $Tr_H^G(w \otimes 1) \in Inv_G(W^G)$.

Let $\{x_i\}$ be a cross section of H in G with x_1=1. Suppose that $v = \sum w_i \otimes x_i \in Inv_G(W^G)$ for $w_i \in W$. Then for each j and each $y \in H$

$$v = vx_j^{-1} y \otimes 1 + \sum_{i>1} w_i ' \otimes x_i$$

for suitable $w_i ' \in W$. Thus $w_j y = w_1$ for all $y \in H$. Hence $w_1 \in Inv_H(V)$ and $v = \sum_i w_1 \otimes x_i = Tr_H^G(w_1 \otimes 1)$. The lemma is proved. □

Lemma 10: Let H be a subgroup of G. Let V, W be finitely generated $R[G]$ modules where V is $R[H]$-projective. Then

(i) $H^0(G, H, V) = (0)$.

(ii) $H^0(G, H, Hom_R(V, W)) = H^0(G, H, Hom_R(W, V)) = (0)$

Proof: (i) By Reference [10] $V \mid (V_H)^G$. Hence by Lemma 9

$$H^0(G, H, V) \mid H^0(G, H, (V_H)^G) = (0)$$

(ii) Immediate by (i) and Lemma 8. The lemma is proved. □

Lemma 11: Theorem: Let H be a subgroup of G and let V be a finitely generated $R[G]$ module. The following are equivalent.

(i) V is $R[H]$-projective.

(ii) $V|(V_H)^G$.

(iii) $V|W^G$ for some finitely generated $R[H]$ module W.

(iv) $Hom_R(V, V)$ is $R[H]$-Projectice.

(v) $H^0(G, H, Hom_R(V, V))=(0)$.

(vi) There exists $f \in Hom_{R[H]}(V, V) = Inv_H\{Hom_R(V, V)\}$ such that $Tr_H^G(f) = 1$.

(vii) V is $R[H]$-injective.

Proof: By Lemma 3 (i), (ii) and (iii) are equivalent.

(iii)→(iv). Clear by Lemma 8.

(iv)→(v). Clear by Lemma 10.

(v)→(vi). Immediate by definition and Lemma 6.

(vi)→(vii). suppose that W is an $R[G]$ module with $V\subseteq W$ such that $V_H|W_H$. Thus there exists a projection e of W onto V which is an $R[H]$-homomorphism. Hence $Tr_H^G(ef)\in Hom_{R[G]}(W, W)$.Let $\{x_i\}$be a cross section of H in G. If $w\in W$ then

$$wTr_H^G(ef)=\sum_i\{(wx_i^{-1})ef\}x_i\in\sum_i(Wef)x_i\subseteq V$$

And if $v\in V$ then

$$vTr_H^G(ef)=\sum_i\{[(vx_i^{-1})e]f\}x_i=\sum_i\{(vx_i^{-1})f\}x_i=v\sum(fx_i)=vTr_H^G(f)=v$$

Hence $Tr_H^G(ef)$ is a projection of W onto V and so $V|W$ as required.

(vii)→(ii). Let$\{x_i\}$ be a cross section of H in G with $x_1=1$. Define g: $V\to(V_H)^G$ by $gv=\sum_i vx_i^{-1}\otimes x_i$.

Thus $g=Tr_H^G(h)$ where h:$V\to V\otimes 1$ with $hv=v\otimes 1$. If $gv=0$ then $vx_1^{-1}\otimes x_1=v\otimes 1=0$. Hence g is an $R[G]$-monomorphism. Let $W=\{\sum_{i\neq 1}v_i\otimes x_i\}$. Then W is an $R[H]$ module. Clearly $g(V)\cap W=(0)$. If $\sum v_i\otimes x_i\in V^G$ then

$$\sum v_i\otimes x_i=\sum_i v_1x_i^{-1}\otimes x_i+\sum_i(v_i-v_1x_i^{-1})\otimes x_i$$

$$=g(v_1)+\sum_{i\neq 1}(v_i-v_1x_i^{-1})\otimes x_i\in g(V)+W$$

Hence $\{(V_H)^G\}_H=g(V)_H\oplus W$. Therefore $V_H|\{(V_H)^G\}_H$ and so $V|(V_H)^G$ since V is $R[H]$-injective. The theorem is proved. □

Lemma 12: Let P be a finitely generated A module. The following are equivalent.

(i) P is projective.

(ii)$P|V$ for some finitely generated free A module.

(iii)Every diagram

$$\begin{tikzcd} & P \arrow[d] & \\ U \arrow[r] & V \arrow[r] & 0 \end{tikzcd}$$

with U, V finitely generated A modules in which the row is exact can be completed to a commutative diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[d] & \\ U \arrow[r] & V \arrow[r] & 0. \end{tikzcd}$$

Proof: It is clear by definition above.□

Lemma 13: Let B be a subring of A with $1\in B$ such that B satisfies A.C.C., A_B is a finitely generated free B module and ${}_BA$ is a finitely generated left B module, Let V be a finitely generated A module, Then V is projective if and only if V_B is a projective B module and V is B-projective.

Proof: Clear by definition and lemma 12.□

Lemma 14: Suppose that $|G:H|$ has an inverse in R for some subgroup H of G. Then every finitely generated $R[G]$ module is $R[H]$-projective.

Proof: Let $f=(1/|G:H|)1\in Hom_{R[H]}(V, V)$, Then $Tr_H^G(f)=1$ .The result follows from lemma 11.□

Lemma 15: Suppose that $|G|$ has an inverse in R. Then every finitely generated $R[G]$ module is projective and every finitely generated R-free $R[G]$ module is projective. If furthermore V is an indecomposable $R[G]$ module and W is a submodule of V with $W_R|V_R$ then $W=(0)$ or $W=V$.

Proof: Clear by lemma 14 and lemma 13. □

Theorem: Let F be a field whose characteristic dose not divide $|G|$. Then every finitely generated $F[G]$ module is completely reducible.

Proof: By lemma 15 every finitely generated $F[G]$ module is projective. Thus every finitely generated $F[G]$ module is completely reducible. Since $F[G]_{F[G]}$ is completely reducible it follows that $F[G]$ is semi-simple. □

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