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Oscillation for a High Order Nonlinear Difference Equations

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Abstract

In this paper, we consider certain nonlinear difference equations

$$\Delta^2(|\Delta^2 y_n|^{\alpha-1}\Delta^2 y_n) + q_n|y_{\tau(n)}|^{\beta-1}y_{\tau(n)} = 0\ ,$$

where

(a) α, β are positive constants;

(b) $\{q_n\}_{n0}^{\infty}$ are positive real sequences. $n_0 \in N_0 = \{1,2\cdots\}$.

Oscillation and nonoscillation theorems of the above equation is obtained.

Key words: Nonlinear difference equations; Oscillation; Nonoscillation; High order

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INTRODUCTION

In this paper, we consider certain quasilinear difference equations

$$\Delta^2(|\Delta^2 y_n|^{\alpha-1}\Delta^2 y_n) + q_n|y_{\tau(n)}|^{\beta-1}y_{\tau(n)} = 0\ ,$$

where

(a) α, β are positive constants;

(b) $\{q_n\}_{n0}^{\infty}$ are positive real sequences. $n_0 \in N_0 = \{1,2\cdots\}$;

(c) $\tau(n) \leq n$, $\lim_n \to\infty\ , \tau(n) = \infty$.

The equation which can also be expressed as

$$\Delta^2(|\Delta^2 y_n|^{\alpha*}) + q_n|y_{\tau(n)}|^{\beta*} = 0\ , \qquad (1)$$

in terms of the asterisk notation

$$\xi^{\gamma*} = |\xi|^{\gamma}\,\mathrm{sgn}\,\xi = |\xi|^{\gamma}\xi,\ \xi \in R,\ \gamma > 0.$$

It is clear that if $\{y_n\}$ is a eventually positive solution of (1), then $-\{y_n\}$ is a eventually negative solution of (1).

Lemma 1.1. Assume that $\{y_n\}$ is a eventually positive solution of (1). then one of the following two cases holds for all sufficiently large n:

I : $\Delta y_n > 0$, $\Delta^2 y_n > 0$, $\Delta(\Delta^2 y_n)^{\alpha*} > 0$,

II : $\Delta y_n > 0$, $\Delta^2 y_n < 0$, $\Delta(\Delta^2 y_n)^{\alpha*} > 0$.

Proof. From (1.1), we have $\Delta^2(\Delta^2 y_n)^{\alpha*} < 0$ for all large n. It follows that Δy_n, $\Delta^2 y_n$, $\Delta(\Delta^2 y_n)^{\alpha*}$ are eventually monotonic and one-signed.

a) if $\Delta(\Delta^2 y_n)^{\alpha*} < 0$ eventually. Then combining this with $\Delta^2(\Delta^2 y_n)^{\alpha*} < 0$,we see that $\lim_{n\to\infty}(\Delta^2 y_n)^{\alpha*} = -\infty$. That is $\Delta^2 y_n \to \infty$ for all large n. It follows that $\Delta y_n \to -\infty$, $y_n \to -\infty$, which contradicts the positivity of $\{y_n\}$.

b) if $\Delta(\Delta^2 y_n)^{\alpha*} > 0$ eventually. Then combining this with $\Delta^2(\Delta^2 y_n)^{\alpha*} < 0$,we see that $\Delta^2(\Delta^2 y_n)^{\alpha*} \to 0$ or $\to a > 0$ so

$$(\Delta^2 y_n)^{\alpha_*} = (\Delta^2 y_N)^{\alpha_*} + \sum_{N}^{n-1}(\Delta^2 y_n)^{\alpha_*}.$$

If $(\Delta^2 y_n)^{\alpha*} > 0 > 0$, That is $\Delta^2 y_n > 0$ is increasing and $\to C$ or ∞. It follows that $\Delta y_n > 0$; if $(\Delta^2 y_n)^{\alpha*} < 0$, that is $\Delta^2 y_n < 0$ is increasing and $\to d$ or 0. If $\Delta y_n < 0$ Then $y_n \to \infty$, it is impossible, so $\Delta y_n > 0$. This complete the proof of the lemma.

From Lemma (1.1), we know Δy_n, $\Delta^2 y_n$, $\Delta(\Delta^2 y_n)^{\alpha*}$ tend to finite or infinite limits as $n \to \infty$. Let

$\lim_{n\to\infty}\Delta^i y_n = \omega_i$, i = 0,1,2 and $\lim_{n\to\infty}\Delta(\Delta^2 y_n)^{\alpha*} = \omega_3$.

It is that ω_3 is a finite nonnegative number. One can easily show that:

If y_n satisfies I, then the set of its asymptotic values ω_i falls into one of the following three cases:

I_1: $\omega_0 = \omega_1 = \omega_2 = \infty$, $\omega_3 \in (0,\infty)$,

I_2: $\omega_0 = \omega_1 = \omega_2 = \infty$, $\omega_3 \in (0,\infty)$,

I_3: $\omega_0 = \omega_1 = \infty$, $\omega_2 \in (0,\infty)$, $\omega_3 = 0$.

If y_n satisfies II, then the set of its asymptotic values ω_i falls into one of the following three cases:

II_1: $\omega_0 = \infty$, $\omega_1 \in (0,\infty)$, $\omega_2 = \omega_3 = 0$,

II_2 : $\omega_0 = \infty$, $\omega_1 = \omega_2 = \omega_3 = 0$,
II_3 : $\omega_0 \in (0,\infty)$, $\omega_1 = \omega_2 = \omega_3 = 0$.

Equivalent expressions for these six classes of positive solutions of (1) are as follows:

I_1: $\lim_{n\to\infty} \frac{y_n}{n^{2+\frac{1}{\alpha}}} = \text{const} > 0$;

I_2: $\lim_{n\to\infty} \frac{y_n}{n^{2+\frac{1}{\alpha}}} = 0, \lim_{n\to\infty} \frac{y_n}{n^2} = \infty$,

I_3: $\lim_{n\to\infty} \frac{y_n}{n^{2+\frac{1}{\alpha}}} = \text{const} > 0$.

II_1: $\lim_{n\to\infty} \frac{y_n}{n} = \text{const} > 0$,

II_2: $\lim_{n\to\infty} \frac{y_n}{n} = 0, \lim_{n\to\infty} y_n = \infty$,

II_3: $\lim_{n\to\infty} y_n = \text{const}$.

Let y_n be a positive solution of (1), such that $y_n > 0$, $y_{\tau(n)} > 0$ for $n \ge N > n_0$. Summing (1) from n to ∞ gives

$$\Delta(\Delta^2 y_n)^{\alpha_*} = \omega_3 + \sum_{s=n}^{\infty} q_s (y_{\tau(n)})^{\beta}, \quad n \ge N. \tag{2}$$

If y_n is a solution of type $I_i (i = 1,2,3)$, then sum (2) three times over $[N, n-1]$ to obtain

$$y_n = k_0 + k_1(n-N) + \sum_{s=N}^{n-1}(n-s)[k_2{}^{\alpha} + \sum_{r=n}^{s-1}(\omega_3 + \sum_{\sigma=r}^{\infty} q_{\sigma}(y_{\tau(\sigma)})^{\beta})]^{\frac{1}{\alpha}}, \tag{3}$$

for $n \ge N$ where $k_0 = y_N$, $k_1 = \Delta y_N$, $k_2 = \Delta^2 y_N$ are nonnegative constants. The equality (3) gives a representation for a solution y_n of *type*$-I_1$. A *type*$-I_2$ solution y_n of (1) is expressed by (3) with $\omega_3 = 0$

If y_n is a solution of type I_3, then, first summing (1) from n to ∞ and then summing the resulting equation twice times over $[N; n-1]$ to obtain

$$y_n = k_0 + k_1(n-N) + \sum_{s=N}^{n-1}(n-s)[\omega_2{}^{\alpha} - \sum_{r=s}^{\infty}(r-s)q_r(y_{\tau(\sigma)})^{\beta}]^{\frac{1}{\alpha}}, \quad n > N. \tag{4}$$

A representation for a solution y_n of type II_1 is derived by summing (2) with $\omega_3 = 0$. Twice from n to ∞ and then once from N to $n-1$:

$$y_n = k_0 + \sum_{s=N}^{n-1}(\omega_1 + \sum_{r=s}^{\infty}[\sum_{\sigma=r}^{\infty}(\sigma - r)q_{\sigma}(y_{\tau(\sigma)})^{\beta}]^{\frac{1}{\alpha}}), \quad n > N, \tag{5}$$

a representation for a solution y_n of type II_2 is given by (5) with $\omega_1 = 0$. A representation for a solution y_n of type II_3 is derived by summing (2) with $\omega_3 = 0$ three times from n to ∞ yield

$$y_n = \omega_0 - \sum_{s=n}^{\infty}(s-n)[\sum_{r=s}^{\infty}(r-s)q_r(y_{\tau(\sigma)})^{\beta}]^{\frac{1}{\alpha}}, \quad n > N \tag{6}$$

MAIN RESULTS

Theorem 1. The Equation (1) has a positive of *type* $-$ I_1 if and only if

$$\sum_{n=n_0}^{\infty} q_n(\tau(n))^{2+\frac{1}{\alpha}}\beta < \infty. \tag{7}$$

Proof. Neccessary. Suppose that (1) has a positive of *type* $-$ I_1. Then it satisfies (3) for $n \ge N$, which implies that

$$\sum_{n=N}^{\infty} q_n(\tau(n))^{\beta} < \infty.$$

This together with the asymptotic relation $\lim_{n\to\infty} \frac{y_n}{n^{2+\frac{1}{\alpha}}} = \text{const} > 0$ shows that the condition (7) is satisfied.

Sufficently. Suppose now that (7) holds. Let $k > 0$ be any given constant.

Choose $N > n_0$ large enough so that

$$\left(\frac{\alpha^2}{(\alpha+1)(2\alpha+1)}\right)^{\beta} \sum_{n=n_0}^{\infty} q_n(\tau(n))^{2+\frac{1}{\alpha}}\beta \le \frac{(2k)^{\alpha} - k^{\alpha}}{(2k)^{\beta}}. \tag{8}$$

Put $N_* = \min\{N, \inf_{n>N} \tau(n)\}$, and define

$$\begin{array}{ll} G(n,N) = \sum_{s=n}^{n-1}(n-s)(s-n)^{\frac{1}{\alpha}} = \frac{\alpha^2}{(\alpha+1)(2\alpha+1)}(n-N)^{\frac{2}{1+\alpha}} & n \ge N \\ G(n,N) = 0 & n < N \end{array}.$$

Let B_N be the Banach space of all real sequences $Y = \{y_n\}$, with the norm $\|Y\| = \sup_{n>n_0} |y_n| < \infty$, we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega = \{Y = \{y_n\} \in B_N \;\; kG(n,N) \le y_n \le 2kG(n,N), \;\; n \ge N_*\}.$$

Define the map T: $\Omega \to B_N$ as follows:

$$Ty_n = \sum_N^{n-1}(n-s)[\sum_N^{s-1}(k^\alpha + \sum_{\sigma=r}^{\infty} q_\sigma (y_{\tau(\sigma)})^\beta]^{\frac{1}{\alpha}}, \qquad n \ge N$$
$$Ty_n = Ty_N \qquad N_* \le n < N \tag{9}$$

a) T maps Ω into Ω for $yn \in \Omega$, then for $n \ge N$

$$Ty_n \ge K\sum_N^{n-1}(n-s)(s-N)^{\frac{1}{\alpha}} = kG(n,N).$$

and

$$\begin{aligned} Ty_n &\le \sum_N^{n-1}(n-s)[\sum_N^{s-1}(k^\alpha + \sum_{\sigma=r}^{\infty} q_\sigma (2k\tau(\tau(\sigma),N)^\beta))]^{\frac{1}{\alpha}} \\ &\le \sum_N^{n-1}(n-s)[\sum_N^{s-1}(k^\alpha + (\frac{2k\alpha^2}{(\alpha+1)(2\alpha+1)})^\beta)\sum_{\sigma=r}^{\infty} q_\sigma(\tau(\sigma))^{(2+\frac{1}{\alpha})\beta}]^{\frac{1}{\alpha}} \\ &\le 2k\sum_N^{n-1}(n-s)(s-N)^{\frac{1}{\alpha}} = 2kG(n,N). \end{aligned}$$

b) T is continous. Let $y^{(k)} \in \Omega \in \Omega$ such that $\lim_{k\to\infty}\|y^{(k)} - y\| = 0,$

$$\left|(Ty^{(k)})_n - (Ty)_n\right| = \sum_N^{n-1}(n-s)[\sum_N^{s-1}(k^\alpha + \sum_{\sigma=r}^{\infty} q_\sigma (y^{(k)}_{\tau(\sigma)})^\beta)]^{\frac{1}{\alpha}} - \sum_N^{n-1}(n-s)[\sum_N^{s-1}(k^\alpha + \sum_{\sigma=r}^{\infty} q_\sigma (y_{\tau(\sigma)})^\beta)]^{\frac{1}{\alpha}},$$

by using Lebesgue's dominated convergence theorem, we can conclude that

$$\lim_{k\to\infty}\|Ty^{(k)} - Ty\| = 0.$$

c) T is uniformly-cauchy, $\forall n_1, n_2 > N_*$

$$\begin{aligned} \left|Ty_{n_1} - Ty_{n_2}\right| &= \sum_N^{n_2-1}(n-s)[\sum_N^{s-1}(k^\alpha + \sum_{\sigma=r}^{\infty} q_\sigma (y_{\tau(\sigma)})^\beta)]^{\frac{1}{\alpha}} \\ &- \sum_N^{n_1-1}(n-s)[\sum_N^{s-1}(k^\alpha + \sum_{\sigma=r}^{\infty} q_\sigma (y_{\tau(\sigma)})^\beta)]^{\frac{1}{\alpha}} \\ &= \sum_{n_1}^{n_2-1}(n-s)[\sum_N^{s-1}(k^\alpha + \sum_{\sigma=r}^{\infty} q_\sigma (y_{\tau(\sigma)})^\beta)]^{\frac{1}{\alpha}} \end{aligned}$$

Therefore, by the Schauder fixed point theorem, there exists a fixed $Ty = y$, which satisfies (1). This completes the proof.

Theorem 2. The Equation (1) has a positive of *type* − I_3, if and only if

$$\sum_{n=n_0}^{\infty} nq_n(\tau(n))^{2\beta} < \infty. \tag{10}$$

Proof. Neccessary. Suppose that (1) has a positive of *type* − I_3. Then, it satisfies (4) for $n \ge N$, which implies that $\sum_{n=n_0}^{\infty}(n-N)q_n(\tau(n))^\beta < \infty$.

This together with the asymptotic relation $\lim_{n\to\infty}\frac{y_n}{n^2} = \text{const} > 0$ shows that the condition (8) is satisfied.

Sufficently. Suppose now that (8) holds. Let $k > 0$ be any given constant.

Choose $N > n_0$ large enough so that

$$\sum_{n=N}^{\infty} nq_n(\tau(n))^{2\beta} \le \frac{(2k)^\alpha - k^\alpha}{(2k)^\beta}. \tag{11}$$

Put $N_* = \min\{N, \inf_{n>N}\tau(n)\}$, Let B_N be the Banach space of all real sequences

$Y = \{y_n\}$, with the norm $\|Y\| = \sup_{n>n_0}|y_n| < \infty$, we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega = \left\{Y = \{y_n\} \in B_N \quad \frac{2}{k}(n-N)_+^2 \le y_n \le k(n-N)_+^2, \quad n \ge N_*\right\}$$

where $n - N_+ = n - N$ if $n \ge N$, and $n - N_+ = 0$ if $n \le N$. Define the map T: $\Omega \to B_N$ as follows:

$$Ty_n = \sum_N^{n-1}(n-s)[2k^\alpha - \sum_{r=s}^{\infty}(r-s)q_r(y_{\tau(r)})^\beta]^{\frac{1}{\alpha}}, \qquad n \ge N$$
$$Ty_n = Ty_N \qquad N_* \le n < N$$

The proof is similar to that of theorem 1 and there exists an element y such that $y = Ty$, which is a *type* − I_3 solution of (1) with the property that $\lim_{n\to\infty}\Delta^2 y_n = 2k > 0$, this completes the proof.

Theorem 3 The Equation (1) has a positive of *type* − II_1, if and only if

$$\sum_{n=N}^{\infty}\left[\sum_{s=n}^{\infty}(s-n)q_s(\tau(s))^\beta\right]^{\frac{1}{\alpha}} < \infty. \tag{12}$$

Proof. Neccessary. Suppose that (1) has a positive of *type* − II_1. Then, it satisfies (4) for $n \ge N$, which implies that $\sum_{n=N}^{\infty}(n-N)q_n(y_{\tau(n)})^\beta < \infty.$

This together with the asymptotic relation $\lim_{n\to\infty}\frac{y_n}{n}=\text{const}>0$ shows that the condition (12) is satisfied.

Sufficently. Suppose now that (12) holds. Let $k > 0$ be any given constant.

Choose $N>n_0$ large enough so that

$$\sum_{n=N}^{\infty}\left[\sum_{s=n}^{\infty}(s-n)q_s y_{\tau(s)}{}^{\beta}\right]^{\frac{1}{\alpha}}<2^{\frac{-\beta}{\alpha}}k^{1-\frac{\beta}{\alpha}}.$$

Put $N_*=\min\{N,\inf_{n>N}\tau(n)\}$, Let B_N be the Banach space of all real sequences

$Y = \{y_n\}$, with the norm $\|Y\| = \sup_{n>n_0}|y_n| < \infty$, we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega=\left\{Y=\{y_n\}\in B_N \quad kn\le y_n\le 2kn,\quad n\ge N_*\right\}.$$

Define the map T: $\Omega\to B_N$ as follows:

$$\begin{aligned}&Ty_n=kn+\sum_{N}^{n-1}\sum_{s}^{\infty}[\sum_{r}^{\infty}(\sigma-r)q_\sigma(y_{\tau(\sigma)})^{\beta}]^{\frac{1}{\alpha}} && n\ge N\\ &Ty_n=kn && N_*\le n<N\end{aligned} \tag{13}$$

The proof is similar to that of theorem 1 and there exists an element y such that $y = Ty$, which is a *type* − II_1 solution of (1) with the property that $\lim_{n\to\infty}\Delta y_n=k>0,$ this completes the proof.

Theorem 4 The equation (1) has a positive of *type* −II_3 if and only if

$$\sum_{n=n_0}^{\infty}n\left[\sum_{s=n}^{\infty}(s-n)q_s\right]^{\frac{1}{\alpha}}<\infty\cdot \tag{14}$$

Proof. Neccessary. Suppose that (1) has a positive of *type* −II_3. Then, it satisfies (6) for $n\ge N$, which implies that

$$\sum_{n=N}^{\infty}n\left[\sum_{s=n}^{\infty}(s-n)q_s(y_{\tau(s)})^{\beta}\right]^{\frac{1}{\alpha}}<\infty\cdot \tag{15}$$

This together with the asymptotic relation $\lim_{n\to\infty}y_n=\text{const}>0$ shows that the condition (14) is satisfied.

Sufficently. Suppose now that (14) holds. Let $k > 0$ be any given constant.

Choose $N > n_0$ large enough so that

$$\sum_{n=N}^{\infty}n\left[\sum_{s=n}^{\infty}(s-n)q_s(y_{\tau(s)})^{\beta}\right]^{\frac{1}{\alpha}}<\frac{1}{2}k^{1-\frac{\beta}{\alpha}}. \tag{16}$$

Put $N_*=\min\{N,\inf_{n>N}\tau(n)\}$, Let B_N be the Banach space of all real sequences

$Y = \{y_n\}$, with the norm $\|Y\| = \sup_{n>n_0}|y_n| < \infty$,, we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega=\left\{Y=\{y_n\}\in B_N \quad \frac{k}{2}\le y_n\le k,\quad n\ge N_*\right\}$$

Define the map T: $\Omega\to B_N$ as follows:

$$\begin{aligned}&Ty_n=k-\sum_{n}^{\infty}(s-n)\left[\sum_{r=s}^{\infty}(r-s)q_r(y_{\tau(r)})^{\beta}\right]^{\frac{1}{\alpha}} && n\ge N\\ &Ty_n=Ty_N && N_*\le n<N\end{aligned}. \tag{17}$$

The proof is similar to that of theorem 1 and there exists an element y such that $y = Ty$, which is a *type* − II_3 solution of (1) with the property that $\lim_{n\to\infty}\Delta y_n=k>0,$ this completes the proof.

Theorem 5 The Equation (1) has a positive of *type* − I_2 if

$$\sum_{n=n_0}^{\infty}q_n(\tau(n))^{(2+\frac{1}{\alpha})\beta}\le\infty, \tag{18}$$

and

$$\sum_{n=n_0}^{\infty}nq_n(\tau(n))^{2\beta}=\infty. \tag{19}$$

Proof. Suppose now that (18) holds. Choose $N > n_0$ large enough so that

$$\sum_{n=N}^{\infty}q_n(\tau(n))^{(2+\frac{1}{\alpha})\beta}\le\frac{1}{2^{\alpha+1}}\left(\frac{(\alpha+1)(2\alpha+1)}{\alpha^2}\right)^{\alpha}. \tag{20}$$

Put $N_*=\min\{N,\inf_{n>N}\tau(n)\}$, Let B_N be the Banach space of all real sequences

$Y = \{y_n\}$,with the norm $\|Y\| = \sup_{n>n_0}|y_n| < \infty$, we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega=\left\{Y=\{y_n\}\in B_N \quad \frac{1}{2^{\frac{1}{\alpha}+1}}(n-N)_+^2\le y_n\le n^{2+\frac{1}{\alpha}},\quad n\ge N_*\right\}.$$

Define the map T: $\Omega\to B_N$ as follows:

$$\begin{aligned}&Ty_n=\sum_{N}^{n-1}(n-s)\left[\frac{1}{2}\sum_{N}^{s-1}\sum_{\sigma=r}^{\infty}(\sigma)q_\sigma(y_{\tau(\sigma)})^{\beta}\right]^{\alpha} && n\ge N\\ &Ty_n=0 && N_*\le n<N\end{aligned}. \tag{21}$$

The proof is similar to that of theorem 1 and there exists an element y such that $y = Ty$, which is a *type*−I_3 solution of (1) with the property that $\lim_{n\to\infty}\Delta y_n=k>0,$ this completes the proof.

Theorem 6 The Equation (1) has a positive of *type* − II_2 if

$$\sum_{n}^{\infty} n\left[\sum_{n}^{\infty}(s-n)q_s(\tau(s))^{\beta}\right]^{\frac{1}{\alpha}} < \infty, \tag{22}$$

and

$$\sum_{n=n_0}^{\infty}\left[\sum_{n}^{\infty}(s-n)q_s\right]^{\frac{1}{\alpha}} = \infty\cdot \tag{23}$$

Proof. Suppose now that (22) holds. Choose $N > n_0$ large enough so that

$$\sum_{N}^{\infty} n\left[\sum_{n}^{\infty}(s-n)q_s(\tau(s))^{\beta}\right]^{\frac{1}{\alpha}} \le 2^{\frac{-\beta}{\alpha}} k^{1-\frac{\beta}{\alpha}}. \tag{24}$$

Put $N_* = \min\{N, \inf_{n>N}\tau(n)\}$, Let B_N be the Banach space of all real sequences

$Y = \{y_n\}$, with the norm $||Y|| = \sup_{n>n_0}|y_n| < \infty$, we define a closed, bounded and convex subset Ω of B_N as follows:

$$\Omega = \{Y = \{y_n\} \in B_N \quad k \le y_n \le 2kn, \quad n \ge N_*\}.$$

Define the map T: $\Omega \rightarrow B_N$ as follows:

$$\begin{aligned} Ty_n &= k + \sum_{N}^{n-1}\sum_{s}^{\infty}\left[\sum_{\sigma=r}^{\infty}(\sigma-r)q_\sigma(y_{\tau(\sigma)})^{\beta}\right]^{\frac{1}{\alpha}} && n \ge N \\ Ty_n &= k && N_* \le n < N \end{aligned} \tag{25}$$

The proof is similar to that of theorem 1 and there exists an element y such that $y = Ty$, which is a *type*−II_2 solution of (1). This completes the proof.

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