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Control of Chaos in the Dynamics of Enceladus

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Abstract: We estimate a localized control term for the Hamiltonian of spin orbit dynamics of Enceladus. The aim is to modify the perturbation locally by a smaller and simple but robust control term which makes the controlled Hamiltonian more regular. Poincare surface of sections witnesses that the estimated control term is able to recreate invariant (KAM) tori without modifying other parts of phase space.
Keywords: Chaos control; Poincare surface of section; Enceladus (Satellite)

1. INTRODUCTION

The mathematical description of chaos has reached a mature state over the last decade using the tools classical phase space, time series analysis, Poincare sections, Lyapunov exponents, etc. and much attention has been paid to control the chaos since it can be harmful in several contexts. That is why now a day's controlling of chaos has become a key challenge in the chaos existing branches of nonlinear sciences. Most of the methods for controlling chaos in the chaotic systems are done by tilting targeted trajectories. For many body experiments (e.g. the magnetic confinement of plasma, satellite's dynamics and the control of turbulent flows, etc.), successful attempts has been done by Pyragas (1992), Ott et al. (1990) and Tsui & Jones (2000) dealing the high number of trajectories simultaneously.

Here we focus on the strategy to control transport properties without significantly altering neither the original structure of the system under investigation nor its overall chaotic structure which is based on building barriers by adding a small perturbation which is localized in phase space, hence confining all the trajectories. The main motivations for a localized control are the following ones: Very often the control of a physical system can only be performed in some specific regions of phase space. For some purposes it is sometimes desirable to stabilize only a given region of phase space without modifying the major part of phase space in order to preserve some specific features of the system. This method can be used to bound the motion of particles without changing the perturbation inside (and outside) the barrier. Also, using a localized control means that one needs to inject much fewer energy than a global control in order to create isolated barriers of transport. There exist numerous attempts in this direction to control the chaos by Ciraolo et al. (2004a, 2004b, 2004c) and Khan & Shahzad (2008).

Here the meaning of control is that one aims at reducing or suppressing chaos inducing a relevant change in the transport properties by means of a small perturbation so that the original structure of the system under investigation is substantially kept unaltered that has been shown by Ciraolo et al. (2004a, 2004b, 2004c) and Khan & Shahzad (2008) in which a very small and suitably chosen perturbations can indeed be an efficient control for Hamiltonian systems which are perturbations of integrable ones. Khan

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& Shahzad (2008) has studies the Hamiltonian system of Mimas-Tethys system (The natural satellites of Saturn) keeping the original structure of the system unaltered. In their study, they have estimated a robust control term and shown that the chaos in the Mimas-Tethys system is suppressed drastically.

Keeping in view the above discussions, we study the chaos control for the spin-orbit dynamics of Enceladus (Wisdom 2004), which is close to integrable. We consider the class of Hamiltonian system of the spin orbit dynamics of Enceladus that can be written in the form of $H = H_0 + eV$ that is an integrable Hamiltonian H_0 (with an action-angle variable) plus a small perturbation eV. For the small perturbation, we have obtained a control term to control the chaotic diffusion in the dynamics of the system under consideration. The main advantage of the control term is that it is explicit in nature. Due to its explicit nature, we have opportunity to study simultaneously the dynamics of the system with and without control term.

For the perturbed Hamiltonian $H = H_0 + eV$, a control term has been estimated (Ciraolo et al. 2004b, Khan & Shahzad 2008, Vittot 2004) up to $O(e^2)$. The inclusion of this term in the above tested Hamiltonian gives us the more regular dynamics or less diffusion than the uncontrolled Hamiltonian.

2. THE DYNAMICS OF ENCELADUS

In their analysis of Voyager images of Enceladus, Dermott & Thomas (1994) found that the shape of Enceladus was well represented by a triaxial ellipsoid with principal radii $a = 256.3 \pm 0.3$ km, $b = 247.3 \pm 0.3$ km, and $c = 244.6 \pm 0.3$ km. They interpret this figure in terms of hydrostatic models and deduce that Enceladus has a density below 1.12 gm/cm^3, with a preferred value near 1.00 gm/cm^3. The ratio of moments of inertia determines the spin-orbit libration frequency. Assuming uniform density, the implied $(B-A)/C$ is 0.03573 ± 0.0017. More to the point, the implied ratio of the librational frequency to the orbital frequency is $\varepsilon = \{3(B-A)/C\}^{1/2} = 0.3274 \pm 0.0077$. The error is purely formal; the real error is probably larger. This near commensurability has very interesting dynamical consequences. For ε near $\frac{1}{3}$, there is period-three bifurcation in the spin-orbit phase space. On a surface of section, a chain of three secondary islands appears near the center of the synchronous island. Now, if Enceladus is locked in this period three island, then there is a forced libration of the figure of Enceladus relative to Saturn. Depending on the system parameters, the rate of tidal heating due to this forced secondary resonance libration can be enhanced by a factor of 100 to 1000 over the rate of tidal heating without secondary resonance libration. A number of natural satellites are significantly out of round, with ε larger than $\frac{1}{3}$. For instance, Dermott & Thomas (1988) measured the elliptical radii for Mimas. From these, ε for Mimas is estimated to be near 0.44. Such satellites also display secondary resonances on their surfaces of section, but because ε is not near a low-order commensurability, these secondary islands are not near the synchronous island center and so are unlikely to play any role in the dynamics of these satellites. Wisdom (2004) also examined the dynamics of the 3:1 secondary spin-orbit resonance, computed the consequent rate of tidal heating, and discussed mechanisms that might have placed Enceladus into this unique dynamical state.

3. CONTROL TERM FOR HAMILTONIAN OF ENCELADUS

An approximate model for the 3:1 secondary resonances for synchronous rotation in the spin-orbit problem using the standard techniques of Hamiltonian perturbation theory (Sussman & Wisdom 2001) was developed by Wisdom (2004). It is based on approximation for a fixed orbit the planet-to-satellite distance and the true anomaly is periodic. Expanding these as Fourier series, the spin-orbit Hamiltonian is

$$H(p,\theta,t)=\frac{p^2}{2C}-\frac{\varepsilon^2 n^2 C}{4}\sum_k \mathrm{H}_k(e)\cos(2\theta-knt) \tag{3.1}$$

Where $\varepsilon=\sqrt{3(B-A)/C}$, the moments of inertia are $A<B<C$, n is the orbital frequency, θ measures the orientation of the axis of minimum moment from the line to pericenter, p is the angular momentum conjugate to θ, with the coefficients $\mathrm{H}_2(e)=1-\frac{5e^2}{2}$, $\mathrm{H}_3(e)=\frac{7e}{2}$, and $\mathrm{H}_1(e)=-\frac{e}{2}$, to second order in *e*.

Considering the terms in (3.1) as perturbation that depends on the eccentricity, the Hamiltonian of the spin orbit dynamics of Enceladus is given by

$$H=H_0+H_1, \tag{3.2}$$

Where

$$H_0=\frac{p^2}{2C}-\frac{\varepsilon^2 n^2 C}{4}\cos(2\theta-2nt),$$

$$H_1=\frac{\varepsilon^2 n^2 Ce}{8}\left\{\cos(2\theta-nt)-7\cos(2\theta-3nt)\right\},$$ keeping only the terms linear in *e*.

To apply the control theory described by Vittot (2004), we put Hamiltonian (3.2) in an autonomous form. Consider that *t* is an additional angle whose conjugate action is *E*. The autonomous Hamiltonian is expressed by

$$H(p,\theta,E,t)=\frac{p^2}{2C}-\frac{\varepsilon^2 n^2 C}{4}\cos(2\theta-2nt)+E+\frac{\varepsilon^2 n^2 Ce}{8}\left\{\cos(2\theta-nt)-7\cos(2\theta-3nt)\right\}. \tag{3.3}$$

Where, the actions are $\mathrm{A}=(p,E)$ and the angles are $\Phi=(\theta,t)$. The unperturbed Hamiltonian which is used to construct the operators Γ, $\Re$ and N is

$$H_0=\frac{p^2}{2C}-\frac{\varepsilon^2 n^2 C}{4}\cos(2\theta-2nt)+E, \tag{3.4}$$

The action of $\{H_0\}$, Γ, $\Re$ and N on function $V(p,\theta,E,t)=\sum_{r_1,r_2\in Z} V_{r_1,r_2}(p,E)e^{i(r_1\theta+r_2 t)}$, $V\in\mathbb{A}$ is given by

$$\{H_0\}V=\sum_{r_1,r_2\in Z} i(r_1 p+r_2)V_{r_1,r_2}(p,E)e^{i(r_1\theta+r_2 t)},$$

$$\Gamma V=\sum_{r_1,r_2\in Z}\frac{\chi(r_1 p+r_2\neq 0)}{i(r_1 p+r_2)}V_{r_1,r_2}(p,E)e^{i(r_1\theta+r_2 t)},$$

$$\Re V=\sum_{r_1,r_2\in Z}\chi(r_1p+r_2=0)V_{r_1,r_2}(p,E)e^{i(r_1\theta+r_2t)},$$

$$\mathrm{N}\ \ V=\sum_{r_1,r_2\in Z}\chi(r_1p+r_2\neq 0)V_{r_1,r_2}(p,E)e^{i(r_1\theta+r_2t)}.$$

When $V(\theta,t)=\dfrac{\varepsilon^2n^2Ce}{8}\left\{\cos(2\theta-nt)-7\cos(2\theta-3nt)\right\}$, is acted upon by the above operators, we obtain

$$\{H_0\}V=\frac{-\varepsilon^2n^2Ce}{8}\left\{\left(\frac{2p}{C}-n\right)\sin(2\theta-nt)-7\left(\frac{2p}{C}-3n\right)\sin(2\theta-3nt)\right\},$$

$$\Gamma V=\frac{\varepsilon^2n^2Ce}{8}\left\{\frac{\sin(2\theta-nt)}{(\frac{2p}{C}-n)}-\frac{7\sin(2\theta-3nt)}{(\frac{2p}{C}-3n)}\right\},$$

$$\Re V=0,$$

$$\mathrm{N}\ \ V=\frac{\varepsilon^2n^2Ce}{8}\left\{\cos(2\theta-nt)-7\cos(2\theta-3nt)\right\},\qquad \text{for } p\neq \tfrac{nC}{2},\tfrac{3nC}{2}.$$

Here, the control term is given by $g(V)=\sum_{s=2}^{\infty}f_s$, where f_s is the order of e^s and given by the formula $f_s=\dfrac{1}{s}\{\Gamma V,\,f_{s-1}\}$, where $f_1=V$ and the term in the series are computed by recursion. In particular, we write the control term

$$g(p,\theta,t)=f_2(p,\theta,t)=-\frac{1}{2}\{\Gamma V,V\},$$

$$g(p,\theta,t)=-\frac{\varepsilon^4n^4e^2C}{32}\left[\frac{\sin^2(2\theta-nt)}{(\frac{2p}{C}-n)^2}-\frac{7\sin(2\theta-nt)\sin(2\theta-3nt)}{(\frac{2p}{C}-n)^2}\right.$$
$$\left.-\frac{7\sin(2\theta-nt)\sin(2\theta-3nt)}{(\frac{2p}{C}-3n)^2}+\frac{49\sin^2(2\theta-3nt)}{(\frac{2p}{C}-3n)^2}\right].$$

The main purpose of the control is to have a control term which is as simple as possible in order to be implemented in the experimental results. A possible simplification is to consider the region in between the two primary resonances located around $p=\frac{nC}{2},\frac{3nC}{2}$. The control term developed around $p=nC$ becomes

$$g(p,\theta,t)=-\frac{\varepsilon^4n^4e^2C}{32}\left[\sin(2\theta-nt)-7\sin(2\theta-3nt)\right]^2.$$

In order to make the control term more effective, we change the amplitude of the control term by introducing β. Thus, the more effective and the finally approximated control term may be written as:

$$g(p,\theta,t)=-\beta\varepsilon^4n^4e^2C\left[\sin(2\theta-nt)-7\sin(2\theta-3nt)\right]^2.$$

Figure 1 : Poincare Surface without control term

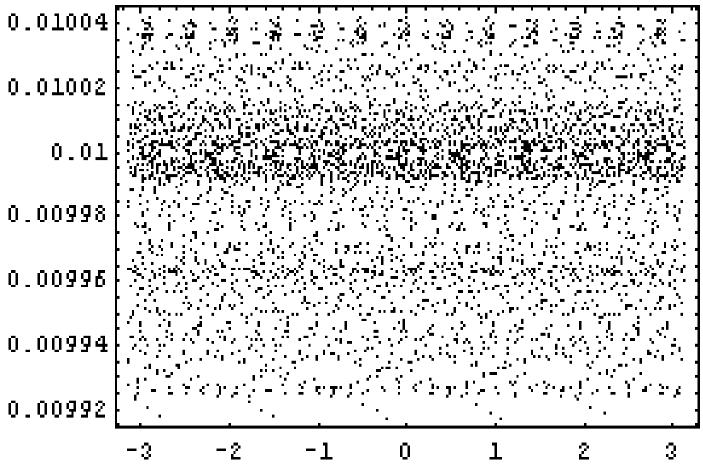
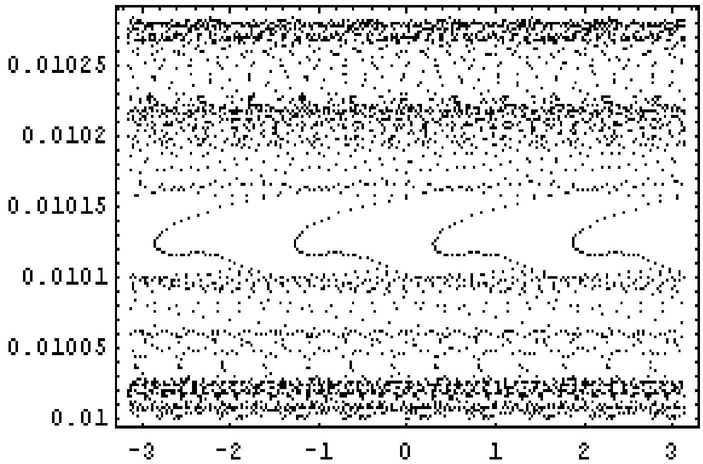


Figure 2 : Poincare Surface with control term



For $\varepsilon = 0.33; e = 0.0045; C = 0.009; n = 0.01; \theta(0) = 0; p(0) = 0.01; \beta = 0.25;$ Poincare surface in figures 1 and 2 depict the dynamics of the system under consideration without and with the estimated control term respectively. In figure 2, it is clear that the dynamics of the Enceladus becomes more regular including the control term (*g*). The Poincare surface in figure 1 depicts the jargon of chaos whereas the Poincare surface in figure 2 depicts the recreation of lots of invariant (KAM) tori.

4. CONCLUSION

We have estimated an explicit and effective control term to reduce the chaotic diffusion in the dynamics of Enceladus using small perturbations. Since the formula of the control term is explicit, we are able to compare the dynamics without and with control. The inclusion of the above estimated control term in the Hamiltonian of the system under investigation gives us more regular dynamics near the actual values of ε and *e*. The computational studies of chaos control is represented through Figures 1 which showcases the Poincare surface of section of the spin orbit dynamics of Enceladus without control term while figure 2 exhibits the Poincare surface of section including the control term. Figure 2 clearly indicates that the control draws the considered system towards regular behavior in the localized phase space. Our computational studies reveal that although the estimated control term is smaller as compared to the perturbation, yet this control is robust which is clear through the Poincare surface of sections.

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