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The Real Numbers System and Why a Negative Number Times a Negative Number Equals a Positive Number

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Abstract

The original purpose of this paper was to provide answers to the question: "Why is a negative number time a negative number equal a positive number". This concept is one of the most mysterious topics taught in any mathematics classroom. Yet this fundamental mathematical idea is listed in most algebra text books as a rule without any justification for the validity of the rule. While researching this issue it became clear that the decimal place value system, and in particular the real value number system was just as mysterious. Hence the decision was taken to broaden the scope of the paper to include some of the issues associated with the real number system; and to outline some of the topics a mathematics student should be acquainted with.

Key words: Negative and positive numbers; Real numbers; System; Negative number times

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INTRODUCTION

According to Research and Development Institute of Sycamore, Illinois (2006):

> It is believed by neuropsychologists that humans are born with "number sense", or an innate ability to perceive, process, and manipulate numbers. It is an intuitive ability to attach meaning to numbers and number relationships, to understand the magnitude of numbers as well as the relativity of measurement of numbers, and to use logical reasoning for estimation.

The State of Ohio Mathematics Standards -Number, Number Sense and Operations: Standard Number: 13

> Grade: K student should be able to: Recognize the number or quantity of sets up to 5 without counting; e.g., recognize without counting the dot arrangement on a domino as 5.

So according to the biologist and other human beings are born with an innate ability to recognize five objects without counting them. So it thus appears that to be able to count beyond the number five an organized symbolic numbers system is required.

A. What is a Number?

Numbers don't exist physically. They were created for handling real world situations to solve every day problems.

In general a number system consists of a set of symbols used to express quantities as the basis for counting, determining order, comparing quantities, performing calculations, and representing values-Number System-Science Zine (2014). In general, a number system is a set of objects (often numbers), operations, and the rules governing those operations. One example is our familiar real number system, which uses base ten numbers; another example is the binary number system. Thus given a set of characters and some mathematical rules, it is possible to create a new number system. Examples of other number systems include the Arabic, Indian, Babylonian, Chinese, Egyptian, Greek, Mayan, and the Roman number systems.

B. Place value or Positional Decimal System

The primary focus of this section is to provide an informative account of the place value system.

The place value system allows us to construct numbers of any size, and most importantly this system helps us determine the value of a number at a glance.

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According to Miller, Heeren and Hornsby (1989, p.159), in their book Mathematical Ideas; They asserted that in a positional numeral, each symbol (called a digit) conveys two things:

a) **Face value** - the inherent value of the symbol.

b) **Place value** - the power of the base which is associated with the position that the digit occupies in the numeral.

The symbolic number system that is in use today is called the decimal number system.

This counting system is composed of ten symbols called digits.

The ten digits are {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

The names of the digits are listed below:

0: Zero, 1: One, 2: Two, 3: Three, 4: Four, 5: Five, 6: Six, 7: Seven, 8: Eight, 9: Nine.

This number system is called the Hindu-Arabic numeration system. This ten digit number system emerged around the year A.D. 800. It is basically the number system that is widely used today. With these ten digits it is possible to construct members of any size, subject to certain conditions.

The place value rule states that--

a) In the decimal number system, the value of a digit depends on its place or position in the number. Thus each place has a value of 10 times the place to its right.

b) Place value decrease from left to right by a power of 10.

c) The position of each digit in a number is important.

The place or position, of a digit in a number determines the actual value of a number.

The (base-10) number system contains numerals (or digits) only, the digits range from 0 through 9, but we often need to use numbers greater than 9. To construct numbers greater than 9 it is necessary to employ the place value positional algorithm, which means that the value of a digit is determined by its place in the entire number. In the base-10 number system, each place has a value that's 10 times the value of the place immediately to its right. Thus the place value system helps us determine the value of numbers.

Consider the number three hundred and twenty one, this number can be written using the placed or position value system, shown pictorially below, where the digits are used in combination to form the number.

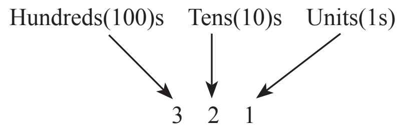

Figure 1
Digits Are Used in Combination to Form the Number

With the Hindu-Arabic number system, ten ones can be replaced by one ten, ten tens can be replaced by one hundred, ten hundreds are replaced by one thousand, 10 one thousand are replaced by 10 thousands, and so on. This scheme is illustrated:

1+1+1+1+1+1+1+1+1+1≡10,
10+10+10+10+10+10+10+10+10+10≡100,
100+100+100+100+100+100+100+100+100+100≡100.

In the case of decimal fraction dot (decimal point) is used to represent decimal numbers. In this scheme, the numbers used in denoting a number take different place values depending upon position , as described above. Decimal numbers or decimal fractions are a proper fraction whose denominator is a power of 10. A decimal point is used to separate the fraction from the integer. Decimal numbers are written without showing denominators. Decimal numbers have three parts: an integer, a decimal point, and a decimal number.

Consider the decimal number 4.25, this number can be represented as shown below:

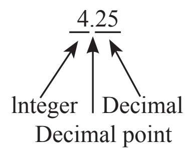

Figure 2
A Representation *f* Decimal Numbers

A decimal number is not necessarily a number with a decimal point in it. For numbers that have a decimal points, numbers are constructed subject to the following conditions:

> One decimal place to the right of the decimal point is the "tenths" place, but one decimal place to the left of the decimal point is the "ones" place. The "tens" place is two places to the left. The decimal point lies between the ones place and the tenths place. Note that all place value positions to the right of the decimal point have a "th" ending.

The figure below shows that number to the left of the decimal place, represent 1's, 10's, 100's, 1000's, and so on; and digits to the right of the decimal point represent 1/10's, 1/100's, 1/1000's, and so forth.

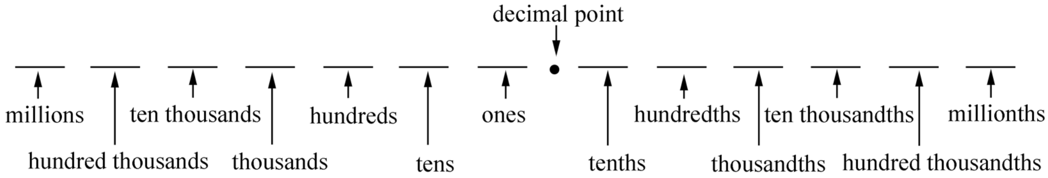

Figure 3
Place Value Chart

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The sequence of numbers 1000, 100, 10, 1, 1/10's, 1/100's, 1/1000's, are shown pictorially:

These numbers can also be written in exponential form as shown below:

$1\times10=10=10^1$ $\quad 0.1=1/10=10^{-1}$

$10\times10=100=10^2$ $\quad 0.01=1/100=10^{-2}$

$10\times10\times10=1000=10^3$ $\quad 0.001=1/1000=10^{-3}$

In the base-10 system the number 178.56 is shown pictorially below:

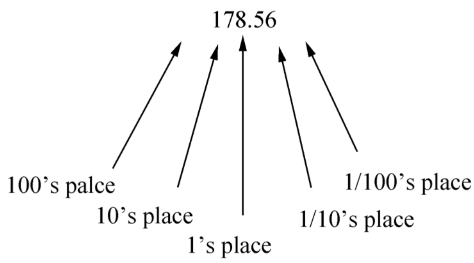

Figure 4
A Representation of Decimal Numbers

According to Wikipedia, the free encyclopedia: Arabic numerals: http://en.wikipedia.org/wiki/Arabic_numerals

> Arabic numerals or Hindu numerals or Hindu-Arabic numerals are the ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9). They are descended from the Hindu-Arabic numeral system developed by Indian mathematicians, in which a sequence of numerals such as "975" is read as a whole number. The Indian numerals were adopted by the Persian mathematicians in India, and passed on to the Arabs further west. They were transmitted to Europe in the Middle Ages. The use of Arabic numerals spread around the world through European trade, books and colonialism. Today they are the most common symbolic representation of numbers in the world.

1. NUMBER SYSTEM CLASSIFICATION

The decimal number scheme is a very complicated system consisting of many subsystems. So in order to make sense of such a complicated construction, it was found necessary to classify the various subsystems according to type. The first type of number is the counting, or "natural" numbers.

1.1 The Natural or Counting Numbers

The first type of number is the counting numbers:

The natural or counting numbers can be represented as follows: $N = \{1, 2, 3, 4, \ldots\}$.

The natural numbers consist of the set of positive non-zero numbers or counting numbers. The set is denoted with the symbol, *N*. There is a first counting number, and for each counting number, there is a next counting number, or a successor. No counting number is its own successor. No counting number has more than one successor. No counting number is the successor of more than one other counting number. Only the number 1 is not the successor of any counting number (Peano axioms for the natural numbers-without proof).

The natural numbers are used to count physical objects in the real world.

The natural numbers system does not support division or negative numbers.

1.2 Whole Numbers

The second type of number is the whole numbers:

$W = \{0, 1, 2, 3, 4, \ldots\}$ the set is denoted with a symbol W. The whole numbers are the natural numbers together with zero. This number system allows us to add and multiply whole numbers. For example the sum of any two whole numbers is also a whole number: 4 + 20 = 24, and the product of any two whole numbers is a whole number ($4\times20 = 80$). This number system does not support subtraction and division.

1.3 The Integers

The third type of number system is the integers:

The integers are the set of numbers consisting of the natural numbers, their additive inverses and zero. This number system is usually symbolized as follows:

$$W=\{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\ldots\}.$$

Thus the sum, product, and difference of any two integers are also integer. However this is not true for division, that is $1 \div 2$ is not an integer.

1.4 The Rational Numbers

The fourth type of number is the rationales:

The rational numbers are those numbers which can be expressed as a ratio between two integers.

$\because = \{ p/q \mid p, q \in Z \text{ and } q \neq 0 \}$ **or equivalently** $Q = Z \times (Z \setminus \{0\})$.

Where p and q are integers; the set of fractions (pairs of integers) with non-zero denominator. with the following relation $\because: p_1/q_1 \square p_2/q_2$ if and only $p_1 q_2 = p_2 q_1$.

The set of rational numbers are ratios of integers (where the divisor is non-zero), such as 2/5, -2/11, etc.. Note that the integers are included among the rational numbers, for example, the integer 3 can be written as 3/1, or even 8/2 . Additional examples of rational numbers are 1/2 -4/7, 6, and 0. Any rational number may be (written as a terminating number, like 0.4 or a repeating decimal number, like 0.3333…). A rational number is a number that can be used to do mathematics: that is calculations, solve equations that do not involve radicals, and used to represent measurements. In general arithmetic operations that involve the sum, product, quotient, and the difference of any two integers are also a rational. However this is not true for irrational, that is, $\sqrt{2}$ is not a rational number.

1.5 The Irrational Number

The fifth type of number is the irrationals:

$L = \{ x \mid$ is a real number, but x cannot be written as a quotient of integers$\}$

An irrational number is a number that cannot be written as a ratio (or fraction). In decimal form, this number system never ends or repeats. The Pythagorean

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discovered that not all numbers are rational; there are equations that cannot be solved using ratios of integers.

It turns out that the rational numbers are not enough to describe the world. Consider, for example, a right triangle whose two short sides are each 1 unit long. Then by the Pythagorean Theorem the longest side of the triangle, the hypotenuse, has a length whose square equals 2. This length is usually referred to as the square root of 2 ($\sqrt{2}$).

Prove that: $\sqrt{2}+\sqrt{3}$ is an irrational number. We will prove the statement, by appealing to the technique of proof by contradiction. Assume that p and q are integers and $p\neq 0$ with no common integer factors, then $\sqrt{2}+\sqrt{3}=\frac{p}{q}$, assume that this fraction is in its simplest form.

Rearrange: $\sqrt{3}=\frac{p}{q}-\sqrt{2}$- squaring both sides:

$$3 = \frac{p^2}{q^2}+2-2\sqrt{2}\,\frac{p}{q} \text{ Rearrange } 2\sqrt{2}\,\frac{p}{q} = \frac{p^2}{q^2}-1$$

$$2\sqrt{2}\,\frac{p}{q} = \frac{p^2}{q^2}-1 \Rightarrow 2\sqrt{2}\,\frac{p}{q} = \frac{p^2-q^2}{q^2} \Rightarrow \sqrt{2} = \frac{p^2-q^2}{2pq}$$

$\frac{p^2-q^2}{2pq}$ is a rational number, because p and q are integers, this implies that $\sqrt{2}$ is a rational number, which is impossible, and so $\sqrt{2}+\sqrt{3}$ is an irrational number.

Thus the sum, product, quotient, rational and the difference of any two irrational is also an irrational number. However this is not true for complex number, that is, numbers of the form, $a+bi$ is not an irrational number.

The union of the natural, whole, integers, rational and irrational is usually referred to as the real numbers, symbolized as follows: R=N È W È ZÈ QÈ L. We will have much more to say about the real numbers in late sections

1.6 The Complex Numbers

The sixth type of number is complex numbers:

In mathematics, a complex number is a number of the form $a+bi$,where $\boldsymbol{a}$ and $\boldsymbol{b}$ are real numbers, and $\boldsymbol{i}$ is the imaginary unit, with the property $i^2=-1$. The real number $\boldsymbol{a}$ is called the real part of the complex number, and the real number $\boldsymbol{b}$ is the imaginary part. Real numbers may be considered to be complex numbers with an imaginary part of zero; that is, the real number a is equivalent to the complex number $\boldsymbol{a}+0\boldsymbol{i}$. The basis number system can be characterized as followed: N Ì W Ì Z Ì QÌ R ÌLÌ C.

1.7a An Ordered Set

The real numbers have the property that they are ordered, which means that given any two different numbers we can say that one number is greater or less than the other number.

A formal way of saying this is: Symbols are used to show how the size of one number compares to another. These symbols are < (less than), > (greater than), and = (equals.)

For any two real numbers a and b, one and only one of the following three statements is true:

a) a is less than b, (expressed as $a < b$),
b) a is equal to b, (expressed as a $=b$),
c) a is greater than b, (expressed as $a > b$).

1.7b Complex Numbers

The field □ of complex numbers is not an ordered field under any ordering.

Proof: Suppose $i>0$. Squaring both sides gives $i^2>0$ or $-1>0$: Adding 1 to both sides of the inequality gives $0>1$; and so we have a contradiction.

1.8 The Real Numbers Line

The geometrical representation of the real numbers on the number line (or the real number line to be mathematical precise), is depicted in Figure 5. This representation allows us to set up a one-to-one correspondence between real numbers and points on the number line. The ordered nature of the real numbers lets us arrange the numbers along a line. The line is made up of an infinite number of points all packed so closely together that they form a solid line. The points are ordered so that points to the right are greater than points to the left:

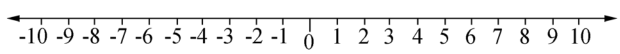

Figure 5
The Real Number Line

Every real number corresponds to a distance on the number line, starting at the center (zero). Negative numbers represent distances to the left of zero, and positive numbers are distances to the right. The arrows on the end of the line indicate that the line keeps on going forever in both directions.

1.9 The Real Numbers

The real number can be represented as follows:

R= { x | x is a number which may be written as a decimal}

□ = {x : $x \in$ □ $\cup L$, □ $\cap L=\varnothing$, where L = {x : $x \notin$ □ , the set of irrational numbers. This set includes all rational and irrational numbers.

The real number system evolved over time by expanding the notion of what we mean by the word "number." At first, "number" meant something one could count, like how many cows a farmer owns. These were called the natural numbers, or sometimes the counting numbers.

The set of real numbers includes all rational and irrationals numbers. The set of real numbers is closed for the operations of addition, subtraction, multiplication and division, and roots of positive numbers. The real number system allows us to carry out mathematical operations involving the number subsystem listed below.

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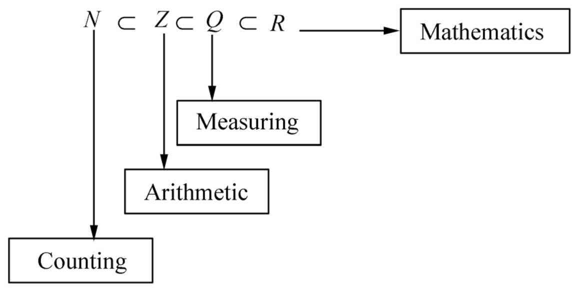

Figure 6
Real Number System

With respect to the rational numbers, odd roots of negative numbers exist, however even roots of negative numbers do not exist.

1.10 Properties of the Real Numbers

We now define the properties of the real number system that allows us to carry out operations such as addition, subtraction, multiplication, and division.

a) Closure property for addition if a and b are real numbers, then $a+ b$ is a unique real number.

b) Closure property for multiplication if a and b are real numbers, then ab is a unique real number.

c) Commutative property of addition if a and b are real numbers, then $a+ b = b+a$.

d) Commutative property of multiplication if a and b are real numbers, then $ab = ba$.

e) Associative property of addition if a, b and c are real numbers, then $(a+ b)+c = a+(b+c)$.

f) Associative property of multiplication if a, b and c are real numbers, then $(ab)c = a(bc)$.

g) Identity property of addition if a is any real number, then $a + 0 = a$.

h) Identity property of multiplication if a is any real number, then $a(1) = a$.

i) Additive inverse property for every real number a, there exists a unique real number a such that $a+(-a)=0$.

j) Multiplication property of zero if a is a real number, then $a(0) = 0$.

k) Multiplication property of negative one if a is a real number, then a $(-1)=-a$.

l) Multiplicative inverse property for every real number a, there exists a unique real number $1/a$ such that $a(1/a) = 1$.

m) Distributive property if $a(b+c)=ab+ac$.

1.11 Absolute Value

The absolute value of a number is the distance from that number to the origin (zero) on the number line. That distance is always given as a non-negative number. If a number is positive (or zero), the absolute value function does nothing to it, for example $|5| =5$. If a number is negative, the absolute value function makes it positive, for example $|-5| = 5$.

Absolute value problems involve inequality can be resolved by rewritten the problem as a combination of inequalities as shown below:

Let a be a positive real number then: $|x| < a$: if and only if $-a < x < a$.

and: $|x| > a$: if and only if $x < -a$ or $x > a$.

The absolute value of a number is the distance the number is from 0 on the real number line. Thus the inequality $|x| < a$ is satisfied by numbers whose distance from 0 is less than a. The set of numbers between $-a$ and a, is depicted below:

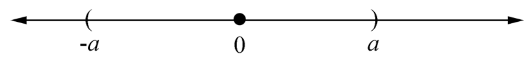

Figure 7
Absolute Value

The inequality $|x| > a$ is satisfied by numbers whose distance from 0 is larger than a. So the solution is for numbers that are either larger than a, or less than $-a$.

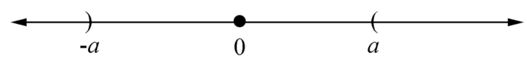

Figure 8
Absolute Value

1.12 The Rational Numbers Field Theorem

This section is partially based on books: Keenan and Gantert (1991, pp.23-25), Integrated Mathematics Course III, and Bumby, Klutch, Collins and Egbers (1996, pp.694 -698); Integrated Mathematics Course II, and Lay (2000, p.95)

A field is a mathematical system consisting of a set F and two operations, usually addition and multiplication, which satisfies eleven properties:

1-5: The set F is a commutative group under the operation of addition, satisfying five properties: closure, associativity, the existence of an identity for addition (usually 0), the existence of inverses under addition, and commutatively.

6-10: The set without the additive identity (usually $F/\{0\}$ is a commutative group under the operation of multiplication, satisfying five properties: closure, associativity, the existence of an identity for multiplication (usually 1), the existence of inverses under multiplication, and commutatively.

11: The second operation, multiplication, is distributive over the first operation, addition.

Definition. $(F, +, \bullet)$ is a field if and only if

a) $(F, +)$ is commutative group,

b) $(F/\{0\})$ is a commutative group,

c) Multiplication distributes over addition.

1.13 Properties of the Rational Numbers

The field axioms are generally written in additive and multiplicative pairs.

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Table 1
The Field Axioms

Names	Addition	Multiplication
Commutatively	$a+b = b+a$	$ab=ba$
Associativity	$(a+b)+c= a+(b+c)$	$(ab)c=a(bc)$
Distributivity	$a(b+c)=ab+ac$	$(a+b)c= ac+bc$
Identity	$a+0= a= 0+a$	$a\bullet 1= a=1\bullet a$
Inverses	$a+(-a) = 0$	$aa^{-1}=1=a^{-1}a$ if $a\neq 0$

2. MATHEMATICAL SYSTEMS

There are many competing definition as to what constitute a mathematical system: the two main candidates are listed below.

2.1 Case 1

This structure is partially due to Occhiogrosso (1992), Reviewing Integrating Mathematics course II.

A mathematical system consists of the following six statements:

a) a set of undefined concepts, or set of elements,

b) a universal set, or set of elements,

c) a set of relations,

d) one or more operations defined for this set,

e) definitions and rules concerning this set and its operations,

f) conclusions concerning this set and its operations, definitions, and rules.

2.2 Case 2

This structure is partially due to Miller, Heeren and Hornsby (1989)

A mathematical system is made up of three things:

a) a set of elements,

b) one or more operations for combining the elements,

c) one or more relations for combining the elements.

Where a set is defined as: A collection of objects. The objects belonging to the set are called the elements, or members, of the set. For example the set of positive integers from 1 to 6: { 1, 2, 3, 4 5, 6}. The ways of combining the elements (called operations) and ways of comparing the elements (called relations). An example of a mathematical system is the set of whole numbers {0, 1, 2, 3, …}, along with the operation of addition and the relation of equality. In mathematics, the concept of a relation or relationship, such as the relation of equality, denoted by the sign "=" in a statement like "4+8=12," or the relation of order, denoted by the sign "<" in a statement like "6 < 12". Relations that involve two objects are called binary relation.

2.3 Equality

From Kaufmann (1992, p.6) an equality is a statement in which two symbols are grouped, or group of symbols, are named for the same number. The symbol = is used to express an equality. From Lay (2000, p.47) a relation R on a set S is an equivalence relation if it has the following properties for all x, y, z in S.

a) Reflexive property — xRx: For any real number a: $a = a$

Example: $5=5$; $x = x$; $a+b = a+b$.

b) Symmetric property –If xRy, then yRx: For any real number a and $b \Rightarrow$ if $a=b$ then $b=a$

Example: If $13+1=14$, then $14=13+1$ and if $3= x+ 2$ then $x + 2=3$.

c) Transitive property If xRy and yRz, then xRz: For any real number a, b, and $c \Rightarrow$ if $a=b$ and $b=c$ then $a=c$.

Example: If $3+4=7$ and $7=5+2$, then $3+4=5+2$, and if $x+1=y$ and $y=5$, then $x+1=5$.

2.4 Quantifiers

To create more complex mathematical statements, use the quantifiers "there exists", written $\exists$, and "for all", written $\forall$. If $P(x)$ is a predicate, then

* \$$x$: $P(x)$ means, "There exists an x such that $P(x)$ holds."
* $\forall x$: $P(x)$ means, "For all x, it is the case that $P(x)$ holds."

For example, if x denotes a real number, then

* \$$x$: $x^2 = 9$ is true, since 3 is an x for which $x^2= 9$.

However "x:$x^2= 9$, is false; not all numbers, when squared, are equal to 9.

* $\forall x$: $x^2+1 > 0$ is true, but $\forall x$: $x^2> 2$ is false, since for example $x = 0.5$ doesn't satisfy the predicate. On the other hand, \$$x$: $x^2> 3$ is true, since $x = 2$ is an example that satisfies it.

3. ARCHIMEDEAN PRINCIPLE FOR THE REAL NUMBERS

Archimedean principle states that the set of real numbers is not bounded above.

For any real number x, there exists an integer n such that $x<n$.

Proof: For any real number $x \in$ □ , $\exists$n, st $n > x$

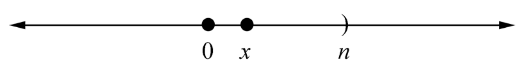

Figure 9
Real Number Line

The scenario is depicted in the figure above. Suppose $x=3.55$, then there exist a natural number, say 5 that is larger than 3.55, that is $5>3.55$.

Suppose the Archimedean Principle is false, and that $x>n$, for all integer n. Then the set of integer □ is bounded above, and by the Completeness Axiom the set has a least upper bound M . Since M is the least upper bound, M-1 cannot be an upper bound. Suppose there is an integer n such that , $M-1<n$. That is, since M is an upper bound, □ has a supermum M. Since ever subset of □ that is bounded above has a supermum. Then for

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M-1, $\exists n \in \mathbb{N}$,st M-1<$n \leq M$. By characterization of supernum(stated not proven). Thus $M < n+1 \in \mathbb{N}$, since M is a supernum of $\mathbb{N}$. This is a contradiction of what we have assumed, and so there exists an integer n such that $x<n$.

4. DENSITY OF THE RATIONAL NUMBERS: NAIVE PROOF

A subset S of rational number ($\mathbb{Q}$) is said to be dense in $\mathbb{R}$ if between any two real numbers a and b, there are a rational number. The density of the rational numbers ($\mathbb{Q}$) in the real numbers means that $\mathbb{R}$ is a closure of $\mathbb{Q}$ in the sense that $\mathbb{R}$ is a set of all limit of the rational numbers. That is, a sequence of rational numbers converges to the real numbers.

The proof is based on Archimedean Principle and on the Completeness Property of the real numbers. If $x < y$, then $\exists\, r \in \mathbb{Q}$ st, $x<r<y$. That is between any two real numbers there exist a rational number. Proof : Assume without loss of generality : $x > 0$ and $y > x$; Then $y\text{-}x > 0$.

By Archimedean Principle $\exists\, n \in \mathbb{N}$ st, $0<\frac{1}{n} < y - x$

$$\Rightarrow ny\text{-}nx>1$$

$$\Rightarrow nx+1<ny \qquad (1)$$

For $nx, \exists m$ st

$$m \leq nx < m+1 \qquad (2)$$

Since $m \leq nx$

$$\text{Then } m+1 \leq nx+1 \qquad (3)$$

Then combining (1), (2) and (3):

$$nx<m+1<nx+1<ny$$

Then $nx<m+1<ny$

$$\Rightarrow x < \frac{m+1}{n} < y$$

Thus $\frac{m+1}{n}$ is a rational number.

Thus between the real number x and y there is a rational $\frac{m+1}{n}$.

5. GROUP THEORY: WHAT IS GROUP THEORY?

Before I answer this question, I am going to solve 2 equations; Problems may be regarded as trivially simple.

Question 1: solve for x: $2x=3$.

Question 2: solve for x: $bx^2=ab$.

Solution 1: Before solving this problem, it is essential that we ask the question, what number system would we like x to represent:

This problem can be solved by dividing both sides of the equation by 2: to give $x=3/2=1.5$

a) If x is a natural number or a negative integer then this problem has no solution.
b) On the other hand if x is a rational number then this problem has a solution.

Solution 2: This problem can be solved by dividing both sides of the equation by b, then taking the square root of both sides of the equation: To give $x=\pm\sqrt{a}$

a) If a is a positive number then the problem has a solution.
b) It a is a negative number then this problem has no solution.

To gain an understanding of equation of the form: $a \cdot x=b$, it will be necessary to consider what types of numbers a , b and x are, and to further consider the type of operation the dot ($\cdot$) symbolized.

With regard to group theory this equation has a unique solution. Group theory does not concern itself with the category to which the variables a, b and x belongs, nor with the operation symbolized by dot ($\cdot$). Thus group theory can handle many different mathematical systems simultaneously; and then finds properties common to all such system subject to four rules called the group axioms:

5.1 Group Axioms

Definition: Let G be a set and let $\bullet$ be a binary operation defined on G. Then (G, $\bullet$) is a group if the following four axioms $G1$-$G4$ holds:

***G*1: Closure:** If g_1 and g_2 are in the group, then $g_1 \cdot g_2$ also belongs to the group.

***G*2: Associativity:** If g_1, g_2 and g_3 are in the group then $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_2)$ also belongs to the group.

***G*3: Identity:** There exist an identity element $e \in G$ such that, for all $g \in G$, $g \cdot e = e \cdot g = e$.

***G*4: Inverse:** For each $g \in G$, there exists an inverse element $g^{-1} \in G$, such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

5.2 Problems Involving Cyclic Subgroup of a Group

I have worked out two problems in great detail that students usually have difficulties understanding:

Problem1: Find the generators for multiplication modulo 7.

Consider the situation: $\{\mathbb{Z}_7/\{0\}, X_7\}=\{1,2,3,4,5,6\}$.

To make sense of this problem, it is best to construct a Caley Table.

Table 2
Caley Table

X_7	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

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This table was completed using modular arithmetic.

For a positive integer n, and integers a and b are said to be *congruent modulo n*, written as:

$a \equiv b(\text{mod } n)$. If the difference between $a - b$ is an integer multiple of n (that is n divides $a - b$). The number n is called the *modulus* of the congruence. Integers congruent to a modulo n form a set called *congruence class*, *residue class* or simply *residue* of the integer a, modulo n.

Table 1 was constructed using modular arithmetic. Column two were constructed as follows:

$2\times1 \text{ (mod 7)} = 2\text{(mod7)} \Rightarrow \frac{2}{7}$=0 remainder 2

$2\times2 \text{ (mod 7)} = 4\text{(mod7)} \Rightarrow \frac{4}{7}$=0 remainder 4

$2\times3 \text{ (mod 7)} = 6\text{(mod7)} \Rightarrow \frac{6}{7}$=0 remainder 6

$2\times4 \text{ (mod 7)} = 8\text{(mod7)} \Rightarrow \frac{8}{7}$=1 remainder 1

$2\times5 \text{ (mod 7)} = 10\text{(mod7)} \Rightarrow \frac{10}{7}$=1 remainder 3

$2\times6 \text{ (mod 7)} = 12\text{(mod7)} \Rightarrow \frac{12}{7}$=1 remainder 5

The rest of the table was completed in a similar manner.

To determine the group theory property associated with Table 1, we must first check to see if this proposed group satisfies the four group axioms:

*G*1: Closure; *G*2: Identity; *G*3: Inverse and *G*4: Associative.

***G*1: Closure.** No new elements were needed to complete the table. So the closure property is satisfied.

***G*2: Identity.** The first row and first column repeat the elements in order. Therefore the identity element is 1.

From the table it can be sees that: $1x_7a=ax_71=a \Rightarrow 1=e$ the identity element.

***G*3: Inverse.** Table 3 was completed with information from Table 1. It can be seen that each element have an inverse.

Table 3
Inverse Values

Element *x*	1	2	3	4	5	6
Inverse *x*-1	1	4	5	2	3	6

***G*4: Associative.** Multiplication is associative.

Since all four properties are satisfied the elements of the table form a group.

The order of the group: The numbers of elements of a group (finite of infinite) is called the order.

The order of a group G is usually symbolized as:$|G|$

The order of the element: The order of an element can be defined as g being an element of G.

$g \in G \Rightarrow g \cdot g \cdots g = e$

$\Rightarrow g^n = e$

And $|g|=n$

Now consider the order of the elements:

The element 1: $1^1=1$ stop when identity is reached.

The element 2: $2^1=2$

$2^2=2x_72=4$: $4\equiv4$(mod7)

$2^3=2x_72=2=8$: $1\equiv7$(mod7) stop when identity is reached.

So the order of the element 2 is 3: That is $2^3=e$(the identity element of the group).

Thus $2^3=e \Rightarrow \{2,4,1\}$

The element 3:$3^1=3$

$3^2=3x_73=9$: $2\equiv9$(mod7)

$3^3=3x_73x_73=27$: $6\equiv27$(mod7)

$3^4=3x_73x_73x_73=81$: $4\equiv81$(mod7)

$3^5=3x_73x_73x_73x_73=243$: $5\equiv243$(mod7)

$3^6=3x_73x_73x_73x_73x_73=729$: $1\equiv729$(mod7) stop when identity is reached.

So the order of the element 3 is 6: That is $3^6=e$ (is the identity element of the group).

The element 4: $4^1=4$

$4^2=4x_74=16$: $2\equiv16$(mod7)

$4^3=64=4x_74x_74x_7=1\equiv64$(mod7) stop when identity is reached.

So the order of the element 4 is 3: That is $4^3=e$ (is the identity element of the group).

The element 5: $5^1=5$

$5^2=5x_75=25$: $4\equiv25$(mod7)

$5^3=5x_75x_75=125$: $2\equiv625$(mod7)

$5^4=5x_75x_75x_75=625$: $2\equiv625$(mod7)

$5^5=5x_75x_75x_75x_75=3125$: $3\equiv3125$(mod7)

$5^6=5x_75x_75x_75x_75x_75=15625$: $1\equiv15625$(mod7) stop when identity is reached.

So the order of the element 5 is 6: That is $6^5=e$ (is the identity element of the group).

The element 5: $6^1=6$

$6^2=6x_76=36$: $1\equiv36$(mod7) stop when identity is reached

So the order of the element 6 is 2: That is $6^2=e$ (is the identity element of the group).

To make sense of what has been done so far, it will be helpful to tabulate the results in a table as follows: Listed in the table are the successive powers of an element until the number one is reached.

From the table it can be seen that the elements 3 and 5 generate all the elements of the group.

Table 4
Order of Elements

Elements: *g*	**Order of the elements: *g*(mod7)**
1^1	{1}
2^3	{2,4,1}
3^6	{3, 2, 6, 4, 5,1}
4^3	{4, 2, 1}
5^6	{ 5, 4, 6, 2, 3, 1}
6^2	{ 6, 1}

It can also be seen that there are only 4 distinct groups: {1}, {2,4,1}, {3, 2, 6, 4, 5,1}, { 6, 1}.

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Elements such as 3 and 5 that generate all the elements in the group are called cyclic subgroup.

A group G is called cyclic if there exists an element g in G such that G=<g>={g^n|n is an integer }. It can be shown that any group generated by an element in a group is a subgroup of that group.

Problem 2

For this exercise the table G is given ***G*=({1, 2, 3, 4, 5, 6, 7, 8}, *)**

a: Identify the identity element of this group?

b: If the group G is cyclic, identify the possible subgroup?

Table 5
Caley Table

Elements: g								
*	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	5	4	7	6	1	8	3
3	3	8	5	2	7	4	1	6
4	4	3	6	5	8	7	2	1
5	5	6	7	8	1	2	3	4
6	6	1	8	3	2	5	4	7
7	7	4	1	6	3	8	5	2
8	8	7	2	1	4	3	6	5

We must first check to see if this group satisfies the four group axioms:

***G*1:Closure.** No new elements were needed to complete the table. So the closure property is satisfied.

***G*2: Identity.** The first row and first column repeat the element in order. Therefore the identity element is 1.

From the table it can be sees that: $1\times a=a\times 1=a \Rightarrow 1=e$ the identity element.

***G*3: Inverse.** Table 6 was completed with information from Table 5. It can be seen that each element has an inverse.

Table 6
Inverses

Element *x*	1	2	3	4	5	6	7	8
Inverse x^{-1}	1	6	7	8	5	2	3	4

***G*4: Associative.**

Multiplication is associative

Since all four properties are satisfied the elements of the table form a group.

Solution: part (*a*)

Since $1\times 1=1 \Rightarrow 1x\ x=x$ and $x\times 1=x$, therefore the identity element is 1.

Part *b*: According to John Farleigh (2003, p.59) in his book, A first Course in Abstract Algebra.

A cyclic subgroup<g> generated by element g can be defined as follows:

H=<g>={g^n|$n\in$□}.

We now carry out the following test to find subgroups generated by the elements of the group G:

For element 1 as generator: <1>=1×1=1

The subgroup generated by the element <1>={1}

For element 2 as generator: $2^1=2$

$2^2=2\times 2=5$

$2^3=2\times 5=6$

$2^4=2\times 6=1$ Identity found:

Thus subgroup generated by the element <2>={2, 5, 6, 1}

Note that $2\times 6=1 \Rightarrow 6^{-1}=2$ so the subgroup generated by the element <6>≡<2>

Thus subgroup generated by the elements <2>=<6>={2, 5, 6, 1 }

For element 3 as generator : $3^1=3$

$3^2=3\times 3=5$

$3^3=3\times 5=7$

$3^4=3\times 7=1$ Identity found

Note that $3\times 7=1 \Rightarrow 7^{-1}=3$ so the subgroup generated by the element <7>≡<3>

Thus subgroup generated by the elements <3>=<7>={3, 5, 7, 1 }

For element 4 as generator $4^1=4$

$4^2=4\times 4=5$

$4^3=4\times 5=8$

$4^4=4\times 8=1$ Identity found.

Note that $4\times 8=1 \Rightarrow 8^{-1}4$ so the subgroup generated by the elements <8>≡<4>

The subgroup generated by the elements <4>=<8>={4, 5, 8, 1 }

For element 5 as generator: $5^1=5$

$5^2=5\times 5=1$ Identity fount

The subgroup generated by the element: <5>={5, 1 }

Listed in the table are the successive powers of an element until the number one is reached.

Table 7
Order of Elements

Elements:	**Order of the elements**
1^1	{1}
2^4	{2, 5, 6, 1 }
3^4	{3, 5, 7, 1 }
4^4	{4, 5, 8, 1 }
5^2	{5, 1 }
6^4	{2, 5, 6, 1 }
7^4	{3, 5, 7, 1 }
8^4	{4, 5, 8, 1 }

Since none of the eight elements generated the entire group, it is therefore clear that the group is not cyclic. That is since |G|=8 and |<g>|={1,2,4}

Therefore $G\neq$<g>That is the group is not cyclic.

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5.3 Proof of Subgroup of Cyclic Group Is Cyclic

This proof is based on Farleigh book (2003, p.61), A first Course in Abstract Algebra, and Pinter (1990, pp.114-115) book, A book of Abstract Algebra:

Let G be a cyclic group, that is G has a generator, let's call this generator g. This means that

$$G=<g>=\{g^n|n\in\mathbb{Z}\}.$$

Let H be a subgroup of G , this can be written as: $H\leq G$. We want to prove that H is cyclic.

Since G is cyclic, it therefore has a generator g; likewise since H is assumed to be cyclic , H also has a generator b, and so H consist of all the powers of b.

To prove that H is cyclic it is necessary to find the generator of $H$and then the check that every element of H is a power of this generator.

We know that $G=<g>$ and $H\leq G$, and $H\in G \Rightarrow$ so every element of H is some power of g.

Now if $H=\{e\}$, then H is cyclic. Then $g^n\in H$ for some $n\in\mathbb{Z}^+$. Let m be the smallest integer $\mathbb{Z}^+$ such that $g^m\in H$. We want to show that every element of H is a power of g^m , hence g^m is a generator of H. We claim that $c=g^m$generates H: that is $H=\{g^m\}=<c>$.

We now want to show that every $b\in H$ is a power of c. Since $b\in H$ and $H\leq G$, this implies that $b=g^n$ for some n. If m is a positive integer, and n is any integer, then there exist unique integer q and r such that : $n=mq+r$ for $0\leq r<m$: where $\boldsymbol{r}$ is the remainder, q is the quotient. This relationship is called the Division Algorithm.

Since $b\in H$ and $H\leq G$ so $b=g^n$ for some $n\in\mathbb{Z}$. Using the Division Algorithm

$n=mq+r$ for $0\leq r<m$ we can write

$b=g^n$ as $b=g^{n=mq+r}=(g^m)^p g^r$,

so $g^n=(g^m)^p g^r \Rightarrow g^r=(g^m)^{-p} g^n$.

Since $g^n\in H$ and $g^m\in H$, and H is a group, it therefore follows that both $(g^m)^{-q}$ and g^n are in H. It follows that $(g^m)^{-p}g^n\in H$, and so $g^r\in H$.

Since m was the smallest positive integer such that $g^m\in H$ and $0\leq r<m$, then it must be that $r=0$, and then $n=mq$, and so: $b=g^n(g^m)^p=c^q$. It therefore follows that b is a power of c.

6. ZERO FACTORS

If $ab=0$, then either $a=0$ or b=0

Proof by contradiction: Assume that both a and b are nonzero. That is a must have multiplicative inverse, that is a^{-1}. Then multiply both of $ab=0$ by a^{-1}.

To give $a^{-1}ab=a^{-1}0$. Then $b=0$.

Since $b=0$, this is a contradiction, and the assumption was false. It follows that both a and b cannot be nonzero, or at least one of them is zero.

7. THE ZERO CASES: DIVISION BY ZERO

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are real numbers, then there are two cases to consider for the ratio of the two numbers $\frac{a}{b}$, namely when $b=0$, then $x=\frac{a}{0}$, then i.f.f x is a unique real number such that $a=0x$, then by the multiplication property of real numbers, $0\,x=0$, therefore, $a=0=0x$

Since $a=0$ is impossible, this contradicts the assumption that $b\neq 0$, so $\frac{a}{0}$ is undefined.

We can therefore say that dividing a real number by zero is undefined. The other case under consideration is when both $a=b=0$, that is when $x=\frac{0}{0}$.

Given that $x=\frac{0}{0}$, then i.f.f x is a unique real number such that $\mathbf{0=0}\boldsymbol{x}$. Now by the zero multiplication property of real numbers; $0=0x$ for all values of x. Therefore x is not unique and so $\frac{0}{0}$ is undefined, since for example if $x=$ 1, 2, 3… then

$0=0\times 1=0$,
$0=0\times 2=0$,
$0=0\times 3=0$,

since the value of x is not unique.

8. BRIEF HISTORY OF NEGATIVE NUMBERS

This section of the paper provides a brief historical account of negative numbers. According to numerous sources, among them Rajapakse (2012), who claimed that the concept of negative numbers was first introduced by the Indian mathematician Brahmagupta (c. 598–c. 670); and that Brahmagupta also devised the four rules of arithmetic (addition, subtraction, multiplication and division) using real numbers. In addition Brahmagupta **i**ntroduced many fundamental concepts to basic mathematics, including the use of zero in the decimal number system, and the use of algebra to describe and predicting astronomical events.

There is disagreement among mathematics historians as to who first introduced the concept of negative number into mathematics. According to a listing in Wikipedia, Struik (1987, cited in Wikipedia):

> Negative numbers appear for the first time in history in the Nine Chapters on the Mathematical Art (Jiu zhang suan-shu), which in its present form dates from the period of the Han Dynasty (202 BC – AD 220).

According to Frey (2012) and others, Fibonacci and Cardano were both accredited with introducing the concept of negative numbers to Europe. *Frey* and others claimed that Fibonacci's book *Liber Abaci* contained

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problems with negative solutions, interpreted as debts (13th century); and that Cardano's *Ars Magna* included negative solutions of equations and also had basic laws of operating with negative numbers (16th century).

It took mathematicians centuries to accept the notion of negative numbers, where the concept was regarded as "absurd" or "fictitious." Consider the paper by Rogers (1997), who asserted that:

> Although the first set of rules for dealing with negative numbers was stated in the 7th century by the Indian mathematician Brahmagupta, it is surprising that in 1758 the British mathematician Francis Maseres was claiming that negative numbers
>
> ... darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple.

This part of the paper is divided into two sections. The first section is entitled Graphical Method. In this section, several graphical techniques were used to show that a negative times a negative equals a positive, and for the second section entitled Algebraic Method several algebraic techniques were used to show that (-1)*(-1) =1.

8.1 Graphical Method -Section 1a

Probable, the simplest way to show that (-1)*(-1) =1, is to appeal to the method of opposite of a real number. The method of opposite asserts that for a real number: If two numbers are the same distance from zero on the number line, then for real number *a*; *-a* is the additive inverse of *a*. To demonstrate this method, consider the following situations:

Number	**Opposite**
1	**-1**
-1	**-1(-1) =1**

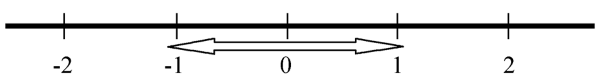

Figure 10
Real Number Line

Since -1 is a negative number, |-1|=-(-1)=1, given that the additive inverse of -1 is +1

It can also be seen from Figure 9, that -1 is opposite to +1, and that +1 is opposite to -1.

This demonstration shows that, the opposite of the negative of a number is that number.

8.2 Section 1b

Consider the following sequence of numbers: -16, -12, -8, -4, 0, 4, 8, 12, 16…..

This sequence of numbers can be rewritten in the form of a table, and on the number line as follows:

4(-4)=-16
3(-4)= -12
2(-4)= - 8
1(-4)= -4
0(-4)= 0
-1(-4)= 4
-2(-4)= 8
-3(-4)= 12
-4(-4)= 16

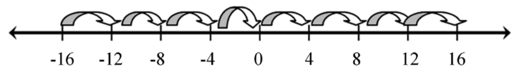

Figure 11
Real Number Line

This type of pattern recognition reasoning suggests that the product of two negative real numbers is a positive real number.

This result can be generalized by incorporate the concept of absolute values to describe multiplication as follows:

a) The product of two positive real numbers or two negative real numbers is the product of their absolute values.

b) The product of a positive and a negative number (order not important) is the opposite of the product of their absolute values.

c) The product of a zero and any real number is zero.

The following examples illustrate the concept of multiplication:

a) (-1)(-1)=|-1|·|-1|=1·1=1,
b) (1)(-1)=-(|1|·|-1|)=-(1·1)=-1,
c) (-1)(1)=-(|-1|·|1|)=-(1·1)=-1,
d) (-1)(0)=0 and (0)(-1)=0.

8.3 Section 1c

The sequence of numbers from section 1b: -16, -12, -8, -4, 0, 4, 8, 12, 16… can be presented in tabular form as shown in Table 9, below:

The constant factor of 4 in Table 9, can be replaced with the real variable *k* (or *k* = 4) as shown in Table 10.

The coefficient of *k* can be replaced with the real variable ***x***: -4, -3, -2, -1, 0, 1, 2, 3, 4,… as shown in Table 9.

Table 8
Sequence of Numbers in Tabular Form

-16	-12	-8	-4	0	4	8	12	16	…
(-4)(4)	(-3)(4)	(-2)(4)	(-1)(4)	(0)(4)	(1)(4)	(2)(4)	(3)(4)	(4)(4)	…..

Table 9
The Coefficient of *k*

-4*k*	-3*k*	-2*k*	-1*k*	0*k*	1*k*	2*k*	3*k*	4*k*	….

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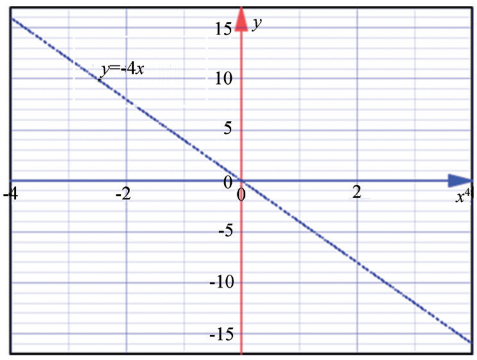
Table 10
Rearrangement of Tables 8 and 9

x	-4	-3	-2	-1	0	1	2	3	4
***k*=4**									
y*= - *kx	16	12	8	4	0	-4	-8	-12	-16

The constant multiplier of ***k*=4** represents the slope of the line in the ***x-y*** plane passing through the origin; the array of numbers can be represented by the formula ***y* = - *kx*** or ***y* = - 4*x***. A plot of this situation is given in Figure 11.



y* =- 4*x
if *x*= -1
theny =- 4*x*
***y*=- 4(-1)=4**

Figure 12
Graphical Representation

This result suggests that the product of two negative real numbers is a positive real number.

Section 2: Using algebraic method to show that (-1)*(-1) =1.

Using the properties of the real numbers to show that **(-1)*(-1) =1.**

a) $\forall a \in$ □, $\exists$ an element $-a$ such that $a+(-a)=0$, additive inverse.

b) Distributive property, $a(b+c)=ab+ac$.

Please see Appendix A and B, for an explanation of Ordered Fields and Equality.

Example 1

$$\text{If}(-1)(-1)=1. \quad (1)$$

Show that the left and right hand side of equation 1 is equal to zero.

Then (-1)(-1)-1=0.

Let -1= +(-1)(1) (-1)(-1)+(-1)(1)=0 Multiplicative Property.

Factoring out (-1) (-1)[(-1)+(1)]=0 Distributive Property.

Assume (-1)+(1)=0 Additive inverse property.

It thus follows that (-1)[0]=0.

0=0 Multiplicative property of zero.

Result as required 0=0 Reflective property of equality.

Example 2

Show that (-1)(-1) =1, using the distributive and the additive inverse property of the real numbers. Assume that the real number properties are true, for $a(b+c)=ab+ac$.

Given that $a=-1$, $b=1$, $c=-1$ and $1+(-1)=0$

Using distributive property $a(b+c) = ab + ac$

Substitution (-1)[1+(-1)]=(-1)(1)+(-1)(-1) (2)

Additive inverse on the left hand side (-1)[0] = -1 + (-1)(-1)

0 = -1 + (-1)(-1), adding 1 to both sides

Required result 1=(-1)(-1)

Next consider the case where (-1)(-1) ≠1.

Proof by contradiction

Assume that $a=-1$, $b=1$, $c=-1$ and (-1)(-1) = -1

Using distributive property: $a(b+c)=ab+ac$.

Substitution -1[1+ (-1)] = (-1)(1)+(-1)(-1) (3)

Additive inverse -1[0]=-1-1

Result 0=-2, contradiction

It there follows that 1 = (-1)(-1).

General Proof 1, show that $(-a)(-b)=ab$.

This proof relies on the distributional properties, and the additive inverse property of the real numbers.

Let a, b and x be any real numbers.

Consider the number x defined as follows:

$$x=ab + (-a)(b)+(-a)(-b). \quad (4)$$

Factor out ***- a*** from eqn.(4) as follows:

$$x=ab + (-a)[(b) + (-b)].$$

Additive inverse

$$x=ab+(-a)[0], \text{ since}(b)+(-b)=0.$$

Thus $x=ab$.

Factor out b from eqn.(4) as follows:

$$x=[a+(-a)](b) + (-a)(-b).$$

Additive inverse

$$x=[0](b)+(-a)(-b), \text{ since } a+(-a)=0.$$

Thus $x= (-a)(-b)$.

Since $ab=x$ and $x=(-a)(-b)$.

Transitivity of equality property $ab=(-a)(-b)$.

General Proof 2, show that (-1)(-*a*)=*a*.

$(-a)+(-1)(-a)=0= a+(-a)$		Axiom 5
$(-a)+(-1)(-a)=(-a)+a$		Axiom 2
$a+[(-a)+(-1)(-a)]=a+[(-a)+a]$		Axiom 1
$[a+(-a)]+[(-1)(-a)]=[a+(-a)]+a$		Axiom 3
$0+(-1)(-a)=0+a$		Axiom 5
$(-1)(-a)+0=a+0$		Axiom 2
$(-1)(-a)=a$	Result as required	Axiom 4

CONCLUSION

The main goal of this paper is to introduce the readers to techniques and ideas associated with the real number system and the notion of (-1)(-1) =1. These concepts were introduced in a concrete and elementary way to allow for a wide readership. It is my fervent desire for anyone from a motivated high school student interested in mathematics to college students specializing in mathematics; that they may find these topics sufficiently intriguing that they will want to carry out further research on these topics.

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